7 Analysis of Axisymmetrically Loaded Members:

7.1 Introduction:

There are a lot of practical situations in which the distribution of stress manifests symmetry about an axis. Examples include pressure vessels, compound cylinders, chemical reaction vessels, heat exchanger tubes, flywheels, solid or hollow spherical structures, and turbine disks.

Consider an infinite thin plate having a small circular hole subjected to uniform pressure, as shown in the following figure,

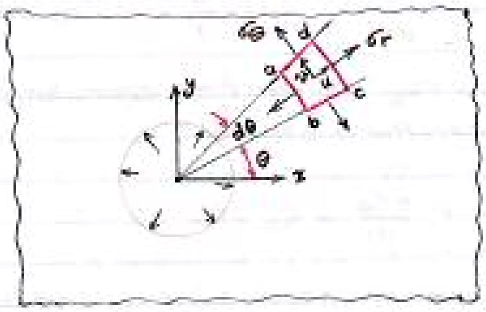


Fig.(1) Infinite thin plate having small circular hole.

Note that axial loading is absent and therefore $\sigma_z = 0$. The stresses are clearly symmetrical about the z axis, and the deformations likewise display θ independence. The symmetry argument also dictates that the shearing stresses $\tau_{r\theta}$ must be zero.
Assuming z independence for this thin plate, the polar equation of equilibrium, reduce to,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0 \qquad (7.1)$$

Here σ_θ and σ_r denote the tangential (circumferential or hoop) and radial stresses acting normal to the side of the element, and F_r represents the radial body force per unit volume. In the absence of body forces, Eqn (7.1) reduces to,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \qquad (7.2)$$

Consider now the radial and tangential displacements, u and v, respectively. There can be no tangential displacement in the symmetrical field, i.e., $v = 0$. A point represented by the element abcd in the figure will thus move radially only as a consequence of loading. On the basis of displacements indicated, the strains given by,

$$\epsilon_r = \frac{du}{dr}, \quad \epsilon_\theta = \frac{u}{r}, \quad \gamma_{r\theta} = 0 \qquad (7.3)$$

Subsituting $u = r\epsilon_\theta$ into the first expression above, a simple compatibility equation is obtained,

$$r\frac{d\epsilon_\theta}{dr} + \epsilon_\theta - \epsilon_r = 0 \qquad (7.4)$$

The equation of equilibrium, and the compatibility relations, and Hooke's Law are sufficient to obtain a unique solution to any axisymmetrical problem with specified boundary conditions.

7.2 Thick-Walled Cylinders.

The circular cylinder is usually divided into thin-walled and thick-walled classifications. A thin-walled cylinder is defined as one in which the tangential stress may be regarded as constant with thickness. The following familiar expression applies to the case of a thin-walled cylinder subjected to internal pressure,

$$\sigma_\theta = \frac{Pr}{t}$$

Here P is the internal pressure, r the mean radius, and t the thickness.

If the wall thickness exceeds the inner radius by more than approximately 10%, the cylinder is generally classified as thick-walled, and the variation of stress with radius can no longer be disregarded.

In the case of a thick-walled cylinder subject to uniform internal or external pressure, the deformation is symmetrical about the z axis. Therefore the equilibrium and strain-displacement relations, apply to any point on a ring of unit length cut from the cylinder.

Assuming that the ends of cylinder are unconstrained, $\sigma_z = 0$, as shall be subsequently demonstrated. Thus according to Hook's law, the strains are given by,

$$\frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta)$$

$$\frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \qquad (7.5)$$

from which σ_r, and σ_θ are as follows,

$$\sigma_r = \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r) = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \qquad (7.6)$$

Substituting the above into Eqn (7.2) results in the following equidimensional equation in radial displacement,

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0 \qquad (7.7)$$

Which having a solution,

$$u = C_1 r + \frac{C_2}{r} \qquad (a)$$

The radial and tangential stresses may now be written in terms of the constants of integration C_1 and C_2 by combining Eqs (a) and (7-6),

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right] \qquad (b)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right] \qquad (c)$$

It is observed that the sum of radial and tangential stresses is constant, regardless of radial position, i.e.,

$$\sigma_r + \sigma_\theta = 2EC_1/(1-\nu)$$

Hence, the longitudinal strain is constant, i.e.,

$$\epsilon_z = -\frac{\nu}{E}(\sigma_r + \sigma_\theta) = \text{Constant}$$

Therefore the plane sections remain plane subsequent to loading. Then $\sigma_z = E\epsilon_z$ = Constant = C. Consider the following figure,

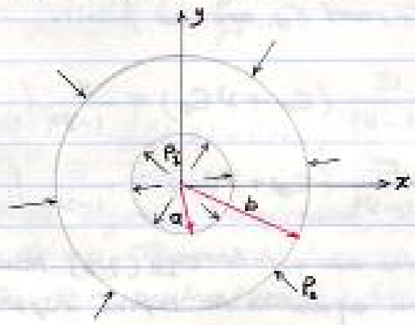


Fig.(2) Thick-walled cylinder with internal and external Pressure loading.

If the ends of the cylinder are free, then,

$$\int_a^b \sigma_z \cdot 2\pi r\,dr = \pi C\,(b^2 - a^2) = 0$$

or $C = \sigma_z = 0$, as already assumed above.

