⑥ Theories of Failure:

From the viewpoint of Mechanical design, it is imperative that some practical guides be available to predict yielding under the conditions of stress as they are likely to exist in service. To meet this need and to understand the basis of material failure, a number of failure theories have been developed. Unfortunatly, no theory can claim to be the final answer. In the development which follows, the yield stress obtained in a simple tension test is denoted by σ_y' and in a simple compression test by σ_y''.

6.1 The Maximum Principle Stress Theory:

According to the maximum principal stress theory, credited by Rankine (1802 – 1872), a material fails by yielding when the maximum principal stress exceeds the tensile yield strength, or when the minimum principal stress exceeds the compressive yield strength. That is, at the onset of yielding.

$$|\sigma_1| = \sigma_y' \quad \text{or} \quad |\sigma_3| = \sigma_y'' \qquad (6.1)$$

The region of failure is shown graphically by reffering to the Mohr's circle representation as shown, below,

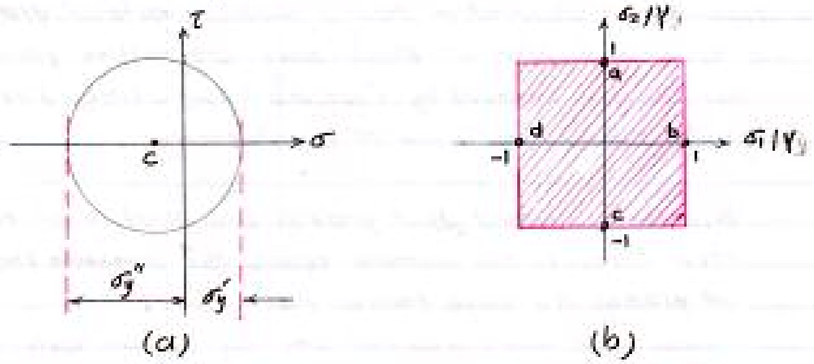


Fig. (1) Mohr's circle representation to the region of failure.

Falicure occur when the circle extends beyond either of the dashed vertical lines.

For materials possessing the same yield stress in tension and compression ($\sigma_y' = \sigma_y'' = Y$) , in the case of plane stress ($\sigma_3 = 0$) , Equ (6.1) becomes ,

$$|\sigma_1| = Y \quad \text{or} \quad |\sigma_2| = Y \qquad (6.2)\,a$$

This may be written as,

$$\frac{\sigma_1}{Y} = \pm 1 \quad \text{or} \quad \frac{\sigma_2}{Y} = \pm 1 \qquad (6.2)\,b$$

The above expressions are shown in Fig(1) (b), where points a, b, and c, d, indicating the tensile and Compressive principal stresses, respectively. For this Case the boundaries represent the onset of failure due to yielding. The area within the boundary of the figure is thus the region of no yielding (Elastic region).

6.2 The Maximum Shear Stress Theory :

The maximum shear stress theory is an outgrowth of the experimental observation that a ductile material yields as a result of slip or shear along crystalline planes. This theory is proposed by Coulomb (1736 – 1806) ; its also reffered to a the Tresca or Guest theory.

This theory predict that yielding will start when the maximum shear stress in the material equals the maximum shear stress at yielding in a simple tension test. Thus ,

$$\frac{1}{2}|\sigma_1 - \sigma_3| = \tau_y = \frac{1}{2}Y$$

or,

$$|\sigma_1 - \sigma_3| = Y \qquad (6.3)$$

In the case of plane stress, $\sigma_3 = 0$, there are two Combinations of Stresses to be Considered.

(a) When σ_1 and σ_2 are of opposite sign, i.e one tensile, the other Compressive, the maximum Shearing stress is $(\sigma_1 - \sigma_2)/2$. Thus, the yield condition is given by,

$$|\sigma_1 - \sigma_2| = Y \qquad (6.4)a$$

which may be restated as,

$$\frac{\sigma_1}{Y} - \frac{\sigma_2}{Y} = \pm 1 \qquad (6.4)b$$

(b) When σ_1 and σ_2 Carry the same sign, the maximum Shearing Stress equals $(\sigma_1 - \sigma_3)/2 = \sigma_1/2$. Then for $|\sigma_1| > |\sigma_2|$ and $|\sigma_2| > |\sigma_1|$, we have the following yield conditions, respectively,

$$|\sigma_1| = Y \quad \text{and} \quad |\sigma_2| = Y \qquad (6.5)$$

The following Figure show a plot of Eqns (6.4) and (6.5),

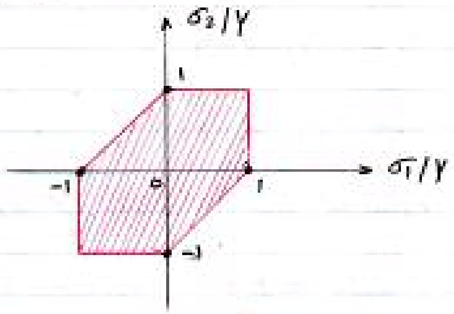


Fig.(2) Yield surface for the Maximum Shear Stress theory.

Note that Equ (6.4) applies to the second and fourth quadrants, while equ (6.5) applies to the first and third quadrants.

