② Analysis of Strain:

2.1 Definition:

When a body is deformed by external forces, points in the body may be displaced from the positions they occupied under no load. Strain is related to such displacements.

Fig.(1) Displacement of an arbitrary length.

Consider the above figure, where the intial length of AB, denoted as l_o is displaced to $A'B'$ (i.e. l') due to external forces. l_o and l' are unequal in length so a state of strain has been induced. When this condition has been satisfied, the strain in AB is defined as,

$$\epsilon \equiv \frac{l' - l_o}{l_o} = \frac{\Delta l}{l_o} \qquad (2\text{-}1)$$

and by chossing small values of these lengths, the concept or mathmatical notion of the state of strain at a point may be invoked in a manner similar to that used with stress.

Tensile strains, caused by extensions, are considered positive whereas compressive strains, due to shortening, are negative. Shear strains, due to angular distorsion, are defined as negative or positive in the next section.

2.2 The Two-dimensional Case for Small Strains:

Consider a two-dimensional case as illustrated in the following figure,

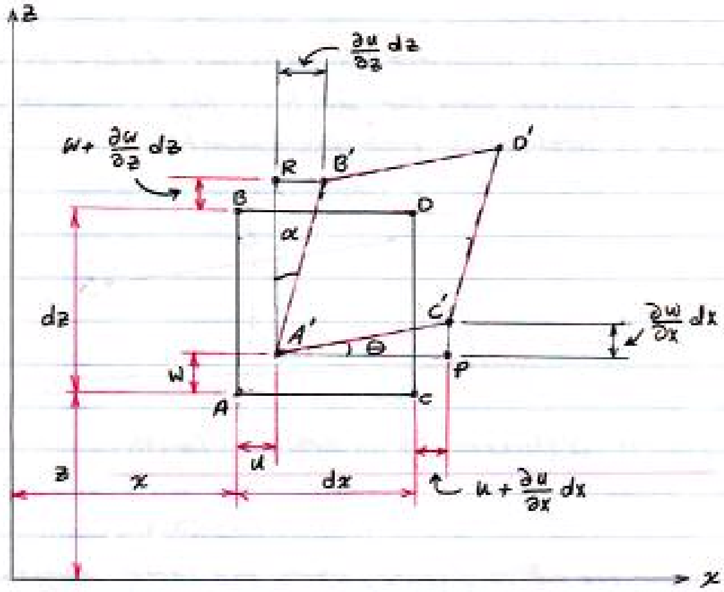


Fig.(2) Description of the Case of small strains in two dimensions.

Initially the element of concern is ABCD where $A(x, z)$, $C(x+dx, z)$, $B(x, z+dz)$ and $D(x+dx, z+dz)$ defines the corners of the element. Due to external effects the element distorts to $A'B'C'D'$ where,

$$A'(x+u,\ z+w), \quad C'\left(x+dx+u+\frac{\partial u}{\partial x}dx,\ z+w+\frac{\partial w}{\partial x}dx\right)$$

defines the ends of the line whose initial length was AC.

Physically, small displacement means $A'P \approx A'C'$ and θ is also small (i.e. $\tan\theta \simeq \theta$).

$$\epsilon_{xx} \equiv \frac{A'C'}{AC} - 1 \simeq \frac{A'P}{AC} - 1 = \frac{dx - u + u + \frac{\partial u}{\partial x} dx}{dx} - 1$$

$$\epsilon_{xx} = 1 + \frac{\partial u}{\partial x} - 1 = \frac{\partial u}{\partial x}$$

and similarly,

$$\epsilon_{zz} = \frac{\partial w}{\partial z}, \quad \epsilon_{yy} = \frac{\partial v}{\partial y}$$

<u>Note:</u>

$$\epsilon_{xx} = \frac{A'C' - AC}{AC} = \frac{A'C'}{AC} - 1$$

Now for shear strain associated with A, this is defined as the change in the angle CAB. This shear strain is denoted as γ_{xz} since its occures in the x-z plane.

The total change in angle CAB is $\alpha + \theta$, recalling that $\tan\theta \simeq \theta$ with our assumptions,

$$\theta = \tan^{-1} \frac{\frac{\partial w}{\partial x} dx}{A'P} = \tan^{-1} \frac{\frac{\partial w}{\partial x} dx}{dx + \frac{\partial u}{\partial x} dx}$$

so,

$$\theta = \frac{\frac{\partial w}{\partial x}}{1 + \frac{\partial u}{\partial x}} = \frac{\partial w}{\partial x} \quad \text{since } \frac{\partial u}{\partial x} \ll 1.$$

Similarly,

$$\alpha = \frac{\partial u}{\partial z}$$

therefore,

$$\gamma_{xz} = \alpha + \theta = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

where u and w are displacements in the x and z directions.

