③ Isotropic Elasticity:

The coupling of stress and strain, leading to what are usually called constitutive relations, has been found to be dependent upon particular properties of the solid in question. When these properties are found to be the same regardless of the direction of measurement, the solid is said to be isotropic. If, however, the value of property differes as a function of direction, the solid is classed as anisotropic with regard to the property of concern. A solid is said to be homogeneous if its properties do not vary with location within the body.

3.1 Model of a Perfectly Elastic Solid:

For a perfectly elastic solid, we consider the model shown below,

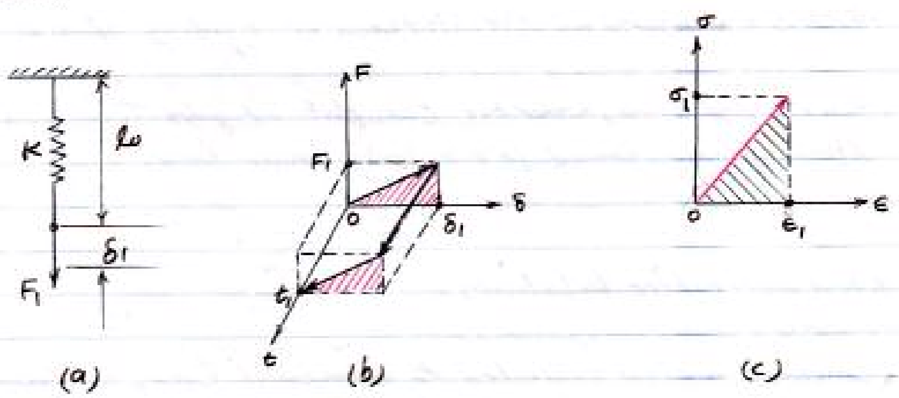


Fig.(1) Description of a perfectly elastic solid.

certain assumptions are pertinent,

1. Prior to the application of force F_1, the solid has a length l_o and property whose resistance to extension under load is defined by K. This property is analogous to the modulus of elasticity and does not vary with time.

2. Under load F_1, the body extends an amount δ_1 and this extension takes place instantaneously and linearly.

3. The energy stored in the body is shown as the shaded area in the $F-\delta$ plane.

4. As F_1 is applied for the time t_1, δ_1 remains constant.

5. Upon release of F_1 instantaneous recovery occurs and all of the stored energy is released.

6. The same analysis would hold if l_0 were shortened rather than extended.

7. The equivalent $\sigma-\epsilon$ curve is as shown in Fig.(1)(C).

The slope of the $\sigma-\epsilon$ plot is the modulus of elasticity E which is a measure of the stiffness or rigidity of a solid.

The shaded area, under the $\sigma-\epsilon$ plot at zero time, is the strain energy stored in the body under load.

3.2 Constitutive Relations:

Consider a solid subjected to a uniaxial force, as described by Fig.(1). Under elastic deformation, the stress and strain are directly related by the elastic modulus. This is known as Hooke's Law and in its simplest form is,

$$\sigma = E\epsilon$$

A generalized form of this relationship, extended to three-dimensions, is found to have wider use. Experiments show

3.3 Elastic Constants:

Note that shear strains cause only a change in shape while any volume change is produced by normal strains. The latter is related to the bulk modulus, B, and the forgoing comments reduce to the following,

1. E = elastic modulus, which relates normal stresses to strains.

2. G = shear modulus, which relates shear stresses to strains.

3. ν = Poisson's ratio, which relates longitudinal and transverse strains.

4. B = bulk modulus, which relates the mean normal stress σ_m to the volume strain or dilation, Δ.

The fractional volume change or dilation is defined as,

$$\frac{dV}{V} = \Delta \equiv \epsilon_1 + \epsilon_2 + \epsilon_3 = \frac{1-2\nu}{E}\left(\sigma_1 + \sigma_2 + \sigma_3\right) \quad (3.4)$$

The mean normal stress is,

$$\sigma_m \equiv \frac{1}{3}\left(\sigma_1 + \sigma_2 + \sigma_3\right) = \frac{\Delta E}{3(1-2\nu)} \quad (3.5)$$

The bulk modulus is defined as,

$$B \equiv \frac{\sigma_m}{\Delta} = \frac{E}{3(1-2\nu)} \quad (3.6)$$

If the case of pure shear is considered, the Mohr's Circle for stress and strain are shown in the following figure,