The boundary of the hexagon in Fig.(2), marks the onset of yielding with points outside the shaded region representing a yielded state.

6.3 The Maximum Principal Strain Theory:

According to the maximum principal strain theory, often referred to as the St. Venant theory in honor of its originator (1797-1886), a material fails by yielding when the maximum principal strain exceeds the tensile yield strain ϵ_y' or when the minimum principal strain exceeds the compressive yield strain ϵ_y''.

Applying the generalized Hook's law, we have after canceling E (the elastic modulus),

$$|\sigma_1 - \nu(\sigma_2 + \sigma_3)| = \sigma_y' \qquad (6.6)a$$

$$|\sigma_2 - \nu(\sigma_1 + \sigma_3)| = \sigma_y'' \qquad (6.6)b$$

For plane stress, $\sigma_3 = 0$, and the yield conditions are describes by,

$$|\sigma_1 - \nu\sigma_2| = \sigma_y'$$

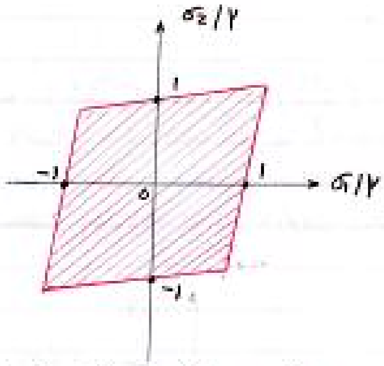
$$|\sigma_2 - \nu\sigma_1| = \sigma_y''$$

These may be restated for $\sigma_y' = \sigma_y'' = Y$ as,

$$\frac{\sigma_1}{Y} - \nu\frac{\sigma_2}{Y} = \pm 1 \qquad (6.7)a$$

$$\frac{\sigma_2}{Y} - \nu\frac{\sigma_1}{Y} = \pm 1 \qquad (6.7)b$$

The above expressions, for $\nu = 0.3$, are plotted in the following figure,



Fig(3) The onset of yielding for the Maximum principal strain theory.

Again the region outside the boundary represents the states for which, yielding may be expected, according to this theory.

6.4 The Maximum Distorsion Energy Theory :

The maximum distorsion energy theory was proposed by Huber in 1904 and further developed by Von Mises (1913) and Hencky (1925). In this theory, failure by yielding occurs when, at any point in the body, the distorsion energy per unit volume in a state of Combined stress becomes equal to that associated with yielding in a simple tension test. Thus,

$$(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{xz}^2) = 2Y^2 \quad (6.8)a$$

or, in terms of principal stresses,

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 = 2Y^2 \quad (6.8)b$$

For plane stress, $\sigma_3 = 0$, and the criterion for yielding becomes,

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = Y^2 \qquad (6.9)\,a$$

or, alternatively,

$$\left(\frac{\sigma_1}{Y}\right)^2 - \left(\frac{\sigma_1}{Y}\right)\left(\frac{\sigma_2}{Y}\right) + \left(\frac{\sigma_2}{Y}\right)^2 = 1 \qquad (6.9)\,b$$

The above expression defines the ellipse shown in the following figure,

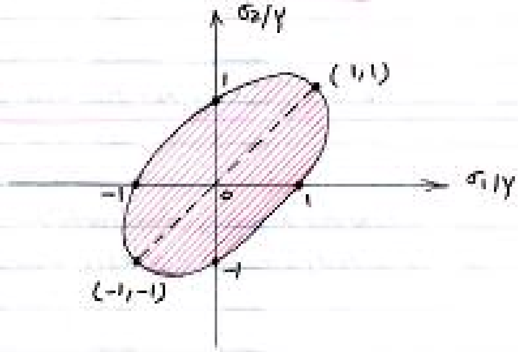


Fig.(4) The onset of yielding for the Maximum Distortion Energy Theory.

Points within the surface of the ellipse represents states of nonyielding.

6.5 The Octahedral Shearing Stress Theory,

The octahedral shearing stress theory, also referred as the Mises-Hencky criterion, predicts failure by yielding when the octahedral shearing stress at a point achieves a particular value. This value is determined by the relationship of τ_{oct} to Y in a simple tension test,

$$\tau_{oct} = 0.47\,Y \qquad (6.10)\,a$$

The Mises - Hencky Criterion may also be viewed in terms of distortion energy as,

$$W_d = \frac{3(1+\nu)}{2E} \tau_{oct}^2 \qquad (4.10)b$$

It is now asserted that the octahedral shearing stress theory enables us to apply the distortion energy theory while dealing with stress rather than energy.

Example (6.1):

A circular bar of tensile yield strength $Y = 360$ MPa is subjected to a combined state of loading defined by bending moment $M = 8$ kN.m and torque $M_t = 24$ kN.m. Calculate the diameter d which the bar must have in order to achieve a factor of safety $N = 2$. Apply the following theories; (a) maximum principal stress, (b) maximum shearing stress, (c) maximum principal strain, (d) maximum energy of distorsion.