The full expressions for the strain components at a point could be defined by the strain matrix, that is,

$$\epsilon_{ij} = \begin{vmatrix} \epsilon_{xx} & \epsilon_{yx} & \epsilon_{zx} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{zy} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{vmatrix} \qquad (2.2)$$

Here, however,

$$\epsilon_{xz} \equiv \frac{1}{2}\gamma_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\epsilon_{xy} \equiv \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad (2.3)$$

$$\epsilon_{yz} = \frac{1}{2}\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

The γ form is the shenz strain associated with an applied shenz stress and the shenz modulus; it is usually referred to as engineering shenz strain. They are given as,

$$\gamma_{xy} = \frac{du}{dy} + \frac{\partial v}{dx}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad (2.4)$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

In general case, the normal strains are expressed by the displacement u, v, and w in the x, y, and z directions as follows,

$$\epsilon_x = \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \qquad (2.5)$$

$$\epsilon_z = \frac{\partial w}{\partial z}$$

2.3 Strain Transformations:

Transformation of strain from one set of refrence axes to another set follows a development identical to that of stresses. The resulting expressions take the following forms,

$$\epsilon_{x'x'} = l^2_{x'x} \epsilon_{xx} + l^2_{x'y} \epsilon_{yy} + l^2_{x'z} \epsilon_{zz} + l_{x'z} l_{x'x} \epsilon_{zx}$$

$$+ l_{x'x} l_{x'y} \epsilon_{xy} + l_{x'y} l_{x'z} \epsilon_{yz} + l_{x'z} l_{x'y} \epsilon_{zy}$$

$$+ l_{x'x} l_{x'z} \epsilon_{xz} + l_{x'y} l_{x'x} \epsilon_{yx} \qquad (2.6)$$

and,

$$\epsilon_{y'z'} = l_{y'z} l_{z'z} \epsilon_{zz} + l_{y'y} l_{z'y} \epsilon_{yy} + l_{y'x} l_{z'x} \epsilon_{xx}$$

$$+ l_{y'y} l_{z'z} \epsilon_{yz} + l_{y'z} l_{z'x} \epsilon_{zx} + l_{y'x} l_{z'x} \epsilon_{xy} + l_{y'z} l_{z'y} \epsilon_{zy}$$

$$+ l_{y'x} l_{z'z} \epsilon_{xz} + l_{y'y} l_{z'x} \epsilon_{yx} \qquad (2.7)$$

In the above equations the mathematical definition of shear strain is used since these are tensor transformation,

$$\epsilon_{ij} = l_{im} l_{jn} \epsilon_{mn}$$

Thus $2\epsilon_{ij} = \gamma_{ij}$, the equivalent transformation in terms of γ, become,

$$\epsilon_{x'x'} = l^2_{x'x} \epsilon_{xx} + l^2_{x'y} \epsilon_{yy} + l^2_{x'z} \epsilon_{zz} + l_{x'y} l_{x'z} \gamma_{yz}$$

$$+ l_{x'x} l_{x'y} \gamma_{xy} + l_{x'z} l_{x'x} \gamma_{zx} \qquad (2.8)$$

and,

$$\gamma_{y'z'} = 2(l_{y'x} l_{z'x} \epsilon_{xx} + l_{y'y} l_{z'y} \epsilon_{yy} + l_{y'z} l_{z'z} \epsilon_{zz})$$

$$+ \gamma_{yz} (l_{y'y} l_{z'z} + l_{y'z} l_{z'y}) + \gamma_{zx} (l_{y'z} l_{z'x} + l_{y'x} l_{z'z})$$

$$+ \gamma_{xy} (l_{y'x} l_{z'y} + l_{y'y} l_{z'x}) \qquad (2.9)$$

As with stresses, there exists a set of axes along which the shear strains are zero. These are the principal strain axes and these normal strains are the principal strains.

