④ Macroscopic Plasticity :

4.1 Assumptions of Plasticity :

Since the onset of yielding and the behaviour that might follow is of primary concern, various models will be used in order to illustrate the physical processes involved. With any of the models presented below, several assumptions are invoked,

1. The Solid is Isotropic and homogeneous.
2. The onset of yielding in tension and Compression is identical. This implies there is no Bauschinger effect.
3. Volume changes are negligible. Thus, the dilatation is zero and Poisson's ratio is one-half. Although this ratio is an elastic constant, no confusion should result by extending this meaning to plastic deformation.
4. The magnitude of the mean normal stress or hydrostatic component of the stress state does not influence yielding.
5. Effects of strain rate are negligible.
6. Temperature effects are not Considered.

4.2 Models for Plastic Deformation :

(a) Rigid Perfectly Plastic Solid Model :

Rigid perfectly plastic behaviour has found wide use in many analytical studies. It implies that no deformation occurs until a certain level of stress is reached, then deformation proceeds indefinitely as long as the necessary flow stress is applied.

This model is shown in the following figure,

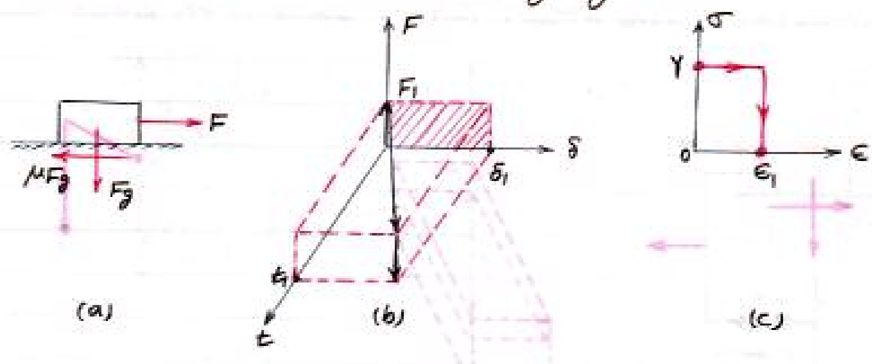


Fig. (1) Description of a rigid perfectly plastic solid showing the model, the force-displacement-time plot, and the stress-strain plot.

Note the following, from the above figure,

1. As the applied load F is increased, no displacement occurs until some critical force F_1 is reached. Once this happens, deformation proceeds continuously with time. The force F_1 is related directly to the yield or flow stress Y.

2. Upon removal of load F_1, there is no recovery of plastic work (shown by the shaded area in the $F-\delta$ plane). Rather, a permanent deformation given by δ_1 remains.

3. The solid does not become stronger during deformation. This implies there is no work-hardening effect.

(b) Rigid Linear Work-Hardening Solid Model:

A rigid linear work-hardening model is somewhat more realistic than the previous model since it incorporates the influence of work hardening observed in many solids, especially ductile metals. Again, a certain critical stress level must be reached before plastic deformation commences, but continued deformation demands an increasing applied stress. This is shown in the following figure,

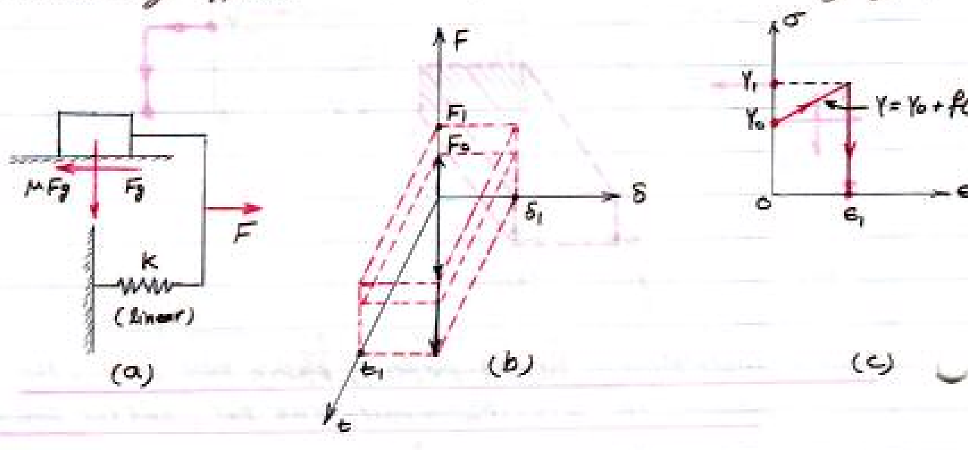


Fig. (2) Description of a rigid linear work-hardening solid.

The following effects are noted,

1. As F is applied, displacement begins only when a critical force F_0 is reached and produces the initial flow stress, Y_0.
2. Displacement continues only under an increase in applied stress Y where,

$$Y = Y_0 + f(\epsilon)$$

   and $f(\epsilon)$ is related to the slope of the line. In this model, work-hardening occurs and implies that plastic deformation causes an increase in the stress required for further deformation.
3. Upon removal of load F_1, no recovery occurs.

(C) Rigid Nonlinear Work-Hardening Solid Model,

A rigid work-hardening model where such hardening follows a power law form of behaviour provides an even better description for many solids. The following figure portrays this model,

Fig. C3) Description of a rigid work-hardening solid following a power law hardening behaviour.

Here it is noted that,

1. The behaviour is identical with the previous model except that strain hardening occurs at a nonlinear rate, the exponent n being greater than zero but less than unity.