Solution:

For the situation described, the principal stresses are,

$$\sigma_{1,2} = \frac{\sigma}{2} \pm \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}, \quad \sigma_3 = 0 \qquad (a)$$

Where; σ and τ, are

$$\sigma = \frac{My}{I} = \frac{32M}{\pi d^3}, \quad \tau = \frac{M_t r}{J} = \frac{16 M_t}{\pi d^3}$$

Thus, from eqn (a),

$$\sigma_{1,2} = \frac{16}{\pi d^3}\left(M \pm \sqrt{M^2 + M_t^2}\right) \qquad (b)$$

(a) Maximum principal stress theory:

on the basis of Eqns (b) and (6.2)a,

$$\frac{16}{\pi d^3}\left(M+\sqrt{M^2+M_t^2}\right) = \frac{Y}{N}$$

Substituting the data given,

$$\frac{16}{\pi d^3}\left(8+\sqrt{8^2+24^2}\right) = 175 * 10^6$$

from which $d = 0.0989$ m or 98.9 mm.

(b) Maximum Shearing Stress theory:

For the state of stress under consideration it may be observed from a Mohr's circle construction that σ_1 is tensile and σ_2 is compressive. Thus, through the use of Eqns (a) and (6.4)a,

$$\left(\frac{32}{\pi d^3}\right)\sqrt{M^2+M_t^2} = \frac{Y}{N}$$

After substitute the numerical values, the above gives $d = 113.7$ mm.

(c) Maximum principal Strain theory:

On the basis of Hooke's Law,

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2+\sigma_3)\right] = \frac{\epsilon_y}{N} = \frac{Y}{EN}$$

or,

$$\sigma_1 - \nu(\sigma_2+\sigma_3) = \frac{Y}{N}$$

Introducing Eqn (b) and $\nu = 0.3$, inserting the data, and solving, we obtain $d = 103.8$ mm.

(d) Maximum energy of distortion theory:

From equs (6.9)a and (b),

$$\left(\frac{16}{\pi d^3}\right)\sqrt{4M^2+3M_t^2} = \frac{Y}{N}$$

This results may also be obtained from the octahedral shearing stress theory by applying Equs (6.10) and (b). Substituting of the data into the above equation, gives $d = 109$ mm.

Example (6.2):

A steel conical tank, supported at its edges, is filled with a liquid of density ρ. The yield stress Y of the material is known. The cone angle is 2α. Determine the required wall thickness t of the tank, based upon a factor of safty N. Apply (a) the maximum shear stress theory and (b) the maximum energy of distortion theory.

Solution:

The variation of the circumferential and longitudinal stresses in the tank are given by,

$$\sigma_1 = \rho(a-y)y\frac{\tan\alpha}{t\cos\alpha}, \quad \sigma_2 = \rho\left(a-\frac{2}{3}y\right)y\frac{\tan\alpha}{2t\cos\alpha} \qquad (a)$$

The principle stresses have the largest magnitude,

$$\sigma_{1\,max} = \frac{\rho a^2}{4t}\cdot\frac{\tan\alpha}{\cos\alpha} \quad \text{at } y=\frac{a}{2}$$

$$\sigma_{2\,max} = \frac{3\rho a^2}{16t}\cdot\frac{\tan\alpha}{\cos\alpha} \quad \text{at } y=\frac{3a}{4} \qquad (b)$$

(a) As σ_1 and σ_2 are of the same sign and $|\sigma_1| > |\sigma_2|$, we have, from the first equation of (6.5) together with (b),

$$\frac{Y}{N} = \frac{\rho a^2}{4t}\frac{\tan\alpha}{\cos\alpha}$$

The thickness of the tank is found from the above, as,

$$t = 0.25 \frac{\rho a^2 N}{Y} \cdot \frac{\tan\alpha}{\cos\alpha}$$

(b) It is observed in Equ (b) that the largest values of principal stress are found at different locations. We shall therefore locate the section at which the combined principal stresses are at a critical value. For this purpose we insert Equ (a) into Equ (6.9)a,

$$\frac{Y^2}{N^2} = \left[\rho(a-y)\,y\,\frac{\tan\alpha}{t\cos\alpha}\right]^2 + \left[\left(a-\frac{2}{3}y\right)y\,\frac{\tan\alpha}{2t\cos\alpha}\right]^2$$
$$- \left[\rho(a-y)\,y\,\frac{\tan\alpha}{t\cos\alpha}\right]\left[\left(a-\frac{2}{3}y\right)y\,\frac{\tan\alpha}{2t\cos\alpha}\right] \quad (c)$$

upon differentiating equ (c) with respect to y, and equating the result to zero we obtain,

$$y = 0.52\,a$$

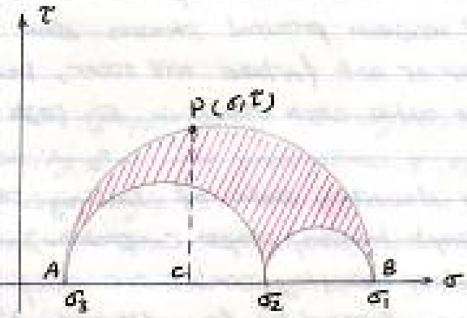
Upon substitution of this value of y into equ (c), the thickness of the tank is,