Where one axis is common to both sets of axes (i.e., z and z' coincide), strain transformations take the form,

$$\epsilon_{xx} = \epsilon_{x'x'} \cos^2\theta + \epsilon_{y'y'} \sin^2\theta + \gamma_{x'y'} \cos\theta \sin\theta \qquad (2.10)$$

$$\epsilon_{yy} = \epsilon_{x'x'} \sin^2\theta + \epsilon_{y'y'} \cos^2\theta - \gamma_{x'y'} \cos\theta \sin\theta \qquad (2.11)$$

$$\gamma_{xy} = 2\epsilon_{x'x'} \cos\theta \sin\theta - 2\epsilon_{y'y'} \cos\theta \sin\theta - \gamma_{x'y'} (\cos^2\theta - \sin^2\theta)$$

$$\text{—} \ (2.12)$$

In terms of principal strains several useful relations are,

$$\epsilon_{xx} = \epsilon_x = \frac{\epsilon_1 + \epsilon_2}{2} + \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\theta \qquad (2.13)$$

$$\epsilon_{yy} = \epsilon_y = \frac{\epsilon_1 + \epsilon_2}{2} - \frac{\epsilon_1 - \epsilon_2}{2} \cos 2\theta \qquad (2.14)$$

$$\gamma_{xy} = (\epsilon_1 - \epsilon_2) \sin 2\theta \qquad (2.15)$$

Note; Eqn (2.6) can be written in the following form,

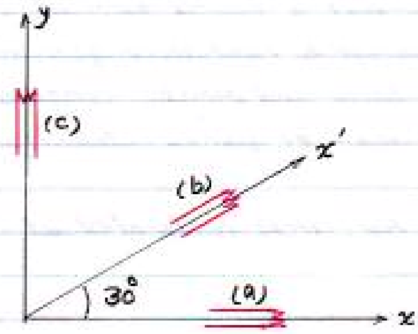
$$\epsilon_{x'x'} = l^2 \epsilon_{xx} + m^2 \epsilon_{yy} + n^2 \epsilon_{zz} + ln\, \epsilon_{xz} + mn\, \epsilon_{yz}$$

$$+ lm\, \epsilon_{xy} + nl\, \epsilon_{zx} + nm\, \epsilon_{zy} + ml\, \epsilon_{yx}$$

Example (2.1):

The sketch below shows three wire resistance strain gages attached to the surface of a part. Under loads, the gages give the following indications of strains,

Gage(a) = 0.002 Cm/Cm.
Gage(b) = 0.0025 Cm/Cm.
Gage(c) = 0.0005 Cm/Cm.



Find the magnitude of γ_{xy} and of the principal strains in the x-y plane.

Solution:

Using Eq (2.8), where $l_{x'x} = \cos 30°$ and $l_{x'y} = \cos 60°$

$$0.0025 = 0.002\,(0.866)^2 + (0.0005)(0.5)^2 + \gamma_{xy}(0.866)(0.5)$$

$$\therefore \gamma_{xy} = 0.00202 \text{ Cm/cm.}$$

To find the principal strains, Eq (2.8) and (2.9) can be rearranged in amaner that to give,

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

therefore,

$$\epsilon_{1,2} = \left(\frac{0.002+0.0005}{2}\right) \pm \left[\left(\frac{0.002-0.0005}{2}\right)^2 + \left(\frac{0.00282}{2}\right)^2\right]^{1/2}$$

$\epsilon_1 = 0.00251$ cm/cm , $\epsilon_2 = 0$.

2.4 Mohr's Circle for Strains:

By using the mathematical definition of shear strain it was mentioned earlier that a matrix form of strain results and is identical in concept to that used for stresses. Some useful relationships are noted and may be compared with their stresses except that $\frac{1}{2}\gamma$ must be plotted along the ordinate.

1. The centre of the circle is located at $\frac{1}{2}(\epsilon_1+\epsilon_2)$ or $\frac{1}{2}(\epsilon_x+\epsilon_y)$.

2. The radius of the circle is equal to $\frac{1}{2}(\epsilon_1-\epsilon_2)$.

3. $(\epsilon_1-\epsilon_2) = \left[(\epsilon_x-\epsilon_y)^2+\gamma_{xy}^2\right]^{1/2}$.