For a cylinder subjected to internal and external pressures P_i and P_o, respectively, the boundary conditions are,

$$(\sigma_r)_{r=a} = -P_i\,, \quad (\sigma_r)_{r=b} = -P_o \qquad (d)$$

Where the negative sign denotes compressive stress. The constants are evaluated by substitution of Eqns (d) into (b),

$$C_1 = \frac{1-\nu}{E}\cdot\frac{a^2P_i - b^2P_o}{b^2-a^2}\,, \quad C_2 = \frac{1+\nu}{E}\cdot\frac{a^2b^2(P_i-P_o)}{b^2-a^2} \qquad (e)$$

leading finally to,

$$\sigma_r = \frac{a^2P_i - b^2P_o}{b^2-a^2} - \frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r^2} \qquad (7.8)a$$

$$\sigma_\theta = \frac{a^2P_i - b^2P_o}{b^2-a^2} + \frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r^2} \qquad (7.8)b$$

$$u = \frac{1-\nu}{E}\cdot\frac{(a^2P_i - b^2P_o)r}{b^2-a^2} + \frac{1+\nu}{E}\cdot\frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r} \qquad (7.8)c$$

These expressions were first derived by Lamé, for whom they are named.

Recall that the maximum shearing stress at any point equals one-half the algebraic difference between the maximum and minimum principal stresses. At any point in the cylinder, we may state that,

$$\tau_{max} = \frac{1}{2}(\sigma_\theta - \sigma_r) = \frac{(P_i-P_o)a^2b^2}{(b^2-a^2)r^2} \qquad (7.9)a$$

The largest value of τ_{max} is found at $r=a$, the inner surface. The effect of reducing P_o is clearly to increase τ_{max}. Consequently, the greatest τ_{max} corresponds to $r=a$ and $P_o=0$,

$$\tau_{max} = \frac{P_i b^2}{b^2-a^2} \qquad (7.9)\,b$$

Because σ_r and σ_θ are principal stresses, τ_{max} occurs on planes making an angle of 45° with the plane on which σ_r and σ_θ act. This is quickly confirmed by a Mohr's circle construction.

Special Cases:

(a) Internal pressure only:

If only internal pressure acts, Eqs (7.8) reduce to,

$$\sigma_r = \frac{a^2 P_i}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right) \qquad (7.10)$$

$$\sigma_\theta = \frac{a^2 P_i}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right) \qquad (7.11)$$

$$u = \frac{a^2 P_i r}{E(b^2-a^2)}\left[(1-\nu)+(1+\nu)\frac{b^2}{r^2}\right] \qquad (7.12)$$

Since $b^2/r^2 \geq 1$, σ_r is negative (compressive) for all except $r=b$, in which case $\sigma_r=0$. The maximum radial stress occurs at $r=a$. As for σ_θ, it is positive (tensile) for all radii, and also has a maximum at $r=a$.

(b) External pressure only:

In this case, $P_i=0$, and the equations (7.8) become,

$$\sigma_r = -\frac{P_o b^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right) \qquad (7\text{-}13)$$

$$\sigma_\theta = -\frac{P_o b^2}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) \qquad (7\text{-}14)$$

$$u = -\frac{b^2 P_o r}{E(b^2-a^2)}\left[(1-\nu)+(1+\nu)\frac{a^2}{r^2}\right] \qquad (7\text{-}15)$$

The maximum radial stress occurs at $r=b$ and is compressive for all r. The maximum σ_θ is found at $r=a$ and is likewise compressive.

Example (7-1):

A thick-walled cylinder with 0.3 m and 0.4 m internal and external diameters is fabricated of a material whose elastic limit is 250 MPa. Let $\nu = 0.3$. Determine (a) for $P_o=0$, the maximum internal pressure to which the cylinder may be subjected without exceeding the elastic limit, (b) for $P_i=0$, the maximum external pressure to which the cylinder can be subjected without exceeding the elastic limit, and (c) the radial displacement of a point on the inner surface for case (a).

Solution:

(a) From Equ (7-11), with $r=a$,

$$\boxed{\sigma_{\theta,max} = P_i\,\frac{b^2+a^2}{b^2-a^2}} \qquad (7\text{-}16)$$

$$\text{or} \quad P_i = \sigma_{\theta,max}\,\frac{b^2-a^2}{b^2+a^2} = (250\times10^6)\,\frac{0.2^2-0.15^2}{0.2^2+0.15^2} = 70 \text{ MPa}$$

(b) From Equ (7-14), with $r=a$,

$$\boxed{\sigma_{\theta,max} = -2P_o\,\frac{b^2}{b^2-a^2}} \qquad (7\text{-}17)$$

Then,

$$P_o = -\sigma_{\theta,max} \frac{b^2-a^2}{2b^2} = -(-250\times10^6)\frac{0.2^2-0.15^2}{2\times0.2^2} = 54.7 \text{ MPa}$$

(C) using Eqn (7-12), we obtain,

$$(u)_{r=a} = \frac{0.15^3\times70\times10^6}{E(0.2^2-0.15^2)}\left[0.7+1.3\frac{0.2^2}{0.15^2}\right] = 4.065\times10^7/E \quad m.$$

Closed-ended Cylinder:

In the case of a closed-ended cylinder subjected to internal and external pressures, longitudinal or z-direction stresses exist in addition to the radial and tangential stresses.
For a transvers section some distance from the ends, this stress may be assumed uniformly distributed over the wall thickness.
The magnitude of σ_z is then determined by equating the net force acting on an end attributable to pressure loading, to the internal z-direction force in the cylinder wall,

$$P_i \pi a^2 - P_o \pi b^2 = (\pi b^2 - \pi a^2)\sigma_z$$

The resulting expression for longitudinal stress, applicable only away form the ends, is,

$$\sigma_z = \frac{P_i a^2 - P_o b^2}{b^2-a^2} \qquad (7.18)$$

Clearly, here it is again assumed that the ends of cylinder are not constrained, $\epsilon_z \neq 0$.

