



University of Basrah - College of Engineering
Materials Engineering Department



Subject: *Numerical Analysis*

Stage: *Third*

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Syllabus of Numerical Analysis

1. Roots of Equations
2. Solution of Linear Simultaneous Equations
3. Finite Difference, Interpolation and Extrapolation
4. Numerical Integration and Differentiation
5. Curve Fitting
6. Numerical Solution of Ordinary Differential Equations
7. Numerical Solution of Ordinary Partial Equations
8. Introduction to Finite Element Method.

1. Roots of Equations

Years ago, you learned to use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (1)$$

To solve

$$f(x) = ax^2 + bx + c = 0 \quad (2)$$

The values calculated with eq.(2) are called the "**roots**" of eq.(1). They represent the values of x that make eq.(2) equal to zero.

Root of equation: The value of x that makes $f(x)=0$. The roots are sometimes are called **zeros** of equation.

There are many other functions for which the root cannot be determined so easily such as $f(x) = e^{-x} - x$. For these cases the numerical methods efficient means to obtain answer.

Before the advent of digital computers, there were several ways to solve for roots of **algebraic** and **transcendental** equations. Such as **Graphical technique method** is to plot the function and determine where it crosses the x-axis. This method is useful for obtaining rough estimated of roots, this is limited because of their lack precision.

The other method is **Trial and error**. This method consists of guessing a value of x and evaluating whether $f(x)$ is zero. If not, another guess is made, and $f(x)$ is again evaluated to determine whether the new value provides a better estimate of the root.

Note: **transcendental function** is one that is nonalgebraic. These include trigonometric, exponential, logarithmic, and other, less familiar, functions. Ex.

$$f(x) = \ln x^2 - 1, \quad \text{or} \quad f(x) = e^x + \sin(3x)$$

1.1. The Bisection Method: (Bracketing methods)

One of the first numerical methods developed to find the root of a nonlinear equation $f(x)=0$ was the bisection method (also called binary-search method).

In general, an equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_1 and x_2 if $f(x_1)f(x_2) < 0$.

Algorithm for the bisection method

The steps to apply the bisection method to find the root of the equation $f(x) = 0$ are

1. Choose x_l and x_u as two guesses for the root such that $f(x_l)f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_l and x_u .

2. Estimate the root, x_m , is determined by :

$$x_m = \frac{x_l + x_u}{2}$$

3. Now check the following:

- a) If $f(x_l)f(x_m) < 0$, then the root lies between x_l and x_m ; then $x_l = x_l$ and $x_u = x_m$.
- b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u ; then $x_l = x_m$ and $x_u = x_u$.
- c) If $f(x_l)f(x_u) = 0$; then the root is x_m . Stop the algorithm if this is true.

4. Find the new estimate of the root

$$x_m = \frac{x_l + x_u}{2}$$

Find the absolute relative approximate error as

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

where

x_m^{new} = **estimated root from present iteration**

x_m^{old} = **estimated root from previous iteration**

5. Compare the absolute relative approximate error $|\epsilon_a|$ with the pre-specified relative error tolerance ϵ_s . If $|\epsilon_a| > \epsilon_s$, then go to Step 3, else stop the algorithm.

Ex.1

The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water. The equation that gives the depth x to which the ball is submerged under water is given by:

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the bisection method of finding roots of equations to find the depth x to which the ball is submerged under water. Find the absolute relative approximate error at the end of each iteration.

Sol.