The following figure presents illustration of the Mohr's circle plot for standard uniaxial tension,

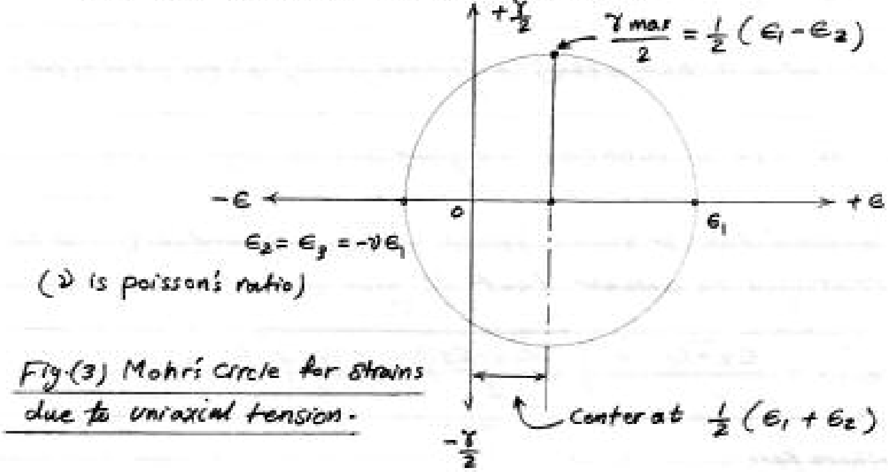
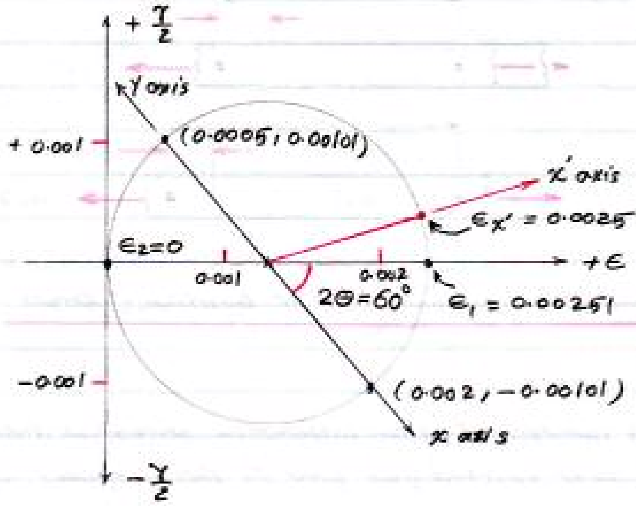


Fig.(3) Mohr's circle for strains due to uniaxial tension.

Example (2.2):

With the values of ϵ_x and ϵ_y given in Example (2.1) and using the computed value of γ_{xy}, find the principle strain in the x-y plane using Mohr's circle construction

$$\epsilon_x = 0.002 \text{ cm/cm}$$
$$\epsilon_y = 0.0005 \text{ cm/cm}$$
$$\gamma_{xy} = 0.00202 \text{ cm/cm}$$



Solution:

Noting that γ_{xy} is positive by the convention used in regard to the stress matrix, it must act counterclockwise and so is negative in regard to a plot Mohr's circle.

2.5 Engineering Strain and True Strain:

Under large plastic strain, the condition of elastic strains is no longer satisfied and a more realistic definition of strain is utitized. Several terms are used to discuss small strains, two of them being engineering or nominal.

Such strains will carry the symbol ϵ and are defined as,

$$\epsilon = \frac{\partial u}{\partial x} = \frac{\Delta l}{l_0} = \frac{l - l_0}{l_0} = \frac{l}{l_0} - 1 \qquad (2.16)$$

The concept of true strain (also called logarithmic or natural) is to be introduced,

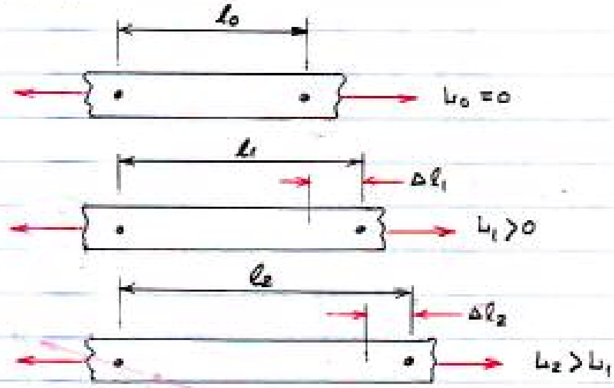


Fig-(3) Basis for defining true or logarithmic strain.

True strain is interpreted as an incremental elongation divided by the gage length that existed just prior to this increment of change.

