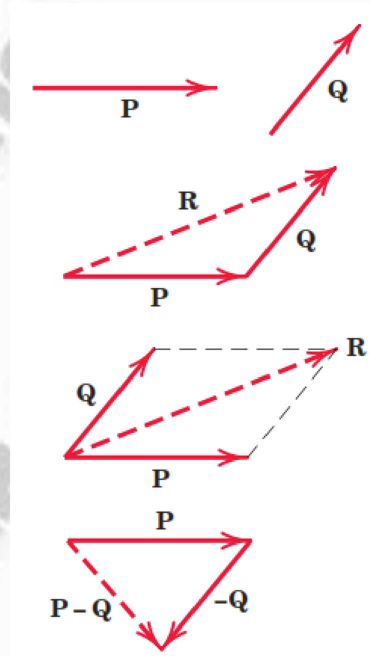
Vector Operations

1. Notation. Vector quantities are printed in boldface type, and scalar quantities appear in lightface italic type. Thus, the vector quantity **V** has a scalar magnitude *V*. In longhand work vector quantities should always be consistently indicated by a symbol such as $\underline{V}$ or $\vec{V}$ to distinguish them from scalar quantities.



2. *Addition*

Triangle addition **P + Q = R**

Parallelogram addition **P + Q = R**

Commutative law **P + Q = Q + P**

Associative law **P + (Q R) = (P + Q) + R**

3. *Subtraction*

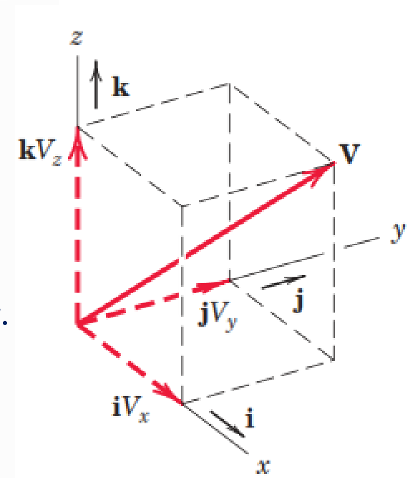
P - Q = P + (-Q)

4. *Unit vectors i, j, k*

$$\mathbf{V} = V_x\,\mathbf{i} + V_y\,\mathbf{j} + V_z\,\mathbf{k}$$

where

$$|\mathbf{V}| = V = \sqrt{V_x^2 + V_y^2 + V_z^2}$$



5. *Direction cosines*

l, m, n are the cosines of the angles between **V** and the *x*-, *y*-, *z-axes.*

$$l = V_x/V \qquad m = V_y/V \qquad n = V_z/V$$

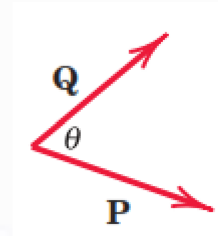
so that

$$\mathbf{V} = V(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$$

and

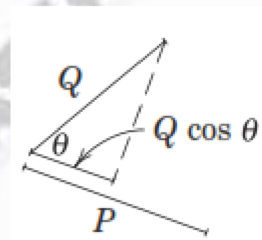
$$l^2 + m^2 + n^2 = 1$$

Dot or scalar product



$$\mathbf{P}\cdot\mathbf{Q} = PQ\ cos$$

This product may be viewed as the magnitude of **P** multiplied by the component $Q\ cos\theta$ of **Q** in the direction of **P,** or as the magnitude of **Q** multiplied by the component $P\ cos\theta$ *of* **P** in the direction of **Q.**



Commutative law $\mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P}$

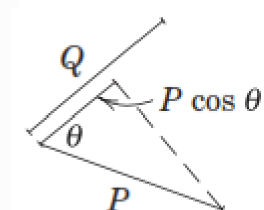
From the definition of the dot product

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

$$\mathbf{P}\cdot\mathbf{Q} = (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k})\cdot(Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})$$
$$= P_x\cdot Q_x + P_y\cdot Q_y + P_z\cdot Q_z$$

$$\mathbf{P}\cdot\mathbf{P} = P_x^2 + P_y^2 + P_z^2$$



It follows from the definition of the dot product that two vectors **P** and **Q** are perpendicular when their dot product vanishes, $\mathbf{P}\cdot\mathbf{Q} = 0$

The angle θ between two vectors $\mathbf{P_1}$ and $\mathbf{P_2}$ may be found from their dot product expression , $\mathbf{P_1}\cdot\mathbf{P_2} = P_1 P_2\cos\theta$, which gives

$$cos\theta = \frac{\mathbf{P_1}\cdot\mathbf{P_2}}{P_1P_2} = \frac{P_{1_x}P_{2_x} + P_{1_y}P_{2_y} + P_{1_z}P_{2_z}}{P_1P_2} = l_1l_2 + m_1m_2 + n_1n_2$$

where *l, m, n* stand for the respective direction cosines of the vectors. It is also observed that two vectors are perpendicular to each other when their direction cosines obey the relation $l_1l_2 + n_1n_2 + m_1m_2 = 0$

Distributive law $\mathbf{P}\cdot(\mathbf{Q}+\mathbf{R}) = \mathbf{P}\cdot\mathbf{Q} + \mathbf{P}\cdot\mathbf{R}$

Cross or vector product: The cross product **P x Q** of the two vectors **P and Q** is defined as a vector with a magnitude

$$|\mathbf{P \times Q}| = PQ\sin\theta$$

and a direction specified by the right-hand rule as shown. Reversing the vector order and using the right-hand rule give **Q x P = -P x Q.**