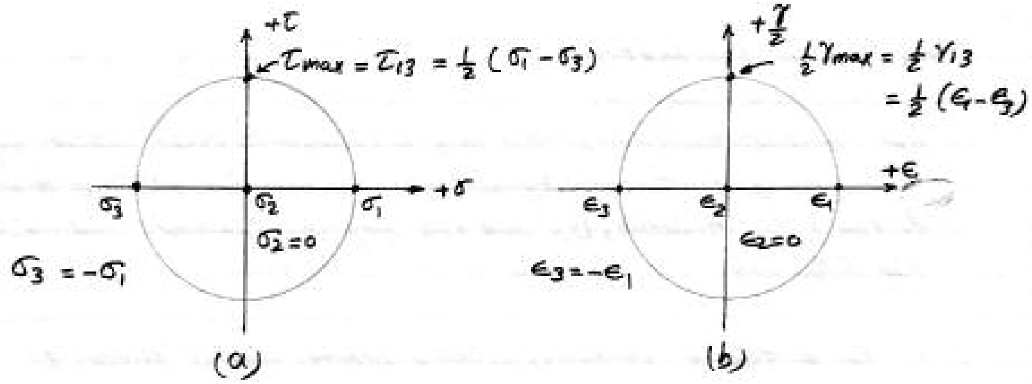


Fig. (2) Mohr's circles of stress and strain for the case of pure shear.

From the above figure,

$$\gamma_{13} = \epsilon_1 - \epsilon_3 = 2\epsilon_1$$

From equ (3.1),

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] = \frac{\sigma_1}{E}(1+\nu)$$

Since $\sigma_3 = -\sigma_1$, $\sigma_2 = 0$.

there fore,

$$\gamma_{13} = 2\epsilon_1 = \frac{2\sigma_1}{E}(1+\nu) = \frac{\tau_{13}}{G} = \frac{\sigma_1 - \sigma_3}{2G} = \frac{\sigma_1}{G}$$

since $\sigma_3 = -\sigma_1$.

So,

$$\gamma_{13} = \frac{2\sigma_1}{E}(1+\nu) = \frac{\sigma_1}{G} \quad \text{or} \quad G = \frac{E}{2(1+\nu)} \qquad (3.7)$$

that as the tensile stress σ_1 is applied it is coupled with a tensile strain ϵ_1 and with contructions in two perpendicular planes in the direction at right angles to the line of force applications.

Use either x-y-z or 1-2-3 coordinate system, the strain in the 2 and 3 directions are directly related to the strain ϵ_1 through the parameter called Poisson's ratio ν. Thus,

$$\epsilon_2 = \epsilon_3 = -\nu \epsilon_1$$

Since the contraction cause negative strains.

In more general situation, tensile stresses might also be applied simultaneously in the three directions, each stress causing a negative strain in the other two directions. Because all equations are linear, this means,

$$\epsilon_1 = \frac{\sigma_1}{E} - \nu \frac{\sigma_2}{E} - \nu \frac{\sigma_3}{E} = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \quad (3.1)a$$

Note that in uniaxial tension $\sigma_2 = \sigma_3 = 0$

Identical expressions for ϵ_2 and ϵ_3 result by appropriate change of subscripts,

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \nu(\sigma_1 + \sigma_3)\right] \quad (3.1)b$$

$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \nu(\sigma_1 + \sigma_2)\right] \quad (3.1)c$$

If an x-y-z system is employed, then,

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \quad (3.2)a$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \quad (3.2)b$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \qquad (3.2)\,c$$

Regarding shear stress and shear strains, these are related by,

$$\tau_{xy} = G\gamma_{xy} = 2G\epsilon_{xy} \qquad (3.3)$$

where G is the shear modulus.

Example (3.1):

A material whose elastic modulus is 207 GPa is subjected to three normal stresses $\sigma_1, \sigma_2, \sigma_3$ whose magnitudes are 250, 200, −100 MPa respectively. Determine the normal strains in the three directions. Assume Poisson's ratio is 0.3.

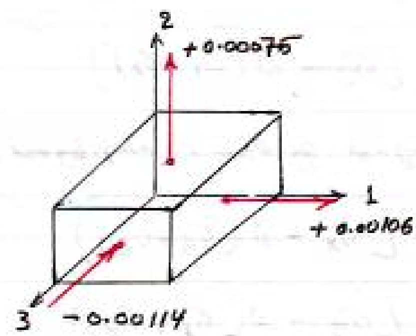
Solution:

Employing eqn (3.1), gives,

$$\epsilon_1 = \frac{1}{207*10^3}\left[250 - 0.3(200-100)\right] = +0.00106$$

$$\epsilon_2 = \frac{1}{207*10^3}\left[200 - 0.3(250-100)\right] = +0.00075$$

$$\epsilon_3 = \frac{1}{207*10^3}\left[-100 - 0.3(250+200)\right] = -0.00114$$



Example (3.2):

(a) With the finding in Example (3.1), find the dilatation using equ (3.4).