2. The use of a power law behavior will be fully interpreted in the macroscopic behaviour of Ductile metals.

Finally, elastic effects can be included in any of the three models by adding on a straight-line section in the initial stages of deformation where the slope would indicate an elastic modulus of an appropriate value that is less than infinity.

Because many situations of concern involve plastic strains that are orders of magnitude greater than the elastic strains, it is convenient to ignore the latter. In doing so, volume changes can only be determined by including elastic effects where ν is less than $\frac{1}{2}$. If elastic and plastic strains are of the same order of magnitude, the above models would not be useful unless the elastic portion were included.

4.3 The Yield Locus and Surface:

A plot of two-dimensional stress space is introduced to indicate some of the results. A plot in $\sigma_1-\sigma_2$ space will be used,

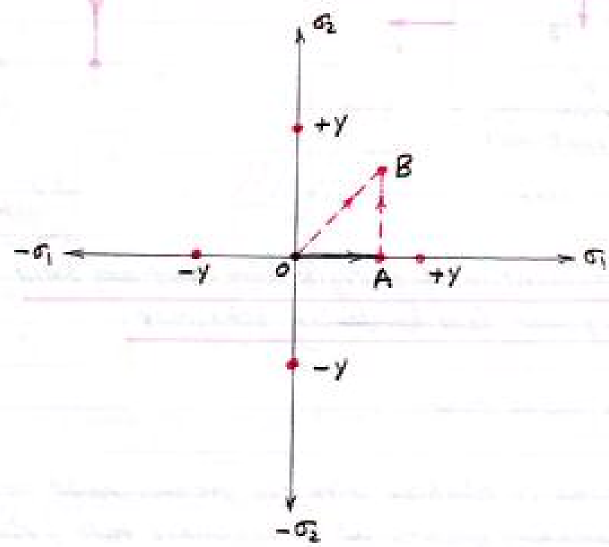


Fig. (4) Yield points in two-dimensional stress space.

Suppose a tensile stress is applied in one direction and $0 < \sigma_1 < Y$ describes elastic behavior only. With equivalence in tension and compression, the elastic range is extended to $-Y < \sigma_1 < Y$ and because of isotropy, $-Y < \sigma_2 < Y$.

Thus, there exist four points in $\sigma_1-\sigma_2$ stress space that indicate the onset of yielding but to develop an acceptable theory of yielding more complex stress states must be included.

The four points shown in the above figure at $\pm Y$ fall on a yield locus in this two-dimensional stress space. Suppose now that the material is stressed to point A as shown and that stress is maintained while a stress σ_2 is added. At some point, such as B, elastic behavior ends and we refer to B as a yield point in stress space.

Thus, to reach B, numerous loading paths might be followed and until that yield point is reached, all behavior is elastic. Using a number of loading paths, the locus described by the resulting yield points divides elastic behavior and the onset of yielding which is the locus itself.

In three-dimensional stress space,

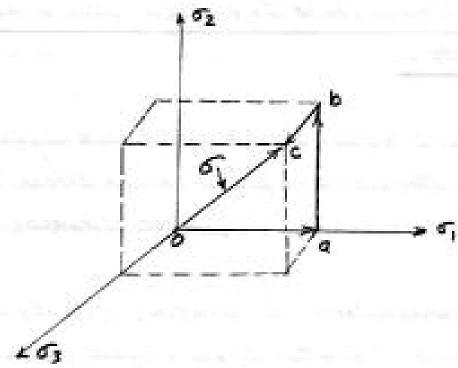
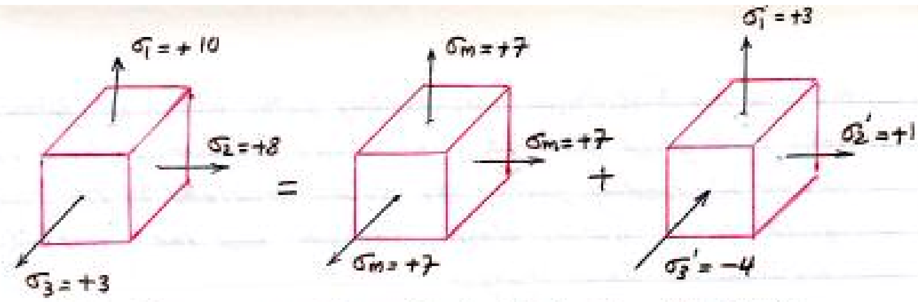


Fig. (5) Stress resultant in 3D stress space.

In the above figure, the combination of stresses a, b, and c, acting in the 1, 2, and 3 direction are assumed to just cause yielding. This total stress state is defined by σ which originates at the origin and its tip in space provides a yield point. If enough experiments were conducted, all such points would be on a yield surface. Note that a yield locus is described by passing a plane through the surface with one of the three principal stresses being a constant.

Considering that the magnitude of the mean normal stress, σ_m, does not influence yielding, the concept of a yield surface can be explained more fully.
Reference to the following figure, will clarify the meaning of σ_m where an applied stress state is indicated on the left element.



Actual = Mean (hydrostatic) + Deviatoric

Fig. (6) Stress state separated into mean and deviatoric components.

As shown, σ_m is equal to on third the algebric sum of the three normal stresses. If the mean stress is subtracted from each applied stress, the deviatoric stresses results.

