where **r** is a position vector which runs from the moment reference point *A* to any point on the line of action of **F.** The magnitude of this expression is given by

$$M = Fr\,sin\alpha = Fd \qquad (2/7)$$

which agrees with the moment magnitude as given by Eq. 2/5. Note that the moment arm $d = r\,sin\alpha$ does not depend on the particular point on the line of action of **F** to which the vector **r** is directed. We establish the direction and sense of **M** by applying the right-hand rule to the sequence **r x F.** If the fingers of the right hand are curled in the direction of rotation from the positive sense of **r** to the positive sense of **F,** then the thumb points in the positive sense of **M.**

We must maintain the sequence **r x F,** because the sequence **F x r** would produce a vector with a sense opposite to that of the correct moment. As was the case with the scalar approach, the moment **M** may be thought of as the moment about point *A* or as the moment about the line *O-O* which passes through point *A* and is perpendicular to the plane containing the vectors **r** and **F.** When we evaluate the moment of a force about a given point, the choice between using the vector cross product or the scalar expression depends on how the geometry of the problem is specified. If we know or can easily determine the perpendicular distance between the line of action of the force and the moment center, then the scalar approach is generally simpler. If, however, **F** and **r** are not perpendicular and are easily expressible in vector notation, then the cross-product expression is often preferable.

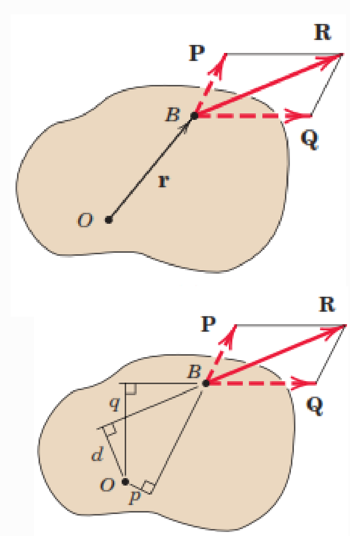
Varignon's Theorem

One of the most useful principles of mechanics is *Varignon's theorem, which states that the moment of a force about any point is equal to* the sum of the moments of the components of the force about the same point. To prove this theorem, consider the force **R** acting in the plane of the body shown in Fig. 2/9*a. The forces* **P** *and* **Q** *represent any two nonrectangular* components of **R.** The moment of **R** about point *O is*

$$\mathbf{M_O} = \mathbf{r \times R}$$

Because **R = P + Q, we may write**

$$\mathbf{r \times R} = \mathbf{r \times (P + Q)}$$



(b)

Figure 2/9

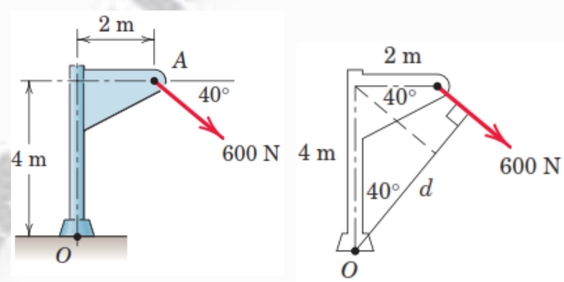
Using the distributive law for cross products, we have

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q} \qquad (2/8)$$

which says that the moment of **R** about *O* equals the sum of the moments about *O* of its components **P** and **Q.** This proves the theorem**.** Varignon's theorem need not be restricted to the case of two components, but it applies equally well to three or more. Thus we could have used any number of concurrent components of **R** in the foregoing proof. Figure 2/9*b illustrates the usefulness of Varignon's theorem. The* moment of **R** about point *O is Rd***.** However, if d is more difficult to determine than *p* and *q, we can resolve **R*** into the components ***P*** and **Q*,*** and compute the moment as

$$M_O = Rd = -pP + qQ$$

where we take the clockwise moment sense to be positive. Sample Problem 2/5 shows how Varignon's theorem can help us to calculate moments.



SAMPLE PROBLEM 2/5

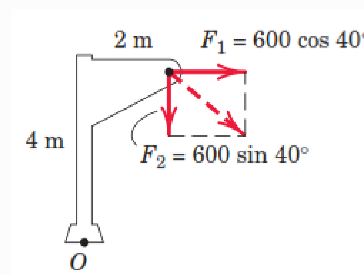
Calculate the magnitude of the moment about the base point *O of the 600-N* force in five different ways.