For a situation in which the maximum tangential stress occurs at $r=b$. Consider a thick-walled cylinder, subjected to P_i and P_o. Denote the ratio b/a by R, P_o/P_i by P, and the ratio of tangential stress at the inner and outer surfaces by S. The tangential stress, given by Eqn (7.8), is written,

$$\sigma_\theta = P_i \frac{1-PR^2}{R^2-1} + P_i b^2 \frac{1-P}{(R^2-1)r^2} \qquad (7\text{-}19)$$

Hence,

$$S = \frac{\sigma_{\theta i}}{\sigma_{\theta o}} = \left[1 + \frac{R^2(1-P)}{1-PR^2}\right] \Big/ \left[1 + \frac{1-P}{1-PR^2}\right] \qquad (7\text{-}20)$$

7.3 Application of Failure Theories:

Unless one is content to grossly overdesign, it is necessary to predict, as best possible, the most probable failure mechanism. To do this, consideration must be given the stresses determined from Lame's equations, the material strength, and an appropriate theory of failure consistent with the nature of the material.

Example (7.2):

A steel cylinder is subjected to an internal pressure four times greater than the external pressure. The tensile elastic strength of the steel is $\sigma_y = 340$ MPa, and the shearing elastic strength $\tau_y = \sigma_y/2 = 170$ MPa. Calculate the allowable internal pressure according to the various theories of failure. The dimensions are $a = 0.1$ m, $b = 0.15$ m, and take $\nu = 0.3$.

Solution:

The maximum stresses occur at the innermost fibers. From Eqns (7.8), for $r = a$ and $P_i = 4P_o$, we have,

$$\sigma_\theta = \frac{P_i(a^2+b^2) - 2P_o b^2}{b^2-a^2} = 1.7P_i \;, \quad \sigma_r = -P_i \qquad (a)$$

The value of internal pressure at which yielding begins is predicted according to the various theories of failure, as follows,

(a) Maximum principal stress theory:

$$1.7 P_i = 340 \times 10^6 \quad , \quad P_i = 200 \text{ MPa}$$

(b) Maximum shearing stress theory:

$$\frac{\sigma_\theta - \sigma_r}{2} = 1.35 P_i = 170 \times 10^6 , \quad P_i = 125.9 \text{ MPa}$$

(c) Energy of distortion theory:

$$\sigma_y = (\sigma_\theta^2 + \sigma_r^2 - \sigma_\theta \sigma_r)^{1/2} = P_i [(1.7)^2 + (-1)^2 - (-1.7)]^{1/2}$$

$$340 \times 10^6 = 2.364 P_i , \quad P_i = 143.8 \text{ MPa}$$

(d) Maximum principal strain theory:

$$\frac{1}{E}(\sigma_\theta - \nu \sigma_r) = \frac{340 \times 10^6}{E} = \frac{2}{E} P_i \quad , \quad P_i = 170 \text{ MPa}$$

(e) Octahedral shearing stress theory:

$$\frac{\sqrt{2}}{3} \sigma_y = \frac{1}{3} \left[(\sigma_\theta - \sigma_r)^2 + (\sigma_r)^2 + (-\sigma_\theta)^2 \right]^{1/2}$$

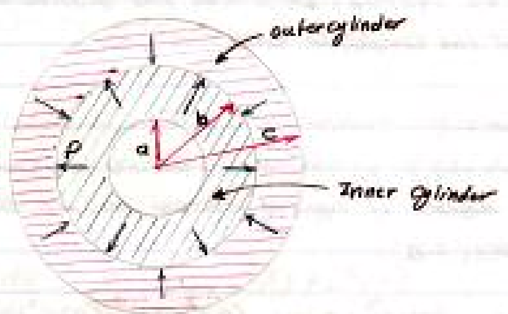
$$\frac{\sqrt{2}}{3}(340 \times 10^6) = \frac{P_i}{3} \left[(2.7)^2 + (-1)^2 + (-1.7)^2 \right]^{1/2} , \quad P_i = 143.8 \text{ MPa}$$

The result found in (c) and (e) are identical as expected.

As the cylinder is made of a ductile material, the onset of inelastic action is governed by the maximum shearing stress. The allowable value of internal pressure is therefore limited to 125.9 MPa, modified by an appropriate factor of safety.

7.4 Compound Cylinders:

If properly designed, a system of multiple cylinders resists relatively large pressures more efficiently, that is, requires less material, than a single cylinder. The cylinders are assembled after the outer cylinder is heated, contact being effected upon cooling. The magnitude of the resulting contact pressure P between members may be calculated by use of the equations developed befor.



Referring to the figure above, assume the external radius of the inner cylinder to be larger, in its unstressed state, than the internal radius of the jacket, by an amount δ. This quantity is the shrinking allowance. Subsequent to assembly, the contact pressure, acting equally on both members, causes the sum of the increase in inner radius of the jacket and decrease in outer radius of the inner member to exactly equal δ. By using Eqns (7.12) and (7.15), we obtain

$$\delta = \frac{bP}{E}\left(\frac{b^2+c^2}{c^2-b^2}+\nu\right) + \frac{bP}{E}\left(\frac{a^2+b^2}{b^2-a^2}-\nu\right) \qquad (7.21)$$

From the above expression, the contact pressure is then found as a function of the shrinking allowance,

$$P = \frac{E\delta}{b}\cdot\frac{(b^2-a^2)(c^2-b^2)}{2b^2(c^2-a^2)} \qquad (7.22)$$

The stresses in the jacket are then determined from Eqns (7.10) and (7.11) by treating the contact pressure as P_i. Similarly, by regarding the contact pressure as P_o, the stresses in the inner cylinder are calculated from Eqns (7.13) and (7.14).

Example (7.3):

A compound cylinder with $a = 150$ mm, $b = 200$ mm, $c = 250$ mm, $E = 200$ GPa, and $\delta = 0.1$ mm is subjected to an internal pressure of 140 MPa. Determine the distribution of tangential stress throughout the composite wall.