Thus, if a combination of stresses $(\sigma_1, \sigma_2, \sigma_3)$ just causes yielding, then $(\sigma_1+\sigma_0, \sigma_2+\sigma_0, \sigma_3+\sigma_0)$ must also cause yielding and all combinations of $(\sigma_1, \sigma_2, \sigma_3)$ for various σ_0 therefore generate a line on the yield surface that is parallel to the line $\sigma_1 = \sigma_2 = \sigma_3$. This line is defined by equal direction cosines with respect to the 1,2,3 system.
Now since isotropy and no Bauschinger affects are assumed, the rotation of such line around the space diagonal $(\sigma_1 = \sigma_2 = \sigma_3)$ must generate prism which is the yield surface.

All planes perpendicular to the space diagonal are defined by the equation,

$$\sigma_1 + \sigma_2 + \sigma_3 = \text{Constant}.$$

This being $3\sigma_m$ for any one group of normal stresses.

If the constant is set to zero, that plane passes through the origin at right angles to the axis of the prism, it is often called the π plane and its intersection with ~~the yield surface is reffered to as the C curve.~~

Finally, consider a stress state as follows,

$$(\sigma_1, \sigma_2, \sigma_3) = (6, -2, 1) \quad \text{and} \quad \sigma_m = \frac{5}{3}$$

If the stress state is reduced by σ_m, then

$$(\sigma_1', \sigma_2', \sigma_3') = \left(\frac{13}{3}, -\frac{11}{3}, -\frac{2}{3}\right) \quad \text{and.}$$

$$\Sigma \sigma_i' = 0$$

Thus, the 3D stress vector is composed of the divatoric stresses that line in the π plane and the mean component that is perpendicular to the π plane.

4.4 Yield Criteria:

As discussed befor, for any 3D stress state there exists a cubic equation whose three roots are the principal stresses, A useful form of this equation is,

$$\sigma_p^3 - I_1 \sigma_p^2 - I_2 \sigma_p - I_3 = 0 \qquad (4.1)$$

where, the invariants, I_1, I_2, and I_3, may be expressed as functions of principal stresses as follows,

$$\begin{aligned} I_1 &= (\sigma_1 + \sigma_2 + \sigma_3) \\ I_2 &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ I_3 &= (\sigma_1\sigma_2\sigma_3) \end{aligned} \qquad (4.2)$$

It may be noted that $I_1 = 3\sigma_m$; thus the first invariant is a function of the mean component and should not influence yielding. Therefore any acceptable yield criterion should not include any refrence to I_1 for those solids whose yield behavior has been found to be independent of σ_m.

Suppose that a yield criterion is proposed as follows,

when $\sigma_1 - \sigma_2 - \sigma_3 = \text{Constant} = +10$

yielding will occur. If this were an acceptable criterion then $\sigma_1 = +5$, $\sigma_2 = -2$, $\sigma_3 = -3$, provides a stress state that would lead to yielding.

The two most widely used criteria both satisfy independence of I_1 and have found best agreement when experiments have utilized ductile metals are Tresca and Von Mises criteria.

4.5 Tresca Criterion:

This criterion proposes that yielding will occur when some function of maximum shear stress reaches a critical value.

Whenever possible, the convention $\sigma_1 > \sigma_2 > \sigma_3$ will be used but there are cases when this relative comparison in not known first.

It is useful, then, to recall that when three Mohr's circles are plotted it is the radius of the largest circle that gives the maximum shear stress. Accounting for algebraic signs, this criterion is written as,

$$\sigma_{max} - \sigma_{min} = \text{Constant} = \sigma_1 - \sigma_3 \qquad (4.3)$$

$$\text{if } \sigma_1 > \sigma_2 > \sigma_3.$$

If such criterion finds reasonably universal acceptance regardless of the applied stress state, then the constant should be readily determined from simple standard tests.

(a) For uniaxial tension, yielding occurs when σ_1 reaches the uniaxial yield stress, Y, thus,

$$\sigma_1 = Y, \quad \sigma_2 = \sigma_3 = 0, \quad \text{and} \quad \tau_{max} = \frac{1}{2} Y$$

Using eq$\underline{u}$ (4.3), this means,

$$\sigma_1 - 0 = \text{Constant} = Y$$

(b) For pure shear, $\sigma_1 = -\sigma_3 = \tau_{max}$, $\sigma_2 = 0$. For convenience, let the maximum allowable shear stress be designated as K, the shear yield stress. Using eq$\underline{u}$ (4.3), we get,

$$\sigma_1 - (-\sigma_1) = \text{constant} = 2\sigma_1 = 2k.$$

Thus, the Tresca Criterion may be expressed as,

$$\sigma_{max} - \sigma_{min} = Y = 2K = \sigma_1 - \sigma_3 \qquad (4.4)$$

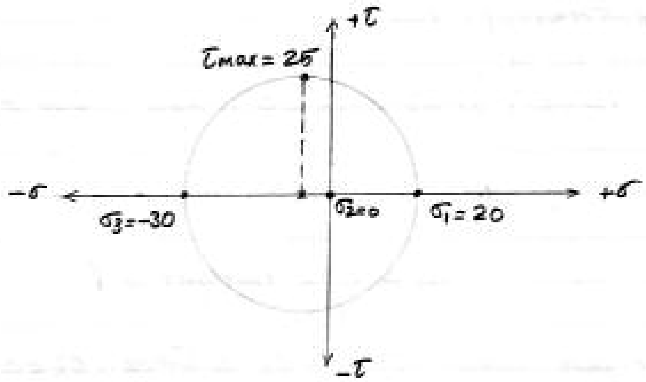
$$\text{if } \sigma_1 > \sigma_2 > \sigma_3.$$

If a solid obeyed this criterion exactly, then the tensile and shear yield stresses would relate in a two-to-one ratio. This does not mean this ratio must be observed, rather, it is predicted by this criterion.