$$t = 0.225 \frac{\rho a^2 N}{Y} \cdot \frac{\tan\alpha}{\cos\alpha}$$

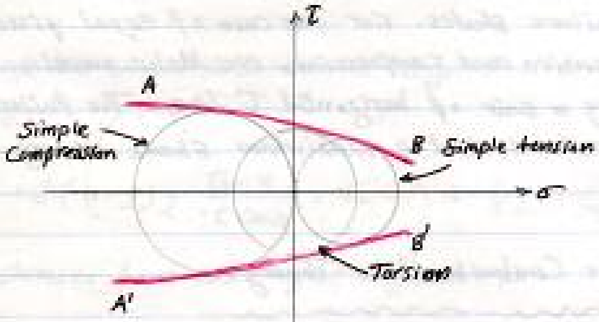
The thickness based upon the maximum shear stress theory is thus 10% larger than that based upon the maximum energy of distortion theory.

6.6 Mohr's Theory:

The Mohr theory of failure makes use of the well-known Mohr Circles of stress, as discussed befor. In Mohr's circle representation, the shear and normal components of stress acting on a particular plane are specified by the coordinates of a point within the shaded area of the following figure,



(a)



(b)

Fig(5) Representation of Mohr's Theory.

Note that τ depends upon σ, i.e.,

$$|\tau| = f(\sigma)$$

The figure indicates that a vertical line such as PC represents the states of stress on planes with the same σ but with deffering τ. It follows that the weakest of all these planes in the one on which the maximum shearing stress acts, designated P.

The same conclusion can be drawn regardless of the position of the vertical line between A and B; the points on the outer circle correspond to the weakest planes. On these planes, the maximum and minimum principal stresses alone are sufficient to decide whether or not failure will occur, because these stresses determine the outer circle shown in Fig. (5)a.

If the data describing states of limiting stress are derived from only simple tension, simple compression, and pure shear tests, the three resulting circles are adequate to construct the envelope, denoted by lines AB and A'B' in Fig. (5)b.

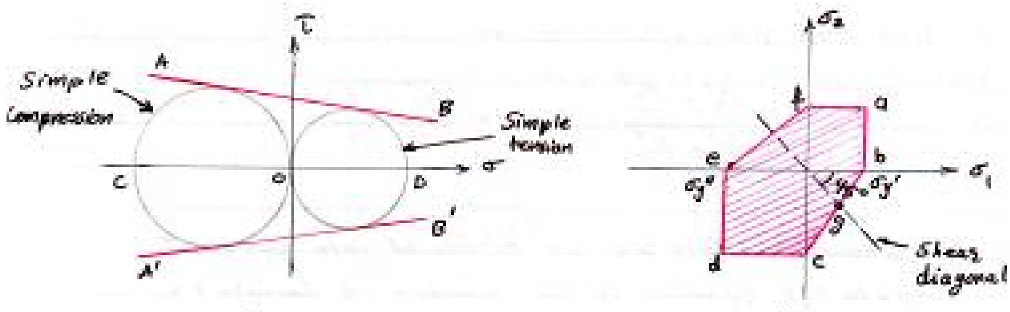
The Mohr envelope thus represents the locus of all possible failure states. For the case of equal yield stresses in tension and compression the Mohr envelope is represented by a pair of horizontal τ lines. The failure theory now reduces to the maximum shear theory.

6.7 The Coulomb-Mohr Theory:

The coulomb-Mohr or internal friction theory assumes that the critical stress is related to internal friction. If the frictional force is regarded as a function of the normal stress acting on a shear plane, the critical shearing stress and normal stress can be connected by an equation of the following form:

$$\tau = a\sigma + b \qquad (a)$$

The constants a and b represents properties of the material.



(a) (b)

Fig.(6) Representation of Coulomb-Mohr Theory.

Expression (a) may also be viewed as a stright line version of the Mohr envelope.

For the case of plane stress, $\sigma_3 = 0$ when σ_1 is tensile and σ_2 is compressive. The maximum shearing stress τ and the normal stress σ acting on the shear plane are,

$$\tau = \frac{\sigma_1 - \sigma_2}{2} \quad , \quad \sigma = \frac{\sigma_1 + \sigma_2}{2} \qquad (b)$$

Substitute the above expressions into Eqn(a), gives,

$$\sigma_1 (1 - a) - \sigma_2 (1 + a) = 2b \qquad (c)$$

In order to evaluate the materiel constants, the following conditions are applied,

$$\begin{aligned} \sigma_1 &= \sigma_y' \quad \text{when} \quad \sigma_2 = 0 \\ \sigma_2 &= \sigma_y'' \quad \text{when} \quad \sigma_1 = 0 \end{aligned} \qquad (d)$$

If now Eqns(d) are inserted into Eqn(c), the results are,

$$\sigma_y' (1 - a) = 2b \quad \text{and} \quad \sigma_y'' (1 + a) = 2b \qquad (e)$$

from which ,

$$a = \frac{\sigma_y' - \sigma_y''}{\sigma_y' + \sigma_y''} \quad , \quad b = \frac{\sigma_y' \sigma_y''}{\sigma_y' + \sigma_y''} \qquad (e)$$

The above constants are now introduced into equ (c) to complete the equation of the envelope of failure by yielding. When this is done, the following expression is obtained, applicable for $\sigma_1 > 0$, $\sigma_2 < 0$,

$$\frac{\sigma_1}{\sigma_y'} - \frac{\sigma_2}{\sigma_y''} = 1 \qquad (6.11)\,a$$

Relationships for the case where the principle stresses have the same sign ($\sigma_1 > 0$, $\sigma_2 > 0$ or $\sigma_1 < 0$, $\sigma_2 < 0$) may be deduced from Fig.(6) a without resort the above procedure.