Solution:

In the absence of applied internal pressure, the contact pressure is from Eqn (7.22),

$$P = \frac{200 \times 10^9 \times 0.0001}{0.2} \cdot \frac{(0.2^2 - 0.15^2)(0.25^2 - 0.2^2)}{2(0.2)^2(0.25^2 - 0.15^2)} = 12.3 \text{ MPa}$$

The tangential stresses in the outer cylinder associated with the pressure are found by using Eqn (7.11),

$$(\sigma_\theta)_{r=0.2} = P \frac{b^2 + c^2}{c^2 - b^2} = 12.3 \times 10^6 \frac{0.2^2 + 0.25^2}{0.25^2 - 0.2^2} = 56.0 \text{ MPa}$$

$$(\sigma_\theta)_{r=0.25} = \frac{2Pb^2}{c^2 - b^2} = \frac{2(12.3 \times 10^6)(0.2^2)}{0.25^2 - 0.2^2} = 43.7 \text{ MPa}$$

The stresses in the inner cylinder are, from Eqn (7.14),

$$(\sigma_\theta)_{r=0.15} = -\frac{2Pb^2}{b^2 - a^2} = \frac{2(12.3 \times 10^6)(0.2^2)}{0.2^2 - 0.15^2} = -56.2 \text{ MPa}$$

$$(\sigma_\theta)_{r=0.2} = -P \frac{b^2 + a^2}{b^2 - a^2} = -12.3 \times 10^6 \frac{0.2^2 + 0.15^2}{0.2^2 - 0.15^2} = -43.9 \text{ MPa}$$

7.5 Rotating Disks of Constant Thickness:

The equation of equilibrium, Eqn (7.1), can be used to treat the case of a rotating disk, provided that the centrifugal (Inertia force) is included as a body force. Again, stresses induced by rotation are distributed symmetrically about the axis of rotation and assumed independent of disk thickness. Thus, application of Eqn (7.1), with the body force per unit volume F_r equated to the centrifugal force $\rho \omega^2 r$, yields,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho \omega^2 r = 0 \qquad (7.23)$$

where ρ is the mass density and ω the constant angular speed of the disk in radians per second. Substituting Eqn (7.6) into Eqn (7.23), we have,

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -(1-\nu^2)\,\rho \omega^2 r / E \qquad (a)$$

requiring a homogeneous and particular solution. It is easily demonstrated that the particular solution is,

$$u_p = -(1-\nu^2)\frac{\rho \omega^2 r^3}{8E}$$

The complete solution is therefore,

$$u = -\frac{\rho \omega^2 r^3 (1-\nu^2)}{8E} + C_1 r + \frac{C_2}{r} \qquad (7.24)\,a$$

Which upon substitution into Eqn (7.6), provides the following expressions for radial and tangential stress,

$$\sigma_r = \frac{E}{1-\nu^2}\left[\frac{-(3+\nu)(1-\nu^2)\rho \omega^2 r^2}{8E} + (1+\nu)C_1 - (1-\nu)\frac{C_2}{r^2}\right] \qquad (7.24)\,b$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\frac{-(1+3\nu)(1-\nu^2)\rho \omega^2 r^2}{8E} + (1+\nu)C_1 + (1-\nu)\frac{C_2}{r^2}\right] \qquad (7.24)\,c$$

The constants of integration may now be evaluated on the basis of the boundary conditions.

Annular Disk :

In the case of an annular disk with zero pressure at the inner and outer boundaries, the distribution of stress is due entirely to rotational effects. The boundary conditions are,

$$(\sigma_r)_{r=a} = 0 \quad , \quad (\sigma_r)_{r=b} = 0 \qquad (b)$$

These conditions, combined with eqs (7-24) b, yield two equations in the two unknown constants,

$$0 = -\rho\omega^2 \frac{a^2}{E} \frac{(1-\nu^2)(3+\nu)}{8} + (1+\nu)C_1 - (1-\nu)\frac{C_2}{a^2}$$
$$0 = -\rho\omega^2 \frac{b^2}{E} \frac{(1-\nu^2)(3+\nu)}{8} + (1+\nu)C_1 - (1-\nu)\frac{C_2}{b^2} \qquad (c)$$

from which,

$$C_1 = \rho\omega^2 \frac{(a^2+b^2)}{E} \cdot \frac{(1-\nu)(3+\nu)}{8}$$
$$C_2 = \rho\omega^2 \left(\frac{a^2b^2}{E}\right) \frac{(1+\nu)(3+\nu)}{8} \qquad (d)$$

The stresses and displacement are therefore,

$$\sigma_r = \frac{3+\nu}{8}\left(a^2+b^2-r^2-\frac{a^2b^2}{r^2}\right)\rho\omega^2 \qquad (7\text{-}25)a$$

$$\sigma_\theta = \frac{3+\nu}{8}\left(a^2+b^2-\frac{1+3\nu}{3+\nu}r^2+\frac{a^2b^2}{r^2}\right)\rho\omega^2 \qquad (7\text{-}25)b$$

$$u = \frac{(3+\nu)(1-\nu)}{8E}\left(a^2+b^2-\frac{1+\nu}{3+\nu}r^2+\frac{1+\nu}{1-\nu}\frac{a^2b^2}{r^2}\right)\rho\omega^2 r \qquad (7\text{-}25)c$$

Applying the condition $d\sigma_r/dr = 0$ to the first of the above equations, it is readily verified that the maximum radial stress occurs at $r = \sqrt{ab}$.

Solid Disk :

In this case, $a = 0$, and the boundary conditions are

$$(\sigma_r)_{r=b} = 0 \ , \quad (u)_{r=0} = 0 \tag{e}$$

In order to satisfy the condition on the displacement it is clear from Eqn (7-24)a that C_2 must be zero. The remaining constant is now evaluated from the first expression of Eqn(d),

$$C_1 = \rho \omega^2 \frac{b^2}{E} \frac{(1-\nu)(3+\nu)}{8}$$

Combining these constants with Eqns (7-24), the following results are obtained,

$$\sigma_r = \frac{3+\nu}{8}\left(b^2 - r^2\right)\rho\omega^2 \tag{7-26)a}$$

$$\sigma_\theta = \frac{3+\nu}{8}\left(b^2 - \frac{1+3\nu}{3+\nu} r^2\right)\rho\omega^2 \tag{7-26)b}$$

$$u = \frac{1-\nu}{8E}\left[(3+\nu)b^2 - (1+\nu)r^2\right]\rho\omega^2 r \tag{7-26)c}$$

The stress and displacement of a solid rotating disk are evaluated using the above equations.