Example (4.1):

A material whose tensile yield strength, Y, is 50 ksi is subjected to a uniaxial compressive stress of 30 ksi. Determine the magnitude of the tensile stress applied at right angles to the initial compressive stress, that would cause yielding according to Tresca Criterion. Plot the Mohr's circle for this situation.

Solution:



$$\sigma_1 - \sigma_3 = Y \ , \text{ where } \sigma_1 > \sigma_2 > \sigma_3$$

Here σ_2 is indicated as zero, σ_3 is negative (compression), and σ_1, the unknown stress is positive.

$$\sigma_1 - (-30) = 50 \ , \text{ so } \sigma_1 = 20 \text{ ksi}$$

Note that the diameter of the circle is 50 ksi.

4.6 Von Mises Criterion:

In its most widely used form, the von Mises criterion, in terms of principal stresses, predicts that yielding occurs when

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = \text{constant} \qquad (4.5)$$

In a more general form,

$$(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = \text{constant} \qquad (4.6)$$

To determine the constant, the same procedure used, i.e.,

(a) For uniaxial tension, yielding occurs when $\sigma_1 = Y$, $\sigma_2 = \sigma_3 = 0$. Using eqn (4.5),

$$2\sigma_1^2 = \text{Constant} = 2Y^2$$

(b) For pure shear, yielding occurs when $\sigma_1 = -\sigma_3 = K$, $\sigma_2 = 0$ and Using Eqn (4.5),

$$\sigma_1^2 + \sigma_1^2 + 4\sigma_1^2 = \text{Constant} = 6\sigma_1^2 = 6K^2$$

Thus the von Mises Criterion may be written as,

$$(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2 = 6K^2 \qquad (4.7)$$

According to this criterion, the tensile and shear yield stresses are related as,

$$Y = \sqrt{3}\,K$$

It is convenient to consider each Criterion as a function of an efficive Stress denoted as $\bar{\sigma}$, where $\bar{\sigma}$ is a function of the applied stresses. Whenever its magnitude reaches the yield strength in uniaxial tension, then that applied Stress State should Cause yielding to occur. Thus,

von Mises,

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]^{1/2} \qquad (4.8)$$

Tresca,

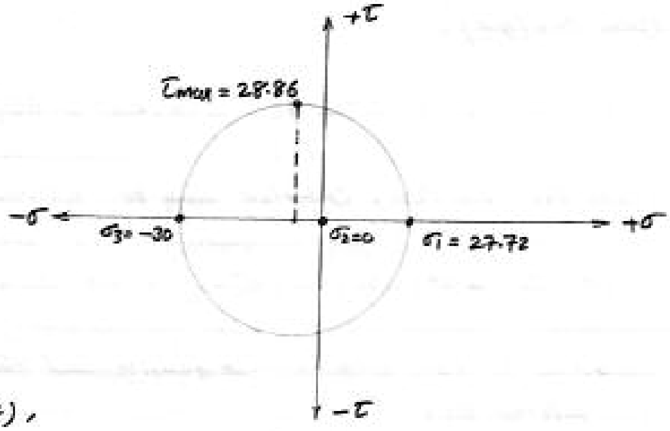
$$\bar{\sigma} = \sigma_{max} - \sigma_{min} \qquad (4.9)$$

When $\bar{\sigma}$ reaches a value of Y because of the effects of σ_1, σ_2, and σ_3 either criterion predicts yielding. However, according to the von Mises criterion when $\bar{\sigma}$ reaches a value of $\sqrt{3}\,K$ yielding is predicted whereas the Tresca criterion requires $\bar{\sigma}$ to reach $2K$ befor yielding is expected.

Example (4.2):

Repeat Example (4.1) using the von Mises Criterion for predictive purpose.

Solution:



Using Eqn (4.7),

$$[(\sigma_1-0)^2+(0-\{-30\})^2+(-30-\sigma_1)^2]=2(50)^2$$

$$\sigma_1^2+900+900+60\sigma_1+\sigma_1^2=5000$$

$$\sigma_1^2+30\sigma_1-1600=0$$

$$\sigma_1=\frac{-30\pm(900+6400)^{1/2}}{2}=\frac{-30\pm85.44}{2}$$

So, $\sigma_1=-57.72$ or $+27.72$ ksi.

In view of the question the tensile value of 27.72 ksi is the correct answer. Note that $\tau_{max}=Y/\sqrt{3}$ here whereas $\tau_{max}=Y/2$ in the previous example.

4.7 Distortion Energy:

One interpretation of the Mises criterion is that yielding occurs when the elastic energy causing distortion reaches a critical value. The total strain energy per unit volume is,

$$W_T = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) \quad (4\text{-}10)$$

or for the case of principal stresses,

$$W_T = \frac{1}{2}(\sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 + \sigma_3 \epsilon_3) \quad (4\text{-}11)$$

To express eqn (4-11) as a function of stresses, the generalized form of Hooke's law given by eqn (3-1) is used to give,

$$W_T = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\nu}{E}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \quad (4.12)$$

Since only normal stresses cause a volume change, the dilatation is,

$$\Delta = \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1-2\nu}{E}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{3}{E}(1-2\nu)\sigma_m \quad (4\text{-}13)$$