(b) Repeat part (a) using equ (3.5).

Solution:

(a) $$\Delta = \epsilon_1 + \epsilon_2 + \epsilon_3 = +0.00181 - 0.00114 = +0.00067$$

(b) $$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(+350) = \frac{\Delta E}{3(1-2\nu)}$$

$$\therefore \Delta = \frac{350(1-0.6)}{207 \times 10^3} = +0.00067$$

The positive sign means an increase in volume of the body subjected to these three normal stresses.

3.4 Relationship of Mohr's circle for Stress and Strain:

It is instructive to show that if the stress circle is known, the corresponding strain circle can be developed. For the general case,

$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right] \quad \text{and} \quad \sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

These two equations when rearranged, give,

$$\epsilon_1 = \frac{\sigma_1}{E}(1+\nu) - \frac{3\nu}{E}\sigma_m$$

and since $G = E/2(1+\nu)$, then

$$\epsilon_1 = \frac{\sigma_1}{2G} - \frac{3\nu\sigma_m}{2G(1+\nu)} = \frac{1}{2G}\left(\sigma_1 - \frac{3\nu\sigma_m}{1+\nu}\right) \qquad (3.8)$$

Thus, the origin or zero of the strain circle is displaced an amount equal to $3\nu\sigma_m/(1+\nu)$ from the origin of the stress circle.

Two examples will illustrate the above findings. Consider uniaxial tension as shown below, where $\sigma_1 = 12$ and $\nu = \frac{1}{3}$. Then,

$$\sigma_m = 4 \quad \text{and} \quad 3\nu\sigma_m/(1+\nu) = 3$$

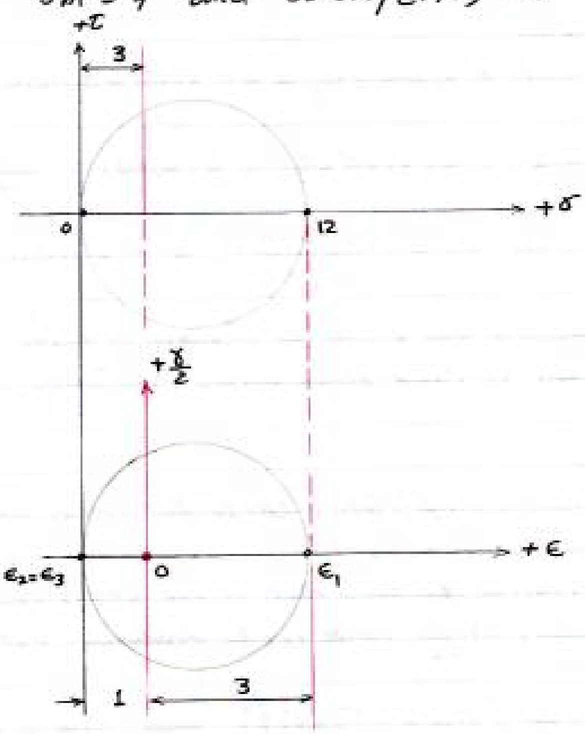


Fig.(3) Mohr's circle of stress and the corresponding strain for uniaxial tension.

By using scaled plots in both circles, it is seen that ϵ_1 is three times ϵ_2 or ϵ_3 in magnitude and opposite in sign. Thus,

$$\epsilon_2 = \epsilon_3 = -\frac{1}{3}\epsilon_1$$

which is must be true since $\nu = \frac{1}{3}$. The physical meaning of the above figure is that the pricipal stress and strain axes conincide.

As a second example consider pure shear where, $\tau_{max} = \sigma_1 = -\sigma_3$. The following figure shows three interpretations,

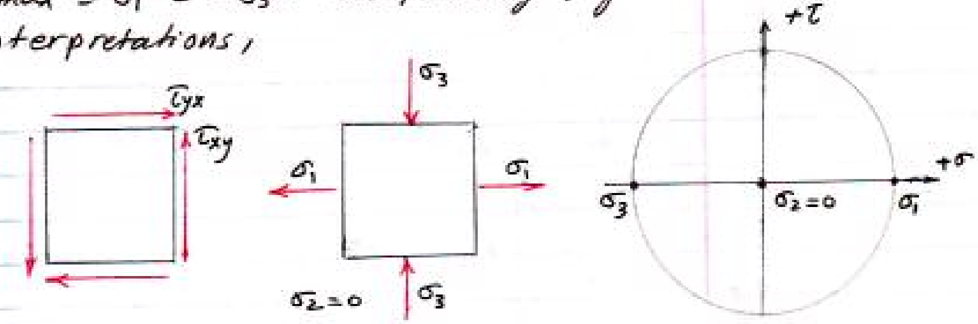


Fig.(4) Equivalent descriptions of pure shear,

Using Eqn (3.1),

$$\epsilon_1 = \frac{\sigma_1}{E}(1+\nu), \quad \epsilon_3 = -\frac{\sigma_1}{E}(1+\nu), \quad \epsilon_2 = 0$$