Example (7-4) :

A flat 0.5 m outer diameter, 0.1 m inner diameter, and 0.08 m thick steel disk is shrunk onto a steel shaft. If the assembly is to run at speeds up to 6900 rpm, determine (a) the shrinking allowance,

(b) the maximum stress when not rotating, and (c) the maximum stress when rotating. The material properties are $\rho = 7.8\ kN\cdot s^2/m^4$, $E = 200\ GPa$, and $\nu = 0.3$.

Solution:

(a) The radial displacements of the disk (u_d) and shaft (u_s) are, from Eqns (7.25) and (7.26),

$$u_d = 0.05 * \frac{3.3 * 0.7}{8E}\left(0.0025 + 0.0625 - \frac{1.3}{3.3} * 0.0025 + \frac{1.3}{0.7} * 0.0625\right) * \rho\omega^2.$$

$$= 0.0026\,\rho\frac{\omega^2}{E}$$

$$u_s = 0.05 * \frac{0.7}{8E}(3.3 * 0.0025 - 1.3 * 0.0025)\,\rho\omega^2$$

$$= 2.1875 * 10^{-5}\,\rho\frac{\omega^2}{E}$$

We observe that u_s may be neglected as it is less than 1% of u_d of the disk at the common radius. The exact allowance is

$$\delta = u_d - u_s = \frac{(0.0026 - 2.1875 * 10^{-5}) * 7.8 * 10^3 (6900 * 2\pi/60)^2}{200 * 10^9}$$

$$= 6.25 * 10^{-5}\ m$$

(b) Applying Eqn (7.22), we have,

$$P = \frac{E\delta}{2b}\cdot\frac{c^2 - b^2}{c^2} = \frac{(200 * 10^9 * 6.25 * 10^{-5})(0.0625 - 0.0025)}{2 * 0.05 * 0.0625}$$

$$= 100.8\ MPa$$

Therefore, from Eqn (7.16),

$$\sigma_{\theta,max} = P\,\frac{c^2 + b^2}{c^2 - b^2} = 100.8 * 10^6\cdot\frac{0.0625 + 0.0025}{0.0625 - 0.0025} = 109.2\ MPa$$

(c) from Eqn (7.25), for $r = 0.05$,

$$\sigma_{\theta,max} = \frac{3.3}{8}\left(0.0025 + 0.0625 - \frac{1.9}{3.3} * 0.0025 + 0.0625\right)\rho w^2$$

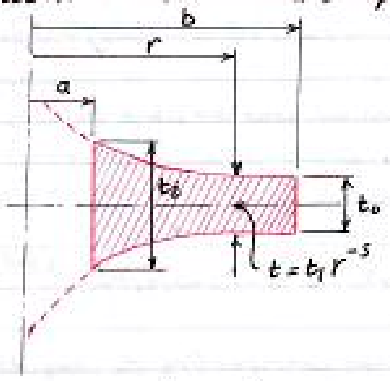
$$= 0.052\,\rho w^2 = 211.78\ MPa.$$

7.6 Rotating Disks of Variable Thickness:

The approach employed in the analysis of flat disks can be extended to variable thickness disks. Let the profile of a radial section be represented by the general hyperbola,

$$t = t_1 r^{-s} \qquad (7.27)$$

Where t_1 represents a constant and s a positive number.



The shape of the curve depends upon the value selected for s, e.g., for $s = 1$, the profile is that of an equivalent hyperbola. The constant t_1 simply the thickness at radius equal to unity. If the thickness at $r = a$ is t_i and that at $r = b$ is t_o, as shown in the figure, the hyperbola curve is fitted by forming the ratio:

$$\frac{t_i}{t_o} = \frac{t_1 a^{-s}}{t_1 b^{-s}}$$

and solving for s. clearly, Eqn (7.27) does not apply to solid disks, as all values of s except zero yield infinite thickness at $r = 0$.

The differential equation of equilibrium, Equ (7.23), must now include $t(r)$ and takes the form,

$$\frac{d}{dr}(t r \sigma_r) - t\sigma_\theta + t\rho\omega^2 r^2 = 0 \qquad (7.28)$$

The above equation satisfied by a stress function of the form,

$$\phi = t r \sigma_r \ , \quad \frac{d\phi}{dr} + t\rho\omega^2 r^2 = t\sigma_\theta \qquad (a)$$

Then the compatibility equ(7.4), using Equ (a) and Hooke's law, becomes,

$$r^2\frac{d^2\phi}{dr^2} + \left(1 - \frac{r}{t}\frac{dt}{dr}\right) r\frac{d\phi}{dr} + \left(\nu\frac{r}{t}\frac{dt}{dr} - 1\right)\phi = -(3+\nu)\rho\omega^2 t r^3 \qquad (b)$$

Introducing Equ (7.27), we have,

$$r^2\frac{d^2\phi}{dr^2} + (1+\delta) r\frac{d\phi}{dr} - (1+\nu S)\phi = -(3+\nu)\rho\omega^2 t_1 r^{3-\delta} \qquad (c)$$

This is an equidimensional equation which the transformation $r = e^{\alpha}$ reduces to a linear differential equation with constant coefficients,

$$\frac{d^2\phi}{d\alpha^2} + \delta\frac{d\phi}{d\alpha} - (1+\nu S)\phi = -(3+\nu) t_1 \rho\omega^2 e^{(3-\delta)\alpha} \qquad (d)$$