Now the normal strains associated with σ_m must be equivalent and since $\Delta = 3\epsilon_m$, then,

$$\epsilon_m = \frac{(1-2\nu)}{E}\sigma_m \quad (4\text{-}14)$$

observing that the work due to dilatation,

$$W_d = 3\left(\frac{1}{2}\sigma_m \epsilon_m\right), \text{ then } W_d = [3(1-2\nu)\sigma_m^2]/2E$$

and finally,

$$W_d = \frac{1-2\nu}{6E}(\sigma_1 + \sigma_2 + \sigma_3)^2 \quad (4\text{-}15)$$

By Substracting Equ (4.15) from equ (4.12) to give the Shear Strain energy, W_s, the result is,

$$W_s = \frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \qquad (4.16)$$

The shear strain energy induced during uniaxial tension where $\sigma_2 = \sigma_3 = 0$ is,

$$W_s = \frac{\sigma_1^2}{6G} \qquad (4.17)$$

The critical value, W_{sc}, that must be developed to cause yielding will result when $\sigma_1 = Y$. Setting equ (4.16) equal to this critical value leads to,

$$\frac{1}{12G}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] = \frac{Y^2}{6G} \qquad (4.18)$$

which is identical to equ (4.7). This explains why the von Mises criterion is often called the distorsion energy theory.

4.8 Octahedral Shear Stress,

A second physical interpretation of the von Mises criterion has also been proposed. For simiplicity consider a coordinate system defined by principal directions and a line from the origin having direction cosines $l = m = n$. The planes normal to this line and equivalent lines in other regions of space are called octahedral planes where the intersection of these eight equivalent planes from an octahedron.

For this phyiscal situation, it was shown earlier that,

$$S_n = l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 \qquad (4.19)$$

since $l = m = n = \cos 54°44' = 1/\sqrt{3}$,

$$S_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3).$$

Thus the stress normal to the octahedral planes is σ_m, and since σ_m has no influence upon yielding, it has proposed that the shear stresses acting on this plane τ_o must reach a critical value for yielding to occur. This stress can be shown as,

$$\tau_o = \frac{1}{3}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]^{1/2} \qquad (4.20)$$

and a comparison with eqn (4.8) shows, that

$$\tau_o = \frac{\sqrt{2}}{3}\bar{\sigma} \qquad (4.21)$$

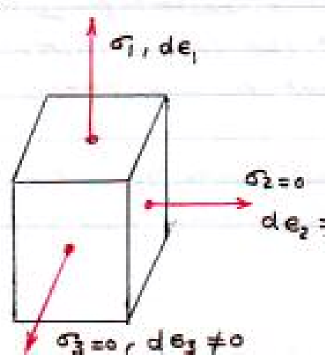
with the use of a proper multiplying factor, this's just another version of the Mises criterion.

4.9 Flow Rules or Plastic Stress-Strain Relationships:

Just as the generalized Hooke's law in the elastic region can be expressed, analogous relations for the plastic region are needed. These relations or flow rules can be developed in a simple way as follows.

Consider plastic flow under uniaxial tension as indicated in the following figure,

Fig (7) Stresses and incremental strains for uniaxial tension.



Now the deviatoric stress in the 1 direction is,

$$\sigma_1' = \sigma_1 - \sigma_m$$

and at the particular instant,

$$\sigma_1' = \sigma_1 - \frac{1}{3}\sigma_1 = \frac{2}{3}\sigma_1$$

and $\sigma_2' = \sigma_3' = 0 - \frac{1}{3}\sigma_1 = -\frac{1}{3}\sigma_1$

so, $\sigma_1' = -2\sigma_2' = -2\sigma_3'$

For volume constancy, the sum of the plastic strain increments must be zero, therefore,

$$d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0$$

and because of symmetry in this instance $d\epsilon_2 = d\epsilon_3$.

Therefore,

$$d\epsilon_1 = -2\,d\epsilon_2 = -2\,d\epsilon_3$$

This leads to $d\epsilon_1 / d\epsilon_2 = -2 = \sigma_1' / \sigma_2'$, and so forth, which can be written as,

$$\frac{d\epsilon_1}{\sigma_1'} = \frac{d\epsilon_2}{\sigma_2'} = \frac{d\epsilon_3}{\sigma_3'} = \text{Constant} = d\lambda \qquad (4.22)$$

i.e. the constant isn't always -2 but these ratios are always in some constant proportion. The implication is that the ratio of the current incremental plastic strain increments to the current deviatoric stresses is a constant.

For greater convenience, the flow rules may be expressed in various forms other than equ (4.22). These are,

(a) $\dfrac{d\epsilon_1 - d\epsilon_2}{\sigma_1 - \sigma_2} = d\lambda$ (4.23)

(b) $d\epsilon_1 = \frac{2}{3} d\lambda \left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3) \right]$ (4.24)

(c) $d\epsilon_1 = \dfrac{d\bar{\epsilon}}{\bar{\sigma}} \left[\sigma_1 - \frac{1}{2}(\sigma_2 + \sigma_3) \right]$ (4.25)

The incremental effective strain, $d\bar{\epsilon}$, requires defenetion and the form ~~to~~ be used is,

$$d\bar{\epsilon} = \frac{\sqrt{2}}{3} \left[(d\epsilon_1 - d\epsilon_2)^2 + (d\epsilon_2 - d\epsilon_3)^2 + (d\epsilon_3 - d\epsilon_1)^2 \right]^{1/2} \quad (4.26)$$