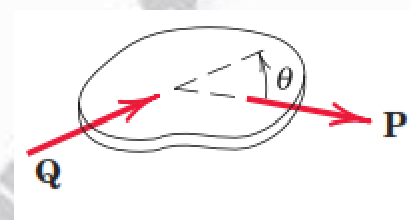
Distributive law **P x (Q + R) = P x Q + P x R**

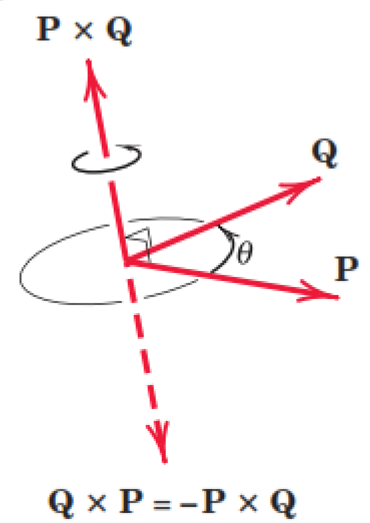
From the definition of the cross product, using a right-handed coordinate system, we get

$$\mathbf{i \times j = k} \qquad \mathbf{j \times k = i} \qquad \mathbf{k \times i = j}$$

$$\mathbf{j \times i = -k} \qquad \mathbf{k \times j = -i} \qquad \mathbf{i \times k = -j}$$

$$\mathbf{i \times i = j \times j = k \times k = 0}$$





With the aid of these identities and the distributive law, the vector product may be written

$$\mathbf{P \times Q} = (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \times (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})$$

$$= (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k}$$

The cross product may also be expressed by the determinant

$$\mathbf{P \times Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

Additional relations

Triple scalar product **(P x Q) · R = R · (P x Q)** . The dot and cross may be interchanged as long as the order of the vectors is maintained. Parentheses are unnecessary since **P x (Q · R)** is meaningless because a vector **P** cannot be crossed into a scalar **Q .R**. Thus, the expression may be written

$$\mathbf{P \times Q \cdot R = P \cdot Q \times R}$$

The triple scalar product has the determinant expansion

$$\mathbf{P \times Q \cdot R} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ Q_x & Q_y & Q_z \\ R_x & R_y & R_z \end{vmatrix}$$

Triple vector product **(P x Q) x R = - R x (P x Q) = R x (Q x P).** Here we note that the parentheses must be used since an expression **P x Q x R** would be ambiguous because it would not identify the vector to be crossed. It may be shown that the triple vector product is equivalent to **(P x Q) x R = R · PQ - R · QP or P x (Q x R) = P · RQ - P · QR.** The first term in the first expression, for example, is the dot product **R · P** , a scalar, multiplied by the vector **Q.**

TWO-DIMENSIONAL FORCE SYSTEMS

Rectangular Components

The most common two-dimensional resolution of a force vector is into rectangular components. It follows from the parallelogram rule that the vector **F** of Fig. 2/5 may be written as:

$$\mathbf{F} = \boldsymbol{F_x} + \boldsymbol{F_y} \qquad (2/1)$$

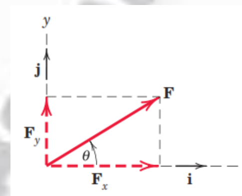


Figure 2/5

where $\mathbf{F}_x$ and $\mathbf{F}_y$ are vector components of **F** in the ***x***- and y-directions. Each of the two vector components may be written as a scalar times the appropriate unit vector. In terms of the unit vectors **i** and **j** of Fig. 2/5, $\mathbf{F}_x = F_x\mathbf{i}$ and $\mathbf{F}_y = F_y\mathbf{j}$, and thus we may write:

$$\mathbf{F} = F_x i + F_y j \qquad (2/2)$$

where the scalars F_x and F_y are the *x* and *y* scalar components of the vector **F.** The scalar components can be positive or negative, depending on the quadrant into which **F** points. For the force vector of Fig. 2/5, the *x* and *y* scalar components are both positive and are related to the magnitude and direction of **F** by**.**

$$F_x = F\cos\theta \qquad F_y = F\sin\theta \qquad F = \sqrt{F_x^2 + F_y^2} \qquad \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right) \qquad (2/3)$$

Determining the Components of a Force

Dimensions are not always given in horizontal and vertical directions, angles need not be measured counterclockwise from the *x-axis*, and the origin of coordinates need not be on the line of action of a force. Therefore, it is essential that we be able to determine the correct components of a force no matter how the axes are oriented or how the angles are measured. Figure 2/6 suggests a few typical examples of vector resolution in two dimensions.

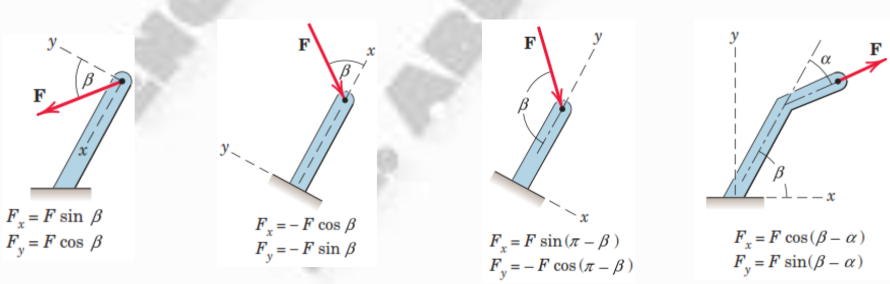


Figure 2/6