The auxiliary equation corresponding to equ(d) is given by,

$$m^2 + \delta m - (1+\nu S) = 0$$

and has the roots,

$$m_{1,2} = -\frac{\delta}{2} \pm \left[\left(\frac{s}{2}\right)^2 + (1+\nu s)\right]^{1/2} \qquad (7.29)$$

The general solution of Equ(d) is then,

$$\phi = C_1 r^{m_1} + C_2 r^{m_2} - \frac{3+\nu}{8-(3+\nu)\delta} t_1 \rho\omega^2 r^{3-\delta} \qquad (e)$$

The stress components for a disk of variable thickness are therefore, from Eqns(a),

$$\sigma_r = \frac{C_1}{t_1} m_1 r^{m_1+\delta-1} + \frac{C_2}{t_1} m_2 r^{m_2+\delta-1} - \frac{3+\nu}{8-(3+\nu)\delta} \rho\omega^2 r^2 \qquad (7\text{-}30)a$$

$$\sigma_\theta = \frac{C_1}{t_1} m_1 r^{m_1+\delta-1} + \frac{C_2}{t_1} m_2 r^{m_2+\delta-1} - \frac{1+3\nu}{8-(3+\nu)\delta} \rho\omega^2 r^2 \qquad (7\text{-}30)b$$

Note that for a flat disk, t = constant; consequently, $\delta = 0$ in Eqn (7-27) and $m = 1$ in Eqn (7-29). Thus Eqns (7-30) reduce to Eqns (7.26), as expected. The C_1 and C_2 constants are determined from the boundary conditions:

$$(\sigma_r)_{r=a} = (\sigma_r)_{r=b} = 0 \qquad (f)$$

Example (7-5):

The cross section of the disk in the assembly given in example (7-4) is hyperbolic with $t_1 = 0.075$ m, $t_0 = 0.015$ m; $a = 0.05$ m, $b = 0.25$ m, and $\delta = 0.05$ mm. The rotational speed is 6900 rpm. Determine (a) the maximum stress owing to rotation and (b) the maximum radial displacement at the bore of the disk.

Solution:

(a) The value of the positive number δ is obtained by the use of Eqn (7-27),

$$\frac{t_i}{t_0} = \frac{t_1 a^{-\delta}}{t_1 b^{-\delta}} = \left(\frac{b}{a}\right)^{\delta}$$

Substituting $t_i/t_0 = 5$ and $b/a = 5$, we obtain $\delta = 1$. The profile will thus be given by $t = t_i/r$. From Eqn (7-29) we have,

$$m_{1,2} = -\frac{1}{2} \pm \left[\left(\frac{1}{2}\right)^2 + (1+0.3\times1) \right]^{1/2}, \quad m_1 = 0.745, \; m_2 = -1.745$$

Hence the radial stresses, using Eqns (7.30) and (f) for $r = 0.05$ and 0.25, are,

$$(\sigma_r)_{r=0.05} = 0 = \frac{C_1}{t_1} 0.05^{0.745} + \frac{C_2}{t_1} 0.05^{-1.745} - 0.00176\, \rho\omega^2$$

$$(\sigma_r)_{r=0.25} = 0 = \frac{C_1}{t_1} 0.25^{0.745} + \frac{C_2}{t_1} 0.25^{-1.745} - 0.0439\, \rho\omega^2$$

from which,

$$\frac{C_1}{t_1} = 0.12829\, \rho\omega^2, \quad \frac{C_2}{t_1} = -6.272\times10^{-5} \rho\omega^2$$

The stress components in the disk, substituting the above values into Eqs (7.30), are therefore

$$\sigma_r = (0.12829\, r^{0.745} - 6.272\times10^{-5}\, r^{-1.745} - 0.70\, r^2)\, \rho\omega^2$$

$$\sigma_\theta = (0.09334\, r^{0.745} + 1.095\times10^{-4}\, r^{-1.745} - 0.40\, r^2)\, \rho\omega^2 \qquad (g)$$

The maximum stress occurs at the bore of the disk, and from Eqns (g) is equal to,

$$(\sigma_\theta)_{r=0.05} = 0.0294\, \rho\omega^2$$

Note that it was $0.052\, \rho\omega^2$ in Example (7-4). For the same speed we conclude that the maximum stress is reduced considerably by tapering the disk.

(b) The radial displacement is obtained from the second equation of (7-5), which together with Eqns (g) gives,

$$(u)_{r=0.05} = (r\sigma_\theta/E)_{r=0.05} = 0.00147\, \rho\omega^2/E$$

Again, this is quite advantageous relative to the value of $0.0026\, \rho\omega^2/E$ found in Example (7-4).

7.7 Thermal Stresses in Thin Disks:

In this section, our concern is with the stresses associated with a radial temperature field $T(r)$ which is idependent of the axial dimension. In this case of plane stress, the applicable equations of stress and strain are obtained as,

$$\sigma_r = \frac{E}{1-\nu^2}\left[\epsilon_r + \nu\epsilon_\theta - (1+\nu)\alpha T\right] \qquad (7.31)a$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\epsilon_\theta + \nu\epsilon_r - (1+\nu)\alpha T\right] \qquad (7.31)b$$

The equation of equilibrium, Equ (7.2), is now,

$$r\frac{d}{dr}(\epsilon_r + \nu\epsilon_\theta) + (1-\nu)(\epsilon_r - \epsilon_\theta) = (1+\nu)\alpha r\frac{dT}{dr} \qquad (a)$$

Introduction of Equ (7.3) into the above expression yields the following differential equation in radial displacement,

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = (1+\nu)\alpha\frac{dT}{dr} \qquad (b)$$

This is rewritten,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = (1+\nu)\alpha\frac{dT}{dr} \qquad (c)$$

The integration of the above equation, gives,

$$u = \frac{(1+\nu)\alpha}{r}\int_a^r Tr\,dr + c_1 r + \frac{c_2}{r} \qquad (d)$$

where a, the inner radius of an annulus disk, is taken as zero for a solid disc, and ν and α have been treated as constants.