Flow rules for any yield criterion may be derived by using the concept of a plastic potential. This method proposes that the incremental strains resulting from a stress σ_{ij} are found by using,

$$d\epsilon_{ij} = \frac{\partial f}{\partial \sigma_{ij}} (d\lambda') \quad (4.27)$$

where f is taken as the yield function. If the von Mises criterion is used,

$$f(\sigma_{ij}) = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = \text{constant}$$

then,

$$\frac{\partial f}{\partial \sigma_1} = 2(\sigma_1 - \sigma_2) - 2(\sigma_3 - \sigma_1) = 4\sigma_1 - 2(\sigma_2 + \sigma_3)$$

but, $3\sigma_m = \sigma_1 + \sigma_2 + \sigma_3$

so, $(\sigma_2 + \sigma_3) = 3\sigma_m - \sigma_1$

Now, $\dfrac{\partial f}{\partial \sigma_1} = 6(\sigma_1 - \sigma_m) = 6\sigma_1'$

Finally,

$$d\epsilon_1 = 6\sigma_1' d\lambda' \quad \text{or} \quad \frac{d\epsilon_1}{\sigma_1'} = d\lambda \quad (4.28)$$

It was earlier stated that equ (4.22) expresses a reduced form of the Prandtl-Russ equations where the elastic strains were omitted. To clarify this point, the following illustration is given. Loading is by uniaxial tension and the normal strain in the direction of loading is to be determined.

$$(d\epsilon)_{total} = (d\epsilon)_{plastic} + (de)_{elastic}$$

$$d\epsilon_{t_1} = d\epsilon_1 + de_1$$

$$d\epsilon_{t_1} = \frac{d\bar{\epsilon}}{\bar{\sigma}}\left[\sigma_1 - \tfrac{1}{2}(\sigma_2+\sigma_3) + \frac{1}{E}\left[d\sigma_1 - \nu(d\sigma_2 + d\sigma_3)\right]\right. \qquad (4.29)$$

Note the plastic strains are related to total stresses whereas elastic strains are associated with incremental stress changes. The following Figure demonstrates this point,

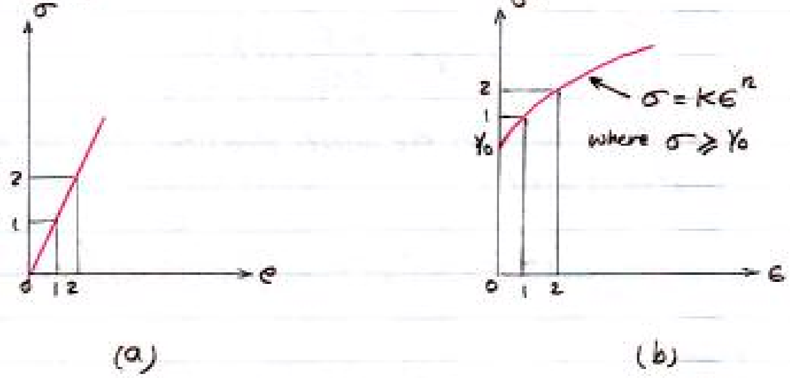


Fig. (8) Stress-Strain Curves for elastic & plastic deformation.

In Fig (8) a an equal incremental change in stress causes the same incremental change in strain, since linear relation exists. However, in Fig. (8) b an equal incremental stress change does not produce the same incremental change in strain; instead, these are related to a changing slope, so a particular increment depends upon the total stress at a given instant.

Example (4.3):

Consider a thin walled tube having closed ends that is internally pressurized to 1000 psi ; it is assumed that this causes plastic deformation to occur and end effects are ignored. The total strain in the hoop direction is to be found. With this loading state, the following stress relations are acceptable,

$$\sigma_\theta = \sigma_1 = \frac{Pr}{t} \; , \; \sigma_l = \sigma_2 = \frac{Pr}{2t} \; , \; \sigma_r = \sigma_3 = 0 \qquad (a)$$

Consider the r/t ratio to be 20 and note that $\sigma_2 = \frac{\sigma_1}{2} = \sigma_m$.

Solution,

With use of Equ (4.25),

$$d\epsilon_1 = -d\epsilon_3 \, , \; d\epsilon_2 = 0 \qquad (b)$$

In incremental form, the total strain in the 1 direction is,

$$d\epsilon_t = d\epsilon + de \quad (\text{i.e. plastic + elastic})$$

Using Equ (4.25) and (3.1),

$$d\epsilon_t = \frac{d\bar{\epsilon}}{\bar{\sigma}}\left[\sigma_1 - \frac{1}{2}(\sigma_2+\sigma_3)\right] + \frac{1}{E}\left[d\sigma_1 - \nu(d\sigma_2 + d\sigma_3)\right]$$

Which reduces to,

$$d\epsilon_t = \frac{d\bar{\epsilon}}{\bar{\sigma}}\left[\frac{3}{4}\sigma_1\right] + \frac{d\sigma_1}{E}\left[1 - \frac{\nu}{2}\right]$$

Using the relations in Equ (a). Substituting the stress relations of equ (a) into Equ (4.8) and the strain relations of Equ (b) into Equ (4.26) gives,

$$\bar{\sigma} = \frac{\sqrt{3}}{2}\sigma_1 \quad \text{and} \quad d\bar{\epsilon} = \frac{2}{\sqrt{3}}d\epsilon_1$$

For the values in the problem statement,

$$\sigma_1 = 20 \text{ ksi} \quad \text{so} \quad \bar{\sigma} = 10\sqrt{3} \text{ ksi}$$