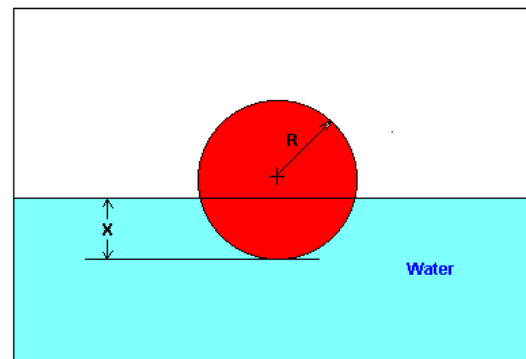
From the physics of the problem, the ball would be submerged between $x = 0$ and $x = 2R$, where, $R =$ radius of the ball

that is

$$0 \leq x \leq 2R$$

$$0 \leq x \leq 2(0.055)$$

$$0 \leq x \leq 0.11$$



Assume $x_l = 0$, $x_u = 0.11$

Check if the function changes sign between x_l and x_u

$$f(x_l) = f(0) = (0)^3 - 0.165(0)^2 + 3.993 \times 10^{-4} = 3.993 \times 10^{-4}$$

$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993 \times 10^{-4} = -2.662 \times 10^{-4}$$

Hence

$$f(x_l)f(x_u) = f(0)f(0.11) = (3.99 \times 10^{-4})(-2.662 \times 10^{-4}) < 0$$

So there is at least one root between x_l and x_u , that is between 0 and 0.11.

Iteration 0

The estimate of the root is

$$x_m = \frac{x_l + x_u}{2} = \frac{0 + 0.11}{2} = 0.055$$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993 \times 10^{-4} = 6.655 \times 10^{-5}$$

$$f(x_l)f(x_m) = f(0)f(0.055) = (3.993 \times 10^{-4})(6.655 \times 10^{-5}) > 0$$

Hence the root is bracketed between x_m and x_u . So, The new bracket is

$$x_l = 0.055, x_u = 0.11$$

At this point the absolute relative approximate error cannot be calculated.

Iteration 1

The estimate of the root is

$$x_m = \frac{x_l + x_u}{2} = \frac{0.055 + 0.11}{2} = 0.0825$$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993 \times 10^{-4} = -1.622 \times 10^{-4}$$

$$f(x_l)f(x_m) = f(0.055)f(0.0825) = (6.655 \times 10^{-4})(-1.622 \times 10^{-4}) < 0$$

Hence the root is bracketed between x_l and x_m . So, The new bracket is

$$x_l = 0.055, x_u = 0.0825$$

The absolute relative approximate error $|\epsilon_a|$ at the end of iteration 1 is:

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100 = \left| \frac{0.0825 - 0.055}{0.0825} \right| \times 100 = 33.33\%$$

Because the absolute relative approximate error is greater than 5%.

Iteration 2

The estimate of the root is

$$x_m = \frac{x_l + x_u}{2} = \frac{0.055 + 0.0825}{2} = 0.06875$$

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993 \times 10^{-4} = -5.563 \times 10^{-5}$$

$$f(x_l)f(x_m) = f(0.055)f(0.0825) = (6.655 \times 10^{-5})(-5.563 \times 10^{-5}) < 0$$

Hence the root is bracketed between x_l and x_m . So, The new bracket is

$$x_l = 0.055, x_u = 0.06875$$

The absolute relative approximate error $|\epsilon_a|$ at the end of iteration 2 is:

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100 = \left| \frac{0.06875 - 0.0825}{0.06875} \right| \times 100 = 20\%$$

Because the absolute relative approximate error is greater than 5%.

Table 1. Root of $f(x)$ as function of number of iteration for bisection method.