Annular Disk :

The radial and tangential stresses in the annular disk may be found by substituting Equ (d) into Equ (7-6),

$$\sigma_r = -\frac{\alpha E}{r^2}\int_a^r Tr\,dr + \frac{E}{1-\nu^2}\left[C_1(1+\nu) - \frac{C_2(1-\nu)}{r^2}\right] \qquad (e)$$

$$\sigma_\theta = \frac{\alpha E}{r^2}\int_a^r Tr\,dr - \alpha E T + \frac{E}{1-\nu^2}\left[C_1(1+\nu) + \frac{C_2(1-\nu)}{r^2}\right] \qquad (f)$$

The constants C_1 and C_2 are determined on the basis of the boundary conditions $(\sigma_r)_{r=a} = (\sigma_r)_{r=b} = 0$. Equ (e) thus gives,

$$C_1 = \frac{(1-\nu)\alpha}{b^2-a^2}\int_a^b Tr\,dr, \quad C_2 = \frac{(1+\nu)a^2\alpha}{b^2-a^2}\int_a^b Tr\,dr$$

The stresses are therefore,

$$\sigma_r = \alpha E\left[-\frac{1}{r^2}\int_a^r Tr\,dr + \frac{r^2-a^2}{r^2(b^2-a^2)}\int_a^b Tr\,dr\right] \qquad (7\text{-}32)a$$

$$\sigma_\theta = \alpha E\left[-T + \frac{1}{r^2}\int_a^r Tr\,dr + \frac{r^2+a^2}{r^2(b^2-a^2)}\int_a^b Tr\,dr\right] \qquad (7\text{-}32)b$$

Solid Disk :

In the case of a solid disk, the displacement must vanish at $r=0$ in order to preserve the continuity of material. The value of C_2 in Equ (d) must therefore be zero. To evaluate C_1, the boundary conditions $(\sigma_r)_{r=b} = 0$ is employed, and Equ (e) now gives,

$$C_1 = \frac{(1-\nu)\alpha}{b^2}\int_0^b Tr\,dr$$

Substituting C_1 and C_2 into Equs (e) and (f), the stresses in a solid disk are found to be,

$$\sigma_r = \alpha E\left[\frac{1}{b^2}\int_0^b Tr\,dr - \frac{1}{r^2}\int_0^r Tr\,dr\right] \qquad (7\text{-}33)a$$

$$\sigma_\theta = \alpha E\left[-T + \frac{1}{b^2}\int_0^b Tr\,dr + \frac{1}{r^2}\int_0^r Tr\,dr\right] \qquad (7\text{-}33)b$$

Given a temperature distribution $T(r)$, the stresses in a solid or annular disk can thus be determined from Eqns (7-32) or (7-33).

7.8 Thermal Stress in Long Circular Cylinders:

Consider a long cylinder with ends assumed restrained so that $w = 0$. This is another example of plane strain, for which $\epsilon_z = 0$. The stress-strain relations are, from Hooke's law,

$$\epsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_\theta + \sigma_z)\right] + \alpha T \qquad (7\text{-}34)a$$

$$\epsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_r + \sigma_z)\right] + \alpha T \qquad (7\text{-}34)b$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right] + \alpha T \qquad (7\text{-}34)c$$

For $\epsilon_z = 0$, the final expression above yields,

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) - \alpha E T \qquad (a)$$

Substitution of Eqn(a) into the first two of Eqns (7-34) leads to the following forms in which z stress does not appear,

$$\epsilon_r = \frac{1+\nu}{E}\left[(1-\nu)\sigma_r - \nu\sigma_\theta + \alpha E T\right]$$

$$\epsilon_\theta = \frac{1+\nu}{E}\left[(1-\nu)\sigma_\theta - \nu\sigma_r + \alpha E T\right] \qquad (b)$$

Inasmuch as Eqns (7-2) and (7-3) are valid for the case under discussion, the solutions for u, σ_r and σ_θ proceed as,

$$u = \frac{(1+\nu)\alpha}{(1-\nu)r}\int_a^r T r\,dr + C_1 r + \frac{C_2}{r} \quad (c)$$

$$\sigma_r = \frac{E}{1+\nu}\left[-\frac{(1+\nu)\alpha}{(1-\nu)r^2}\int_a^r T r\,dr + \frac{C_1}{1-2\nu} - \frac{C_2}{r^2}\right] \quad (d)$$

$$\sigma_\theta = \sigma_r + r\,\frac{d\sigma_r}{dr} \quad (e)$$

Finally, from equ(a) we obtain,

$$\sigma_z = -\frac{\alpha E T}{1-\nu} + \frac{2\nu E C_1}{(1+\nu)(1-2\nu)} \quad (f)$$

Solid Cylinder:

In order for the radial displacement of a solid cylinder to vanish at $r=0$, the constant C_2 in Equ(c) must clearly be zero. Applying the boundary condition $(\sigma_r)_{r=b}=0$, Equ(d) may be solved for the remaining constant of integration,

$$C_1 = \frac{(1+\nu)(1-2\nu)\alpha}{(1-\nu)b^2}\int_0^b T r\,dr \quad (g)$$

and the stress distributions determined from Equs(d),(e), and (f),

$$\sigma_r = \frac{\alpha E}{1-\nu}\left[\frac{1}{b^2}\int_0^b T r\,dr - \frac{1}{r^2}\int_0^r T r\,dr\right] \quad (7.35)a$$

$$\sigma_\theta = \frac{\alpha E}{1-\nu}\left[-T + \frac{1}{b^2}\int_0^b T r\,dr + \frac{1}{r^2}\int_0^r T r\,dr\right] \quad (7.35)b$$

$$\sigma_z = \frac{\alpha E}{1-\nu}\left[\frac{2\nu}{b^2}\int_0^b T r\,dr - T\right] \quad (7.36)a$$