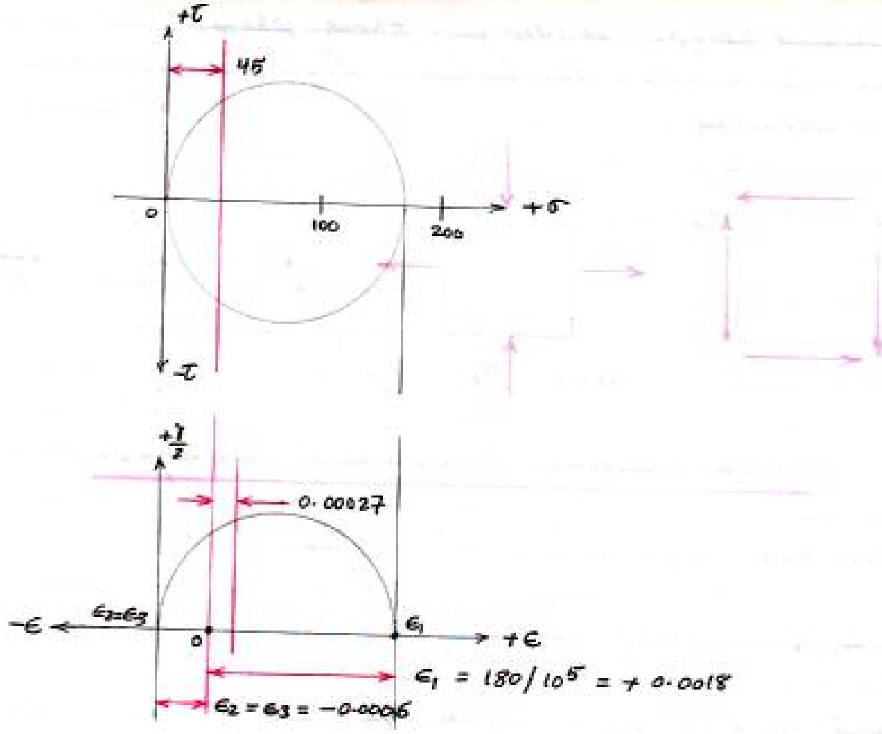
and since $\sigma_m = 0$, $3\nu\sigma_m/(1+\nu) = 0$.

For this situation, the centres of the stress and strain circles coincide. Note that $\Delta = 0$ which means there is no volume change; this should be expected since pure shear produces a shape change only.

Example (3.3):

The elastic modulus of a solid is 100 GPa and it has a Possion's ratio of $\frac{1}{3}$. It is subjected to a uniaxial tensile stress of 180 MPa. Construct the Mohr's circles for stress and corresponding strains.

Solution:



$\sigma_m = 60$, So that the origin of the strain circle is displaced from the origin of the stress circle by,

$$3(\tfrac{1}{3})\cdot 60/(1+\tfrac{1}{3}) = 45$$

$\epsilon_2 = \epsilon_3 = -\nu\epsilon_1 = -0.0006$ which checks the scalled plot.

3.5 Strain Energy due to the work of elastic deformation:

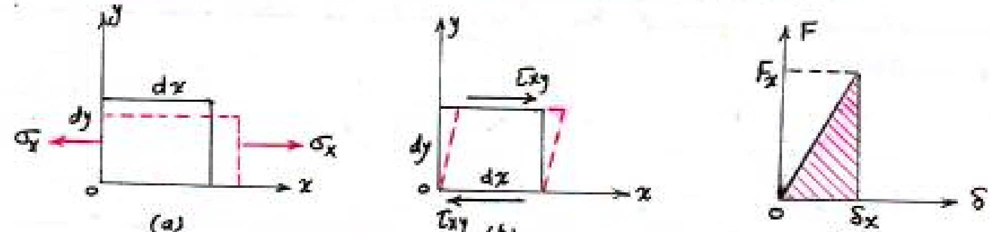


Fig.(15) Dan's for determining Strain energy induced by Applied Stresses and accompanying Strains.