Consider the sequence of events depicted in the above figure,

$$\text{True Strain} = e \equiv \frac{\text{incremental length increase}}{\text{Previous gage length}} \qquad (2.17)$$

As the load is increased from zero L_1, the true strain that exists due to L_1 would be,

$$e_1 = \frac{\Delta l_1}{l_0}$$

Now as the load is increased from L_1 to L_2, the incremental true strain induced would be,

$$e_2 = \frac{\Delta l_2}{l_0 + \Delta l_1} = \frac{\Delta l_2}{l_1}$$

Now the total true strain induced by load L_2 can be thought of as,

$$e_t = e_1 + e_2 = \frac{\Delta l_1}{l_0} + \frac{\Delta l_2}{l_0 + \Delta l_1}$$

In general form, up to the particular load of concern. Thus, dropping the subscript t for total,

$$e = \sum \frac{\Delta l_i}{l_0 + \Delta l_{i-1}} \qquad (2.18)$$

which can be readily expressed as,

$$de \equiv \frac{dl}{l} \qquad (2.19)$$

Assuming the specimen contains no intial strain, there results,

$$\int_0^{e_i} de = \int_{l_0}^{l_i} \frac{dl}{l} \quad \text{or} \quad e = \ln\left(\frac{l}{l_0}\right) \qquad (2.20)$$

Two final points are worth noting,

1. The symbol e or its incremental form de will always be used when plastic strains are involved whereas ϵ will be used when elastic strains are discussed.

2. The tensor concept is valid for small deformations so that incremental true strains (de) are tensor quantities regardless of the overall degree of deformation.

Example (2.3):

The intial length of a work piece is 50 mm when the part is under no load. Under the application of a tensile load, the length is increased to 60 mm. Find the nominal and true strains induced.

Solution:

$$\text{Nominal strain, } \in = \frac{60-50}{50} = 0.200$$

$$\text{True strain, } e = \ln\left(\frac{60}{50}\right) = 0.182$$

Example (2.4):

If the loading in example (2.3) had been compressive instead of tensile, to what length would the 50 mm dimension be reduced to produce the equivalent true strain? What would be the nominal strain under this condition?

Solution:

$$e = -0.182 = \ln\left(\frac{l}{50}\right)$$

$$+0.182 = \ln\left(\frac{50}{l}\right)$$

$$(\text{Natural logarithms})\ e^{0.182} = \frac{50}{l}$$

$$l = 50 / e^{0.182} = 41.67 \text{ mm}$$

As for the nominal strain:

$$\in = \frac{l - l_o}{l_o} = \frac{41.67 - 50}{50} = -0.167$$

Problems :

(2.1) A block of intial dimensions l_o, w_o, t_o is subjected to tensile loading that increases l_o to l while the other dimensions decrease to w and t respectively.

(a) calculate the engineering strains in the other directions.

(b) Calculate the true strains in the three directions.

(2.2) Wire is produced by pulling it through a series of circular dies of ever-decreasing cross-sectional area until the desired wire diameter is reached. If a wire of intial diameter of 0.010 cm, is passed through five dies, each causing an area reduction of 10%.

(a) What is the final wire diameter.

(b) Plot the ratio of d/d_o versus the total nominal and true strains resulting after each die has been passed.

(2.3) Three strain gages are positioned on a solid such that gage (1) lies parallel to the x-axis. Gage (2) is positioned 60° counterclockwise to gage one, while the third gage is positioned 120° counter clockwise to the first. Under loading, the gages indicate the following strains,

Gage (1) = 3000 μ cm/cm.
Gage(2) = 1500 μ cm/cm.
Gage(3) = 1000 μ cm/cm.

With the use of Mohr Circle plot, determine the principal strains in the plane from which these measured values were obtained.

(2.4) A solid is deformed under plane strain conditions (i.e. $\epsilon_2 = 0$). Mesurments indicates that strains in the 1-3 plane (perpendicular to the 2 direction) are.

$\epsilon_x = 0.010$, $\epsilon_y = 0.005$, $\gamma_{xy} = 0.007$

(a) Construct Mohr's circle for this condition.
(b) Determine the magnitudes of ϵ_1 and ϵ_3.

(2.5) Show that true strain, defined as $\ln(l/l_0)$ may be represented by any of the following forms for uniform deformation,

$$e = \ln\left(\frac{l}{l_0}\right) = \ln\left(\frac{A_0}{A}\right) = 2\ln\left(\frac{D_0}{D}\right) = \ln\left(\frac{1}{1-r}\right)$$

where l_0, A_0, and D_0 are original values of length, area and diameter, l, A, and D are instantaneous values, and r is the decimal reduction in area.