The longitudinal stress given by the equ(7.36)a is valid only for the case of a fixed-ended cylinder. In the event the ends are free, a uniform axial stress $\sigma_z = S_0$ may be superimposed to cause the force resultant at each end to vanish:

$$S_0 \pi b^2 - \int_0^b \sigma_z (2\pi r)\, dr = 0$$

This expression together with equ (f) yields,

$$S_0 = \frac{2\alpha E}{b^2(1-\nu)} \int_0^b Tr\, dr - \frac{2\nu E C_1}{(1+\nu)(1-2\nu)} \qquad (h)$$

The longitudinal stress for a free-ended cylinder is now obtained by adding S_0 to the stress given by Equ (f):

$$\sigma_z = \frac{\alpha E}{1-\nu}\left(\frac{2}{b^2}\int_0^b Tr\, dr - T\right) \qquad (7\text{-}36)b$$

Stress components σ_r and σ_θ remain as before. The axial displacement is obtain by adding to the right-hand side of Equ (c), $u_0 = -\nu S_0 r / E$, a displacement due to uniform axial stress S_0.

Cylinder with Central Circular Hole:

When the inner and outer surfaces of a hollow cylinder are free of applied load, the boundary conditions $(\sigma_r)_{r=a} = (\sigma_r)_{r=b} = 0$ apply. Introducing these into Equ. (d), the constants of integration are,

$$C_1 = \frac{(1+\nu)(1-2\nu)\alpha}{(1-\nu)(b^2-a^2)} \int_a^b Tr\, dr - \nu\, \epsilon_z$$

$$C_2 = \frac{(1+\nu)a^2\alpha}{(1-\nu)(b^2-a^2)} \int_a^b Tr\, dr \qquad (i)$$

Equations (d), (e) and (f) thus provide,

$$\sigma_r = \frac{E}{1-\nu}\left[-\frac{\alpha}{r^2}\int_a^r Tr\, dr + \frac{(r^2-a^2)\alpha}{r^2(b^2-a^2)}\int_a^b Tr\, dr\right] \qquad (7\text{-}37)a$$

$$\sigma_\theta = \frac{E}{1-\nu}\left[-T + \frac{\alpha}{r^2}\int_a^r Tr\, dr + \frac{(r^2+a^2)\alpha}{r^2(b^2-a^2)}\int_a^b Tr\, dr\right] \qquad (7\text{-}37)b$$

$$\sigma_z = \frac{\alpha E}{1-\nu}\left[\frac{2\nu}{b^2-a^2}\int_a^b T r\,dr - T\right] \qquad (7.38)a$$

If the ends are free, proceeding as in the case of a solid cylinder, the longitudinal stress is described by,

$$\sigma_z = \frac{\alpha E}{1-\nu}\left[\frac{2}{b^2-a^2}\int_a^b T r\,dr - T\right] \qquad (7.38)b$$

Implementation of the foregoing analysis is dependent upon a knowledge of the radial distribution of Temperature $T(r)$.

Example (7.6):

Determine the stress distribution in a hollow free-ended cylinder, subject to constant temperatures at the inner and outer surfaces.

Solution:

The radial steady-state heat flow through an arbitrary internal cylindrical surface is given by Fourier's law of conduction,

$$Q = -2\pi r K \frac{dT}{dr} \qquad (j)$$

Here Q is the heat flow per unit axial length, and K the thermal conductivity. Assuming Q and K to be constant,

$$r\frac{dT}{dr} = -\frac{Q}{2\pi K} = \text{Constant} = C$$

The above is easily integrated upon separation,

$$T = C_1 \frac{dr}{r} + C_2 = C_1 \ln r + C_2 \qquad (K)$$

Applying the temperature boundary conditions $(T)_{r=a} = T_1$, $(T)_{r=b} = T_2$. Eqn (K) may be written, outer diameter of each is twice the inner diameter. What is the ratio of the pressures for the following cases? (a) The maximum tangential stress has the same absolute value in each cylinder. (b) The maximum tangential strain has the same absolute value in each cylinder. Take $\nu = \frac{1}{3}$.

(7.4) A steel cylinder is subjected to an internal pressure only. (a) obtain the ratio of the wall thickness to the inner diameter, if the internal pressure is three-quarters of the maximum allowable tangential stress. (b) Determine the increase in inner diameter of such a cylinder, 0.15 m in internal diameter, of an internal pressure of 63 MPa. Take $E = 210$ GPa and $\nu = \frac{1}{3}$.

(7.5) A brass solid cylinder is a firm fit within a steel tube of inner diameter $2b$ and outer diameter $4b$ at a temperature T_1 °C. If now the temperature of both elements is increased to T_2 °C find the maximum tangential stresses in the cylinder and in the tube. Take $\alpha_s = 11.7 \times 10^{-6}/°C$, $\alpha_b = 19.5 \times 10^{-6}/°C$, and neglect longitudinal friction forces at the interface.

(7.6) Consider a steel rotating disk of hyperbolic cross section, with $a = 0.125$ m, $b = 0.625$ m, $t_1 = 0.125$ m, and $t_0 = 0.0625$ m. Determine the maximum tangential force that can occur at the outer surface in newtons per meter of circumference if the maximum stress at the bore is not to exceed 140 MPa. Assume that outer and inner edges are free of pressure.

(7.7) A gear of inner and outer radii 0.1 m and 0.15 m, respectively, is shrunk onto a hollow shaft of inner radius 0.05 m. The maximum tangential stress induced in the gear wheel is 0.21 MPa. The length of the gear wheel parallel to shaft axis is 0.1 m. Assuming a coefficient of static friction of 0.2 at the common surface, what maximum torque may be transmitted by the gear without slip?