Figures (a) and (b) above, show a normal and simple shear stress acting on stress elements of dimensions dx, dy, dz. In part (a), the stress σ_x produces a strain ϵ_x and the work done is,

$$d(\text{work done}) = \frac{1}{2} F_x \cdot \delta_x$$

where, $F_x = \sigma_x \, dy \, dz$ and $\delta_x = \epsilon_x \, dx$

Therefore,

$$dW = \frac{1}{2} F_x \cdot \delta_x = \frac{1}{2} \sigma_x \epsilon_x \, dx \, dy \, dz$$

i.e., $dW = \frac{1}{2} \sigma_x \epsilon_x \, dv$

Thus, the work done per unit volume,

$$\frac{dW}{dV} = \frac{1}{2} \sigma_x \epsilon_x \qquad (3.9)$$

Similarly, in Figure (b),

$$dW = \frac{1}{2} (\tau_{xy} \, dx dz)(\gamma_{xy} \, dy) = \frac{1}{2} \tau_{xy} \gamma_{xy} \, dx \, dy \, dz$$

thus,

$$\frac{dW}{dV} = \frac{1}{2} \tau_{xy} \gamma_{xy} \qquad (3.10)$$

In the most general case, and with the use of superposition, the work or strain energy per unit volume is expressed by,

$$W_v = \frac{1}{2} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx})$$

--- --- (3.11)

Where principal direction are involved,

$$W_v = \frac{1}{2} (\sigma_1 \epsilon_1 + \sigma_2 \epsilon_2 + \sigma_3 \epsilon_3) \qquad (3.12)$$

3.6 Plane-Stress and Plane-Strain Physical situation:

Plane-stress (often called biaxial) occurs when,

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0 \quad (z \text{ is a principal direction}) \qquad (3.13)$$

Plane-Strain occurs when displacements are everywhere independent of one direction, that is, the strain in one direction is zero. In effect, all displacement occurs in one plane and is everywhere parallel. If, for instance, $\epsilon_z = 0$, then,

$$\sigma_z = \nu(\sigma_x + \sigma_y) \qquad (3.14)$$

This relationship is always true when plane strain occurs.

Problems:

(3.1) A thin-walled tube with closed ends is subjected to internal pressure P. The tube radius is R and wall thickness is t. The following are adequate for stress calculations,

$$\text{Hoop stress}, \sigma_1 = PR/t$$
$$\text{Axial stress}, \sigma_2 = PR/2t$$
$$\text{Radial stress}, \sigma_3 \simeq 0$$

Derive the above relation from first principles, Taking ν possion's ratio equal to $\frac{1}{2}$.

(3.2) Consider a cube of steel that is 1 in per side subjected to a uniaxial tensile stress of 30 ksi. Determine the magnitude of volume increase or decrease.

(3.3) A thin sheet of aluminum is subjected to the unbalanced biaxial tensile stress state, $\sigma_x = 10$ ksi, $\sigma_y = 5$ ksi, $\sigma_z = 0$ (z is the thickness direction). The thickness before loading is 0.035 in.

(a) Find the three principle strains.

(b) What is the thickness after the stresses are applied?

(3.4) Under plane stress loading (say σ_x and σ_y) the strains $\epsilon_x = +0.001$ and $\epsilon_y = -0.0007$ are measured. If $E = 200$ GPa and $\nu = \frac{1}{3}$, find ϵ_z.

(3.5) A block of metal is subjected to three mutually perpendicular normal stresses of $\sigma_x = 0.14$ GPa, $\sigma_y = -0.2$ GPa, and $\sigma_z = 0.25$ GPa. If the elastic modulus is 200 GPa and Poisson's ratio is 0.3, determine the work or strain energy per unit volume of metal that is induced. Assume all deformation is elastic.

(3.6) Loading on a slab of material induces plane strain deformation. If direction 1 coincide with loading, 2 with the direction zero strain, and no loading occurs in the 3 direction, determine,

(a) the magnitudes of the three normal strains

(b) the strain energy per unit volume that is induced

Here, $\sigma_1 = 50$ ksi, $E = 28 \times 10^6$ psi and $\nu = 0.33$.

(3.7) The elastic modulus of a solid is 10^7 psi and $\nu = 0.36$. If this body is subjected to normal stresses where $\sigma_1 = 500$ MPa, $\sigma_2 = -150$ MPa and $\sigma_3 = 0$

(a) Construct the Mohr's circle of stress.

(b) Construct the corresponding strain circle and by introducing a proper scale to this circle, find the three normal strains.

(c) Check the results in part (b) by using Eqn (3.1).