In the case of biaxial tension (now $\sigma_{min} = \sigma_3 = 0$, σ_1 and σ_2 are tensile), the corresponding Mohr's circle is represented by diameter OD. Therefore, yielding occurs if either of the two tensile stresses achieves the value σ_y'. That is,

$$\sigma_1 = \sigma_y' \quad , \quad \sigma_2 = \sigma_y' \qquad (6.11)\,b$$

For biaxial compression ($\sigma_{max} = \sigma_3 = 0$, σ_1 and σ_2 are compressive), a Mohr's circle of diameter OC is obtained. Failure by yielding occurs if either of the compressive stresses attains the value σ_y'',

$$\sigma_2 = -\sigma_y'' \quad , \quad \sigma_1 = -\sigma_y'' \qquad (6.11)\,c$$

Figure (6) b is a graphical representation of the Coulomb-Mohr theory plotted in the σ_1, σ_2 plane. Lines ab and af represent Equ (6.11) b , and lines dc and de, Equ (6-11) c . The boundary bc, and boundary ef is obtained through the application of Equ (6.11) a . Points lying within the shaded area should not represent yielding, according to the theory.

In the case of pure shear, the corresponding limiting point is g. The magnitude of the limiting shear stress may be graphically determined from the figure or calculated from Equ (6.11) a by letting $\sigma_1 = -\sigma_2$,

$$\sigma_1 = \tau_y = \frac{\sigma_y' \sigma_y''}{\sigma_y' + \sigma_y''} \qquad (6.12)$$

6.8 Comparison of the Yielding Theories:

Two approaches may be employed for the purpose of comparing the theories of yielding for a material with $\sigma_y' = \sigma_y'' = \sigma_y$. The first comparison is made by equating for each theory the critical values corresponding to uniaxial loading and torsion for Poisson's ratio $\nu = 0.3$. We have

Maximum principal stress theory:	$\tau_y = \sigma_y$
Maximum shearing stress theory:	$\tau_y = 0.5\,\sigma_y$
Maximum principal strain theory:	$\tau_y = 0.77\,\sigma_y$
Maximum energy of distortion theory:	$\tau_y = 0.577\,\sigma_y$
Maximum octahedral shearing stress theory:	$\tau_y = 0.577\,\sigma_y$

Experiment shows that for ductile materials, the yield stress obtained in a torsion test is 0.5 to 0.6 times that determined from a simple tension test. Therefore the energy of distortion theory or its equivalent, the octahedral shearing stress theory, is most suitable for ductile materials. The shearing stress theory, which gives $\tau_y = 0.5\,\sigma_y$, is in widespread use, however, because it is simple to apply and offers a conservative result in design.

Example (6.3):

A thin-walled tube is fabricated of a brittle metal having ultimate tensile and compressive strengths $\sigma_y' = 300$ MPa and $\sigma_y'' = 700$ MPa. The outer and inner radii are $b = 105$ mm and $a = 100$ mm. Calculate the limiting torque that can be applied without causing failure by fracture. Apply three criteria; (a) the maximum principal stress theory, (b) the maximum shearing stress theory, (c) the Coulomb-Mohr theory.

Solution;

The torque and maximum shearing stress are related by the torsion formula,

$$M_t = \frac{J}{r}\tau = \frac{\pi(b^4-a^4)}{2b}\tau = \frac{\pi[0.105^4 - 0.1^4]}{2(0.105)}\tau$$

$$= 3.224 \times 10^{-4}\tau .$$

The state of stress is described by $\sigma_1 = -\sigma_2 = \tau$, $\sigma_3 = 0$.

(a) Equs (6-1) are applied with σ_3 replaced by σ_2 because the latter is negative; $\sigma_1 = \sigma_y'$ or $\sigma_2 = \sigma_y''$. As we have $\sigma_1 = \sigma_y' = 300 \times 10^6 = \tau$, then,

$$M_t = 3.224 \times 10^{-4}(300 \times 10^6) = 96{,}720 \quad N\cdot m$$

(b) The theory requires that $\sigma_y' = \sigma_y'' = \sigma_y$. Applying Equ (6.4)a,

$$|\sigma_1 - \sigma_2| = \sigma_y' ; \quad 2\sigma_1 = \sigma_y' = 2\tau \quad \text{or} \quad \tau = \sigma_y'/2$$