What is now essential is an effective stress – effective strain relationship. Assume for now that the form,

$$\bar{\sigma} = K\bar{\epsilon}^n$$

is appropriate for the plastic portion of deformation where $K = 25$ ksi and $n = 0.25$. Then the relation,

$$10\sqrt{3} = 25(\bar{\epsilon})^{0.25}$$

results in,

$$\bar{\epsilon} = (0.693)^4 = 0.23$$

which is from eqn (4-26),

$$\int_0^{\bar{\epsilon}} d\bar{\epsilon}$$

Therefore,
$$\int_0^{\epsilon_1} d\epsilon_1 = \int_0^{\bar{\epsilon}} \frac{\sqrt{3}}{2} d\bar{\epsilon}$$

so,
$$\epsilon_1 = 0.199$$

To compute the elastic portion of strain, note that $d\sigma_1$ equals $20\, dP$ where dP equals 1000 psi. For aluminum (whose values of K and n are reasonably well represented by the numbers used above) take $E = 10^7$ psi and $\nu = \frac{1}{3}$. Since,

$$de_1 = d\sigma_1(1 - \nu/2)/E, \quad e_1 \cong 0.002 \quad \text{so,}$$

~~$\epsilon_t = 0.199 + 0.002 = 0.201$.~~

From the previous example several points are pertinent,

1. Where large plastic strains are encountered, ignoring elastic strains introduces little error and often greatly simplifies an analysis.
2. Many real problems must invoke the use of approximation since the resulting deformation does not follow a simple loading path as used in this example.
3. Some type of $\bar{\sigma}-\bar{\epsilon}$ relationship must be available if numerical answers are required.
4. The lower limit on the above integral were taken as zero. Physically this implies that the pressure was zero at the outset and no elastic or plastic strains had been induced in the material before the application of pressure (i.e no initial elastic or plastic strains).

4.10 Plastic Work:

Consider a bar of length l_0 subjected to a tensile force F acting on an area $(w_0 t_0)$ with a resulting extension dl. The work done is $F\,dl$ and on a unit volume basis is,

$$dW_v = \frac{F\,dl}{w_0 t_0 l_0} = \frac{F}{w_0 t_0} \cdot \frac{dl}{l_0} = \sigma\, d\epsilon \qquad (4\text{-}30)$$

If a shear force caused deformation, a similar argument would show that $\tau\, d\gamma$ expresses the work per unit volume done by that force. These individual contributions could be summed up in a manner that produced Eqn (4-10), noting that the coefficient of one-half does not appear in Eqn (4-30). In terms of principal stresses,

$$dW_v = \sigma_1 d\epsilon_1 + \sigma_2 d\epsilon_2 + \sigma_3 d\epsilon_3 \qquad (4\text{-}31)$$

In terms of effective stress and strain functions, the resulting expression is,

$$dW_p = \bar{\sigma}\, d\bar{\epsilon} \qquad (4.32)$$

To examples will demonstrate this equivalence. Consider uniaxial tension where $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = 0$ and $d\epsilon_2 = d\epsilon_3 = -\frac{1}{2} d\epsilon_1$ (since $d\epsilon_1 + d\epsilon_2 + d\epsilon_3 = 0$). Using eqn (4.31) gives,

$$dW_p = \sigma_1\, d\epsilon_1$$

Since the other terms vanish. The effective stress and strain functions show that,

$$\bar{\sigma} = \sigma_1 \quad \text{and} \quad d\bar{\epsilon} = d\epsilon_1$$

so equivalence results were obtained.

Next consider pure shear where $\sigma_1 = -\sigma_3$, $\sigma_2 = 0$ and $d\epsilon_1 = -d\epsilon_3$, $d\epsilon_2 = 0$. From eqn (4.31),

$$dW_p = \sigma_1\, d\epsilon_1 + (-\sigma_1)(-d\epsilon_1) = 2\sigma_1\, d\epsilon_1$$

The relation in terms of effective values shows that

$$\bar{\sigma} = \sqrt{3}\,\sigma_1 \quad \text{while} \quad d\bar{\epsilon} = \frac{2}{\sqrt{3}}\, d\epsilon_1$$

Thus,

$$dW_p = (\sqrt{3}\,\sigma_1)\left(\frac{2}{\sqrt{3}}\, d\epsilon_1\right) = 2\sigma_1\, d\epsilon_1$$

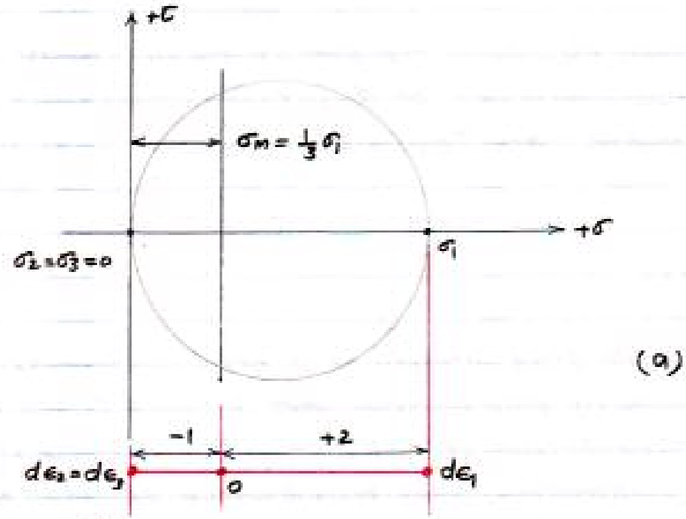
as before, the same results.