Solution: ***(I)*** The moment arm to the 600-N force is

$d = 4 \cos 40° + 2 \sin 40° = 4.35$ m

1 By $M = Fd$ the moment is clockwise and has the magnitude

$M_O = 600(4.35) = 2610$ N m²

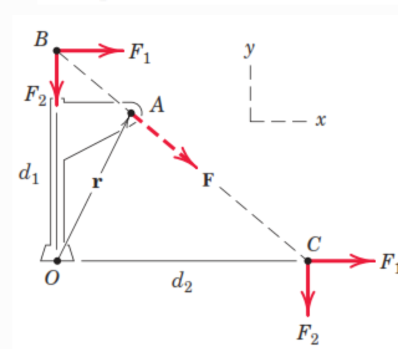


(II) Replace the force by its rectangular components at *A*,

$F_1 = 600 \cos 40° = 460$ N, $\qquad F_2 = 600 \sin 40° = 386$ N

By Varignon's theorem, the moment becomes

2 $M_O = 460(4) + 386(2) = 2610$ N .m



(III) By the principle of transmissibility, move the 600-N force along its line of action to point B, which eliminates the moment of the component F_2. The moment arm of F_1 and the moment is becomes:

$d_1 = 4 + 2 \tan 40° = 5.68$ m

and the moment is

$M_O = 460(5.68)\ \ 2610$ N.m

3 ***(IV)*** Moving the force to point C eliminates the moment of the component F_1. The moment arm of F_2 becomes

$$d_2 = 2 + 4 \cot 40° = 6.77 \text{ m}$$

and the moment is

$$M_O = 386(6.77)\ \ 2610 \text{ N m}$$

(V) By the vector expression for a moment, and by using the coordinate system indicated on the figure together with the procedures for evaluating cross products, we have

4 $$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i} \cos 40° - \mathbf{j} \sin 40°) = -2610\mathbf{k}\ \ N.m$$

The minus sign indicates that the vector is in the negative *z*-direction. The magnitude of the vector expression is

$$M_O = 2610 \text{ N.m}$$

Helpful Hints

1. The required geometry here and in similar problems should not cause difficulty if the sketch is carefully drawn.
2. This procedure is frequently the shortest approach.
3. The fact that points *B* and *C* are not on the body proper should not cause concern, as the mathematical calculation of the moment of a force does not require that the force be on the body.
4. Alternative choices for the position vector $\mathbf{r}$ are $\mathbf{r} = d_1\mathbf{j} = 5.68\mathbf{j}$ m and $\mathbf{r} = d_2\mathbf{i} = 6.77\mathbf{i}$ m.

Couple

The moment produced by two equal, opposite, and non collinear forces is called a *couple. Couples have certain unique properties and* have important applications in mechanics.

Consider the action of two equal and opposite forces **F** and **-F** a distance d apart, as shown in *Fig. 2/10a.* These two forces cannot be combined into a single force because their sum in every direction is zero. Their only effect is to produce a tendency of rotation. The combined moment of the two forces about an axis normal to their plane and passing through any point such as O in their plane is the couple ***M.*** This couple has a magnitude

$$M = F(a+d) - Fa$$

or

$$M = Fd$$

Its direction is counterclockwise when viewed from above for the case illustrated. Note especially that the magnitude of the couple is independent of the distance a which locates the forces with respect to the moment center O. *It follows that the moment of a couple has the same* value for all moment centers.

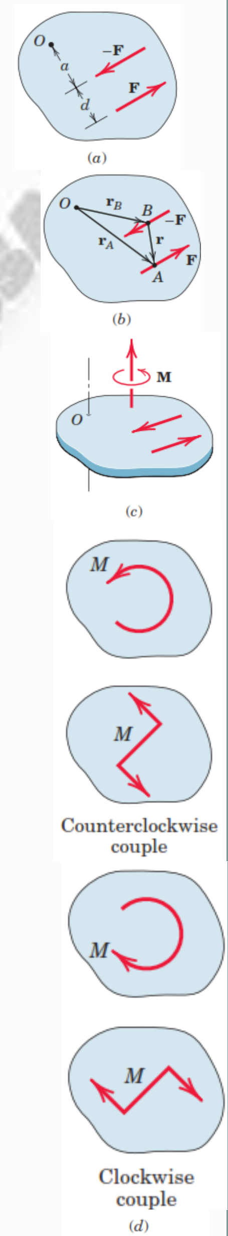


Figure 2/10

Vector Algebra Method

We may also express the moment of a couple by using vector algebra. With the cross-product notation of Eq. 2/6, the combined moment about point O of the forces forming the couple of Fig. 2/10*b is*

$$\mathbf{M} = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F}$$

where $\mathbf{r}_A$ and $\mathbf{r}_B$ are position vectors which run from point O to arbitrary points A and B on the lines of action of **F** and **-F,** respectively. Because $\mathbf{r}_A - \mathbf{r}_B = \mathbf{r}$, we can express **M** as

$$\mathbf{M} = \mathbf{r} \times \mathbf{F}$$

Equivalent Couples

Changing the values of F and d does not change a given couple as long as the product Fd remains the same. Likewise, a couple is not affected if the forces act in a different but parallel plane. Figure 2/11shows four different configurations of the same couple **M.** In each of the four cases, the couples are equivalent and are described by the same free vector which represents the identical tendencies to rotate the bodies.