Substituting into the torque equation, gives;

$$M_t = 1.612 \times 10^{-4}(300 \times 10^6) = 48{,}360 \quad N\cdot m$$

(c) Applying Equ (6-11)a,

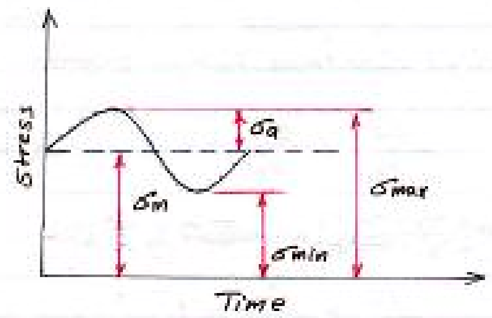
$$\frac{\tau}{300 \times 10^6} - \frac{-\tau}{700 \times 10^6} = 1$$

from which $\tau = 210$ MPa, and from the torque equation

$$M_t = 3.224 \times 10^{-4}(210 \times 10^6) = 67{,}704 \quad N\cdot m$$

6.9 Failure Criteria for Metal Fatigue:

A very common type of loading consists of an alternating sinusoidal stress superimposed upon a uniform stress, as shown below,



Fig(7) Alternating Sinusoidal Stress State.

Variation of stress with time occurs, for example, if a forced vibration of constant amplitude is applied to structural member already transmiting a constant load.

Reffering to the figure, we define the average or mean stress and the alternating, fluctuating, or fatigue stress as follows,

$$\sigma_m = \frac{1}{2}(\sigma_{max} + \sigma_{min}) \qquad (6.13)a$$

$$\sigma_a = \frac{1}{2}(\sigma_{max} - \sigma_{min}) \qquad (6.13)b$$

In the case of complete reversal, it is clear that the average stress equals zero. The alternating stress component is the most important factor in determining the number of cycles of load the material can withstand before fracture; the mean stress level is less important, particularly, if σ_m is negative (compressive). To predict whether the state of stress at a critical point will result in failure, a criterion is employed which is based upon the mean and fluctuating stresses and which utilizes the simple S-N curve relationships.

Uniaxial State of Stress:

Many approaches have been suggested for interpreting fatigue data. The commonly employed criteria, also referred to as meanstress-alternating stress relations are as follows,

① Modified Goodman Fatigue Criterion:

$$\frac{\sigma_a}{\sigma_{cr}} + \frac{\sigma_m}{\sigma_u} = 1$$

② Soderberg Fatigue Criterion:

$$\frac{\sigma_a}{\sigma_{cr}} + \frac{\sigma_m}{\sigma_y} = 1$$

③ Gerber Fatigue Criterion:

$$\frac{\sigma_a}{\sigma_{cr}} + \left(\frac{\sigma_m}{\sigma_u}\right)^2 = 1$$

④ SAE Fatigue Criterion:

$$\frac{\sigma_a}{\sigma_{cr}} + \frac{\sigma_m}{\sigma_f} = 1$$

Notations:

a = alternating, y = tensile yield, m = mean, u = tensile ultimate
cr = completely reversed, f = fracture.

In each case, fatigue strength for complete stress reversal at a specified number of cycles, may have a value between the fracture stress and endurance stress (ie $\sigma_f \leq \sigma_{cr} \leq \sigma_e$). Experience has shown that for steel, the Soderberg or Modified Goodman relations are the most reliable for predicting fatigue failure. The Gerber criterion leads to more libral results and hence less safe to use. ~~For hard steels, the SAE and modified Goodman relations result in identical solutions, since for brittle materials $\sigma_u = \sigma_f$.~~

Example (6.4) :

A square prismatic bar of sides 0.05 m is subjected to an axial thrust (tension) $F_m = 90$ KN. The fatigue strength for completely reversed stress at 10^6 cycles is 210 MPa and the static tensile yield strength is 280 MPa. Apply the Soderberg criterion to determine the limiting value of completely reversed axial load F_a that can be superimposed to F_m at the midpoint of a side of the cross section without causing fatigue failure at 10^6 cycles.

Solution :

The alternating and mean stresses are given by,

$$\sigma_a = \frac{M * C}{I} = \frac{0.025 F_a (0.025)}{(0.05)^4/12} = 1200 F_a$$

$$\sigma_m = \frac{F_m}{A} = \frac{90,000}{0.05 * 0.05} = 36 \text{ MPa}$$

Applying the Soderberg criterion,

$$\frac{1200 F_a}{210 * 10^6} + \frac{36 * 10^6}{280 * 10^6} = 1$$

from which,

$$F_a = 152.5 \text{ KN}.$$

Combined State of Stress :

Structural and machine elements are often subjected to combined fluctuating bending, torsion, and axial loading. Under conditions of a general state of stress that are cyclic, it is common practice to modify the static failure theories for purposes of analysis. The maximum distortion energy theory, is expressed by,

$$(\sigma_{xa}-\sigma_{ya})^2+(\sigma_{ya}-\sigma_{za})^2+(\sigma_{za}-\sigma_{xa})^2+6(\tau_{xya}^2+\tau_{xza}^2+\tau_{yza}^2)=2\sigma_{ea}^2$$

$$(\sigma_{xm}-\sigma_{ym})^2+(\sigma_{ym}-\sigma_{zm})^2+(\sigma_{zm}-\sigma_{xm})^2+6(\tau_{xym}^2+\tau_{xzm}^2+\tau_{yzm}^2)=2\sigma_{em}^2 \qquad (6.14)a$$

or,

$$(\sigma_{1a}-\sigma_{2a})^2+(\sigma_{2a}-\sigma_{3a})^2+(\sigma_{3a}-\sigma_{1a})^2=2\sigma_{ea}^2$$

$$(\sigma_{1m}-\sigma_{2m})^2+(\sigma_{2m}-\sigma_{3m})^2+(\sigma_{3m}-\sigma_{1m})^2=2\sigma_{em}^2 \qquad (6.14)b$$

Here σ_{ea} and σ_{em}, the equivalent alternating stress and equivalent mean stress, respectively.