Thus besides finding use in regard to yielding predictions and the flow rules, the concepts of effective stress and strain provide a convenient way to calculate the work due to plastic deformation.

4.11 Comparison of Mohr's Circles for Stress and Plastic Strain Increments.

Since the hydrostatic component of the total stress causes no plastic flow, this implies that the value of σ_m on a circle plot of stresses should coincide with the origin of the incremental strain circle (i.e. $d\epsilon = 0$). Under uniaxial tension, $\sigma_m = \frac{1}{3}\sigma_1$, $\sigma_2 = \sigma_3 = 0$, while

$$d\epsilon_2 = d\epsilon_3 = -\frac{1}{2}\, d\epsilon_1$$



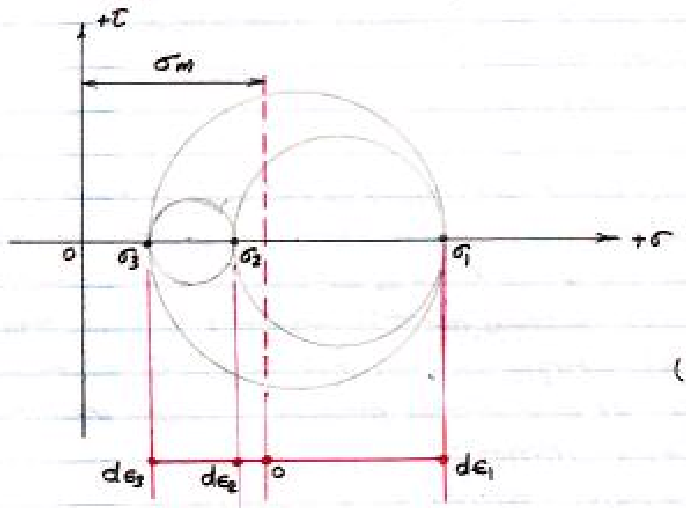


Fig.(9) Relationship of Mohr's circle of stress and incremental plastic strains for uniaxial tension and a general triaxial situation.

Fig.(9)(a) shows the physical meaning that relates the stress circle to the scaled results in terms of strains; note the negative two-to-one correspondence of incremental strains.

Fig.(9)(b) shows a more general case where $\Sigma d\epsilon_i$ must equal zero. It is also instructive to consider plane strain, where from equ (3.13),

$$\sigma_2 = \frac{1}{2}(\sigma_1 + \sigma_3) \quad \text{since } \nu = \frac{1}{2} \text{ for plastic flow.}$$

For this case, $\sigma_m = \sigma_2$ so the zero of the incremental strain circle coincides with σ_2 or $d\epsilon_2 = 0$. This result can be readily checked using Equ (4.25).

Problems:

(4.1) The tensile yield strength of a metal is given as Y ksi. If a specimen of this material were subjected to two normal compressive stresses of $-\frac{1}{4}Y$ and $-\frac{1}{2}Y$ acting along the x and y directions, what tensile stress applied in the z direction would cause yielding according to,

(a) von Mises criterion,

(b) Tresca criterion.

(4.2) A long, thin-walled tube, capped on the ends is made of a metal whose yield strength in uniaxial tension is 40 ksi. The tube is 60 in long, has a wall thickness of 0.015 in and a diameter 2 in. Under service conditions the tube experience an axial tensile load of 1000 lbf, a torque of 1000 lb-in and is to be pressurized internally. At what internal pressure is yielding predicted by,

(a) the Tresca Criterion?

(b) the von Mises Criterion?

(4.3) A thin-walled cylinder whose diameter is 80 mm, and its wall thickness 3.5 mm, just yields when auniform axial stress of 200 MPa is applied. If an identical cylinder is loaded in bending to a maximum axial normal stress of 140 MPa, calculate the internal pressure required to cause yielding, using the Mises criterion.

(4.4) A material whose yield strength is 50 ksi made in the form of a cube and subjected to a tensile stress, σ_1, along one axis and a stress $\sigma_3 = -\sigma_1/2$ along a second set of axes.

(a) Determine the ratio of the principal strain increment $d\epsilon_1 / d\epsilon_2$.

(b) Using the von Mises criterion, determine the magnitude of τ_{max} at the onset of yielding.

(c) Repeat (b) using Tresca criterion.

(4.5) A stress state is described in (ksi) by $\sigma_1 = 80$, $\sigma_2 = 15$, $\sigma_3 = 0$.

(a) Determine the strain ratio $d\epsilon_1 / d\epsilon_3$.

(b) If a hydrostatic stress of 20 ksi is superimposed by fluid pressure upon the intial stress state, how does the ratio $d\epsilon_1 / d\epsilon_3$ change?

(4.6) A thin-walled cylinder just yields when the torsional shear stress reaches 43.8 ksi. If an identical cylinder is loaded to a torsional shear stress of 38.6 ksi, calculate the applied axial compressive stress necessary to just cause yielding. Assume the Tresca criterion holds.

(4.7) A rigid-plastic cube is subjected to σ_1 on one pair of opposite faces, to $\sigma_2 = 0.2\sigma_1$ on a second pair, and to $\sigma_3 = -0.4\sigma_1$ on the third pair of faces. The stress is gradually increased, maintaining the above ratios. Using the Mises criterion (for $Y = 300\ MN/m^2$), calculate the magnitudes ~~of the principle stresses and the ratios of the three principal strain increments at the moment of yielding.~~