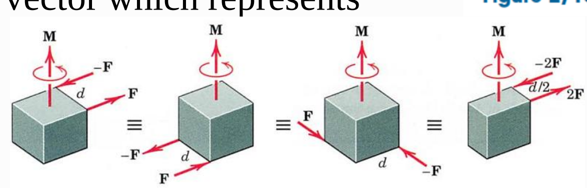


Figure 2/11

Force–Couple Systems

The effect of a force acting on a body is the tendency to push or pull the body in the direction of the force, and to rotate the body about any fixed axis which does not intersect the line of the force. We can represent this dual effect more easily by replacing the given force by an equal parallel force and a couple to compensate for the change in the moment of the force. The replacement of a force by a force and a couple is illustrated in Fig. 2/12, where the given force **F** acting at point *A* is replaced by an equal force **F** at some point *B* and the counterclockwise couple ***M* = *Fd*.**

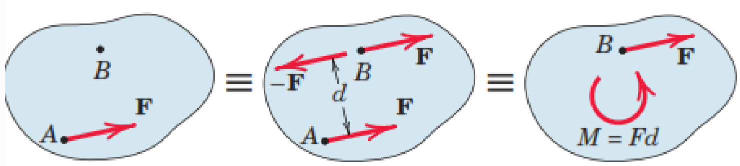


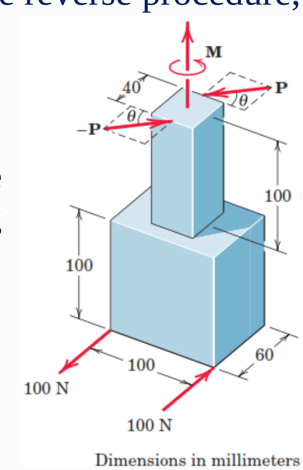
Figure 2/12

The transfer is seen in the middle figure, where the equal and opposite forces **F** and **-F** are added at point *B* without introducing any net external effects on the body. We now see that the original force at *A* and the equal and opposite one at *B constitute the couple M = Fd*, which is counterclockwise for the sample chosen, as shown in the right-hand part of the figure. Thus, we have replaced the original force at *A* by the same force acting at a different point *B* and a couple, without altering the external effects of the original force on the body. The combination of the force and couple in the right-hand part of Fig. 2/12 is referred to as a force–couple system.

By reversing this process, we can combine a given couple and a force which lies in the plane of the couple (normal to the couple vector) to produce a single, equivalent force. Replacement of a force by an equivalent force–couple system, and the reverse procedure, have many applications in mechanics and should be mastered.

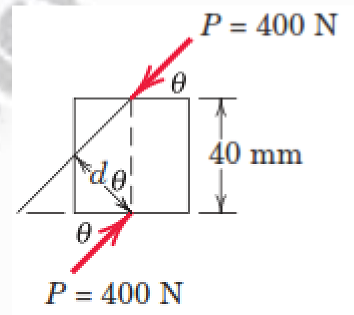
SAMPLE PROBLEM 2/7

The rigid structural member is subjected to a couple consisting of the two100-N forces. Replace this couple by an equivalent couple consisting of the two forces **P** and **P,** each of which has a magnitude of 400 N. Determine the proper angle θ .



Solution: The original couple is counterclockwise when the plane of the forces is viewed from above, and its magnitude is

[M Fd] $M = 100(0.1) = 10$ N m



The forces **P** and **-P** produce a counterclockwise couple

$$M = 400(0.040)\cos\theta$$

Equating the two expressions gives

$$10 = (400)(0.040)\cos\theta$$

$$\theta = \cos^{-1}\frac{10}{16} = 51.3°$$

Helpful Hint

Since the two equal couples are parallel free vectors, the only dimensions which are relevant are those which give the perpendicular distances between the forces of the couples.

SAMPLE PROBLEM 2/8

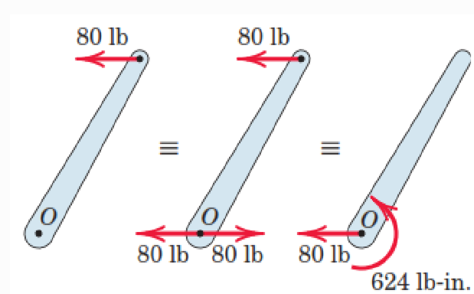
Replace the horizontal 80-lb force acting on the lever by an equivalent system consisting of a force at *O* and a couple.

Solution.

We apply two equal and opposite 80-lb forces at O and identify the counterclockwise couple

.*[M = Fd]* $M = 80(9 \sin 60°) = 624$ lb- in

1 Thus, the original force is equivalent to the 80-lb force at *O and the 624-lb-in.* couple as shown in the third of the three equivalent figures.



Helpful Hint

1 The reverse of this problem is often encountered, namely, the replacement of a force and a couple by a single force. Proceeding in reverse is the same as replacing the couple by two forces, one of which is equal and opposite to the 80-lb force at *O*. *T*he moment arm to the second force would be *M/F* = 624/80 = 7.79 in., which is 9 sin 60, thus determining the line of action of the single resultant force of 80 lb.