The equivalent mean stress – equivalent alternating stress fatigue failure relation are the same relations given before, replacing σ_a and σ_m by σ_{ea} and σ_{em}. These criteria together with modified static failure theories are used to compute fatigue strength under combined loading.

Example (6.5):

Consider a thin-walled cylinder of radius $r = 0.04$ m and thickness $t = 5$ mm, subject to an internal pressure varying from a value of $-P/4$ to P. Employ the octahedral shear theory together with the Soderberg criterion to compute the value of P producing failure after 10^8 cycles. The material tensile yield strength is 300 MPa and the fatigue strength is $\sigma_e = 250$ MPa at 10^8 cycles.

Solution:

The maximum and minimum values of the tangential and axial principal stresses are given by,

$$\sigma_{\theta,max} = \frac{Pr}{t} = 8P, \quad \sigma_{\theta,min} = \frac{(-P/4)r}{t} = -2P$$

$$\sigma_{z,max} = \frac{Pr}{2t} = 4P \quad , \quad \sigma_{z,min} = \frac{(-P/4)r}{2t} = -P$$

The alternating and mean stresses are therefore,

$$\sigma_{\theta a} = \tfrac{1}{2}(\sigma_{\theta max} - \sigma_{\theta, min}) = 5P \, , \, \sigma_{\theta m} = \tfrac{1}{2}(\sigma_{\theta max} + \sigma_{\theta, min}) = 3P$$

$$\sigma_{za} = \tfrac{1}{2}(\sigma_{z,max} - \sigma_{z,min}) = 2.5P \, , \, \sigma_{zm} = \tfrac{1}{2}(\sigma_{z,max} + \sigma_{z,min}) = 1.5P$$

The octahedral shear theory, Equ (6.10), for cyclic combined stress is expressed,

$$\boxed{\begin{aligned} (\sigma_{\theta a}^2 - \sigma_{\theta a}\sigma_{za} + \sigma_{za}^2)^{1/2} &= \sigma_{ea} \\ (\sigma_{\theta m}^2 - \sigma_{\theta m}\sigma_{zm} + \sigma_{zm}^2)^{1/2} &= \sigma_{em} \end{aligned}} \qquad (6.15)$$

In terms of computed alternating and mean stress, the above equation appear as,

$$(25P^2 - 12.5P^2 + 6.25P^2)^{1/2} = \sigma_{ea}$$

$$(9P^2 - 4.5P^2 + 2.25P^2)^{1/2} = \sigma_{em}$$

from which $\sigma_{ea} = 4.33P$ and $\sigma_{em} = 2.60P$,

The Soderberg relation then leads to,

$$\frac{4.33P}{250 \times 10^6} + \frac{2.60P}{300 \times 10^6} = 1$$

Solving the above gives, $P = 38.43$ MPa.

6.10 Impact or Dynamic Loads:

A dynamic force acts to modify the static stress and strain fields as well as the resistance properties of a material. Shock loading is usually produced by a sudden application of force or motion to a member, whereas impact loading results from the collision of bodies. Our concern here will be only with the influence of impact forces upon the maximum stress and deformation of the body.

To idealize an elastic system subjected to an impact force, consider the figure below, in which are shown a weight W, which falls through a distance h, striking the end of a free standing spring.

As the velocity of the weight is zero initially, and is again zero at the instant of maximum deflection of the spring δ_{max}, the change in kinetic energy of the system is zero, and likewise the work done on the system. The total work consists of the work done by gravity on the mass as it falls and the resisting work done by the spring,

$$W(h + \delta_{max}) - \frac{1}{2} K \delta_{max}^2 = 0 \qquad (6.16)$$

where K is known as the spring constant.

It is noted that the weight is assumed to remain in contact with the spring. The deflection corresponding to a static force equal to the weight of the body is simply W/K. This is termed the static deflection, δ_{st}. Then the general expression of maximum dynamic defelection is, from eqn (6.16), therefore,

$$\delta_{max} = \delta_{st} + \sqrt{\delta_{st}^2 + 2\delta_{st} h} \qquad (6.17)$$

or, by rearrangment,

$$\delta_{max} = \delta_{st}\left(1 + \sqrt{1 + \frac{2h}{\delta_{st}}}\right) \qquad (6.18)$$

The impact factor, the ratio of the maximum dynamic deflection to the static deflection, is given by,

$$\frac{\delta_{max}}{\delta_{st}} = 1 + \sqrt{1 + \frac{2h}{\delta_{st}}} \qquad (6.19)$$

Multiplication of the impact factor by W yields an equivalent or so-called dynamic load,

$$P_{dy} = W\left(1 + \sqrt{1 + \frac{2h}{\delta_{st}}}\right) \qquad (6.20)$$

To compute the maximum stress and deflection resulting from impact loading, the above load may be used in the relationships derived for static loading.

