THREE-DIMENSIONAL FORCE SYSTEMS

Many problems in mechanics require analysis in three dimensions , and for such problems it is often necessary to resolve a force into its three mutually perpendicular components. The force **F** acting at point *O* in Fig. 2/16 has the *rectangular components* F_x, F_y, F_z, where

$$\left.\begin{aligned}
F_x &= F\cos\theta_x \\
F_y &= F\cos\theta_y \\
F_z &= F\cos\theta_z \\
F &= \sqrt{F_x^2 + F_y^2 + F_z^2} \\
\mathbf{F} &= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \\
\mathbf{F} &= F\left(\mathbf{i}\cos\theta_x + \mathbf{j}\cos\theta_y + \mathbf{k}\cos\theta_z\right)
\end{aligned}\right\} \quad (2/11)$$

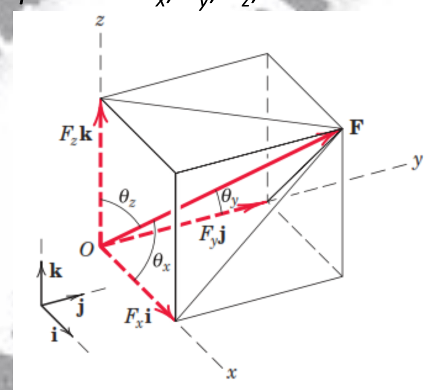

Figure 2/16

The unit vectors **i, j**, and **k** are in the *x-, y-, and z-directions, respectively.* Using the direction cosines of **F,** which are

$l = \cos\theta_x$, $m = \cos\theta_y$,and $n = \cos\theta_z$, where $l^2+m^2+n^2+=1$,

we may write the force as

$$\mathbf{F} = F(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$

We may regard the right-side expression of Eq. 2/12 as the force magnitude *F times a unit vector* $\mathbf{n}_F$ which characterizes the direction of **F**, or

$$\mathbf{F} = F\mathbf{n}_F$$

It is clear from Eqs. 2/12 and 2/12*a that* $\mathbf{n}_F = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$, *which* shows that the scalar components of the unit vector $\mathbf{n}_F$ are the direction cosines of the line of action of **F.** In solving three-dimensional problems, one must usually find the *x, y, and z scalar components of a force. In most cases, the direction of a* force is described (*a) by two points on the line of action of the force or* (*b) by two angles which orient the line of action.*

(a) <u>*Specification by two points on the line of action of the force.*</u>

$$\mathbf{F} = F\mathbf{n}_F = F\frac{\overrightarrow{AB}}{AB} = F\frac{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$

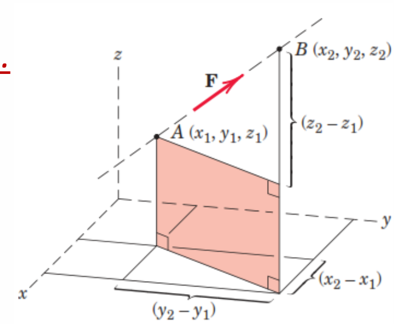

Figure 2/17

(b) Specification by two angles which orient the line of action of the force

Consider the geometry of Fig. 2/18. We assume that the angles *and are known. First resolve* **F** *into horizontal and vertical* components.

$$F_{xy} = F\cos\emptyset \qquad F_z = F\sin\emptyset$$

Then resolve the horizontal component F_{xy} into *x- and y-components.*

$$F_x = F_{xy}\cos\theta \qquad F_y = F_{xy}\sin\theta$$

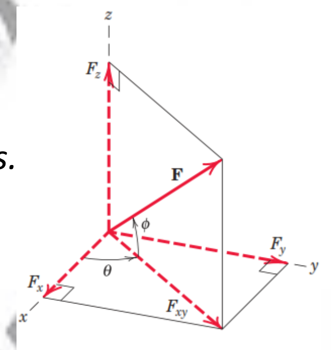

Figure 2/18

Dot Product

The dot product of two vectors **P** and **Q,** Fig. 2/19*a, is defined as the product of their magnitudes* times the cosine of the angle between them. It is written as **P · Q = *PQ cos α***

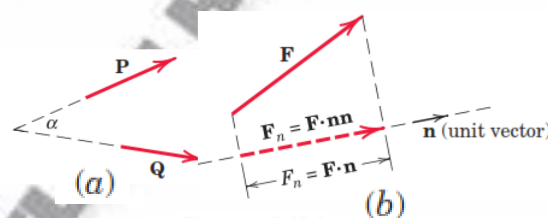

Figure 2/19

We can view this product either as the orthogonal projection *P cosα of* **P** in the direction of **Q** multiplied by ***Q**, or as the orthogonal projection Q cosα of* **Q** *in the direction of* ***P** multiplied by **P.** In either case the dot* product of the two vectors is a scalar quantity. Thus, for instance, we can express the scalar component $F_x = F\cos\theta_x$ of the force **F** in Fig.2/16 as $F_x = \mathbf{F}\cdot\mathbf{i}$, where **i** is the unit vector in the *x-direction.*

In more general terms, if **n is** a unit vector in a specified direction, the projection of **F** in the **n**-direction, Fig. 2/19*b, has the magnitude* $F_n = \mathbf{F}\cdot\mathbf{n}$. If we want to express the projection in the **n**-direction as a vector quantity, then we multiply its scalar component, expressed by **F·n** , by the unit vector **n** to give $\mathbf{F_n} = \mathbf{F}(\mathbf{F}\cdot\mathbf{n})\mathbf{n}$. We may write this as $\mathbf{F_n} = \mathbf{F}\cdot\mathbf{nn}$ without ambiguity because the term **nn** is not defined, and so the complete expression cannot be misinterpreted as **F·nn** . If the direction cosines of **n** are α,β , and γ, then we may write **n** in vector component form like any other vector as $\mathbf{n} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$
where in this case its magnitude is unity. If the direction cosines of **F** with respect to reference axes *x-y-z are l, m, and n, then the projection* of **F** in the **n**-direction becomes

$$F_n = \mathbf{F}\cdot\mathbf{n} = F(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})\cdot(\alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}) = F(l\alpha + \beta m + \gamma n)$$

because $\quad \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1 \quad$ *and* $\quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$

The latter two sets of equations are true because **i, j,** and **k** have unit length and are mutually perpendicular.