Two extreme cases are clearly of particular interest. When $h \gg \delta_{max}$, the work term, $W\delta_{max}$, in eqn (6.16) may be neglected, reducing the expression to,

$$\delta_{max} = \sqrt{2\delta_{st} h}$$

On the other hand, when $h = 0$, the load is suddenly applied and eqn (6.16) becomes $\delta_{max} = 2\delta_{st}$.

An analysis similar to the above may be employed to derive expressions for the case of a weight W in horizontal motion with a velocity v, arrested by an elastic body. In this instant, the kinetic energy $Wv^2/2g$ replaces $W(h+\delta_{max})$, the work done by W, in eqn (6.16). Here g is the gravitational acceleration. By so doing, the maximum dynamic load and deflection are found to be, respectively,

$$P_{dy} = W\sqrt{\frac{v^2}{g\delta_{st}}}, \quad \delta_{max} = \delta_{st}\sqrt{\frac{v^2}{g\delta_{st}}} \qquad (6.21)$$

where δ_{st} is the static deflection caused by a horizontal force W.

Example (6.6):

A weight $W = 180$ N is dropped a height $h = 0.1$ m, striking at midspan a simply supported beam of length $L = 1.16$ m. The beam is of rectangular cross section: 0.025 m width and 0.075 m depth. For a material with modulus of Elasticity $E = 200$ GPa, determine the instantaneous maximum deflection and maximum stress for the following cases: (a) the beam is rigidly supported, (b) the beam is supported at each end by springs of stiffness $K = 180$ kN/m.

Solution:

The deflection of a point at midspan, owing to a statically applied load, is,

$$\delta_{st} = \frac{WL^3}{48EI} = \frac{180 * 1.16^3 * 12}{48 * 200 * 10^9 * 0.025 * 0.075^3} = 0.033 * 10^{-3} \text{ m}$$

The maximum static stress, also occuring at midspan, is calculated from,

$$\sigma_{st,max} = \frac{Mc}{I} = \frac{180 * 1.16 * 0.0375 * 12}{4 * 0.025 * 0.075^3} = 2.23 \text{ MPa}$$

(a) The impact factor is, from eqn (6.19)

$$1 + \sqrt{1 + \frac{2*0.1}{0.033*10^{-3}}} = 78.86$$

We thus have,

$$\delta_{max} = 0.033 \times 10^{-3} \times 78.86 = 2.602 \times 10^{-3}\ m$$

$$\sigma_{max} = 2.23 \times 78.86 = 175.86\ MPa$$

(b) The static deflection of the beam due to its own bending and the deformation of the spring is,

$$\delta_{st} = 0.033 \times 10^{-3} + \frac{90}{180{,}000} = 0.533 \times 10^{-3}\ m$$

The impact factor is thus,

$$1 + \sqrt{1 + \frac{2 \times 0.1}{0.533 \times 10^{-3}}} = 20.40$$

Hence,

$$\delta_{max} = 0.533 \times 10^{-3} \times 20.40 = 0.011\ m$$

$$\sigma_{max} = 2.23 \times 20.40 = 45.49\ MPa$$

It is observed from a comparison of the results that dynamic loading increases the value of deflection and stress considerably. Also noted is a reduction in stress with increased flexibility attributable to the springs added to the supports.

Problems:

(6.1) A steel circular cylindrical bar of 0.1 m diameter is subjected to compound bending and tension at its ends. The material yield strength is 221 MPa. Assume failure to occur by yielding and take the value of the applied moment to be $M = 17$ kNm. Determine, using the octahedral shear stress theory, the limiting value of P which can be applied to the bar without causing permanent deformation.

(6.2) At a point in a structural member, yielding occured under a state of stress given by

$$\begin{vmatrix} 0 & 40 & 0 \\ 40 & 50 & -60 \\ 0 & -60 & 0 \end{vmatrix} \text{ MPa}$$

Determine the uniaxial tensile yield strength of the material according to (a) maximum principal strain theory, (b) maximum shear stress theory, and (c) octahedral shear stress theory.

(6.3) A circular shaft of 0.12 m diameter is subjected to end loads $P = 45$ kN, $M = 4$ kN.m, and $M_t = 11.2$ kN.m. Let $Y = 280$ MPa. What is the factor of saftey, assuming failure to occur in accordance to the octahedral shear stress theory?

(6.4) The state of stress at a point is given by,

$$\begin{vmatrix} 2.8 & 7 & 14 \\ 7 & 21 & 35 \\ 14 & 35 & 28 \end{vmatrix} \text{ MPa}$$

Taking $Y = 82$ MPa, $\nu = 0.3$, and a factor of saftey of 1.2, determine whether failure takes place at the point, using (a) the maximum principle strain theory and (b) the maximum distortion energy theory.

(6.5) A weight W is dropped from a height $h = 0.75$ m onto the free end of a cantilever beam of length $L = 1.2$ m. The beam is of 50 mm by 50 mm square cross section. Determine the value of W required to result in yielding. Take, $Y = 28$ MPa and $E = 200$ GPa. Neglect the weight of the beam.