N	x_l	x_u	x_m	$f(x_l)$	$f(x_m)$	$f(x_l)f(x_m)$	New Interval ($x_l - x_u$)	$ \epsilon_a $ %
0	0	0.11	0.055	3.993×10^{-4}	6.655×10^{-5}	> 0	0.055 - 0.11	-
1	0.055	0.11	0.0825	6.655×10^{-5}	-1.622×10^{-4}	< 0	0.055 - 0.0825	33.3
2	0.055	0.0825	0.06875	6.655×10^{-5}	-5.563×10^{-5}	< 0	0.055 - 0.06875	20
3	0.055	0.06875	0.06188	6.655×10^{-5}	4.484×10^{-6}	> 0	0.0618 - 0.6875	11.11
4	0.06188	0.06875	0.06531	4.484×10^{-6}	-2.593×10^{-5}	< 0	0.0618 - 0.0653	5.26
5	0.06188	0.06531	0.06359	4.484×10^{-6}	-1.0804×10^{-5}	< 0	0.0618 - 0.0635	2.70
6	0.06188	0.06359	0.06273	4.484×10^{-6}	-3.176×10^{-6}	< 0	0.0618 - 0.0627	1.37
7	0.06188	0.06273	0.0623	4.484×10^{-6}	6.497×10^{-7}	> 0	0.0623 - 0.0627	0.69
8	0.0623	0.06273	0.06252	6.497×10^{-7}	-1.265×10^{-6}	< 0	0.0623 - 0.0625	0.343
9	0.0623	0.06252	0.06241	6.497×10^{-7}	-3.0768×10^{-7}	< 0	0.0623 - 0.0624	0.172

1.2. Iteration Method (Fixed point method)

(open method)

We put $x=g(x)$, assume initial value for x_0 then calculating next value as follows

$$x_{n+1} = g(x_n)$$

and continue in this iteration. The absolute relative approximate error as

$$|\epsilon_a| = \left| \frac{x_{new} - x_{old}}{x_{new}} \right| \times 100 = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| \times 100$$

Ex1. Solve the equation $f(x) = x^2 - 3x + 1 = 0$ using iteration method.

Since we know the solution $x = 1.5 \pm \sqrt{1.25}$ thus 2.618 and 0.3819

Sol. The equation $f(x) = x^2 - 3x + 1 = 0$ may be written

$$x = g_1(x) = \frac{1}{3}(x^2 + 1), \quad \Rightarrow \quad x_{n+1} = \frac{1}{3}(x_n^2 + 1)$$

If Choose $x_0 = 1$

N	x_n	$g_1(x_n)$	$ \epsilon_a $
0	1	0.667	50%
1	0.667	0.481	39%
2	0.481	0.411	17%
3	0.411	0.39	5.3%
4	0.39	0.38	2.6%
Convergence			

If Choose $x_0 = 3$

N	x_n	$g_1(x_n)$	$ \epsilon_a $
0	3	3.33	10%
1	3.33	4.037	17.5%
2	4.037	5.766	30%
3	5.766	11.415	50%
4	11.415	Divergence	

**** To Find other roots may be rewritten above equation (divided by x)**

$$f(x) = x^2 - 3x + 1 = 0 \quad \Rightarrow \quad x - 3 + \frac{1}{x} = 0$$

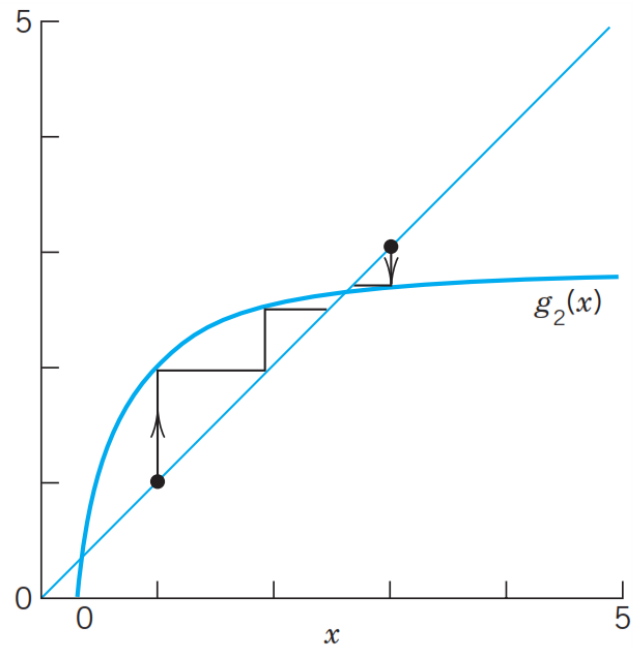
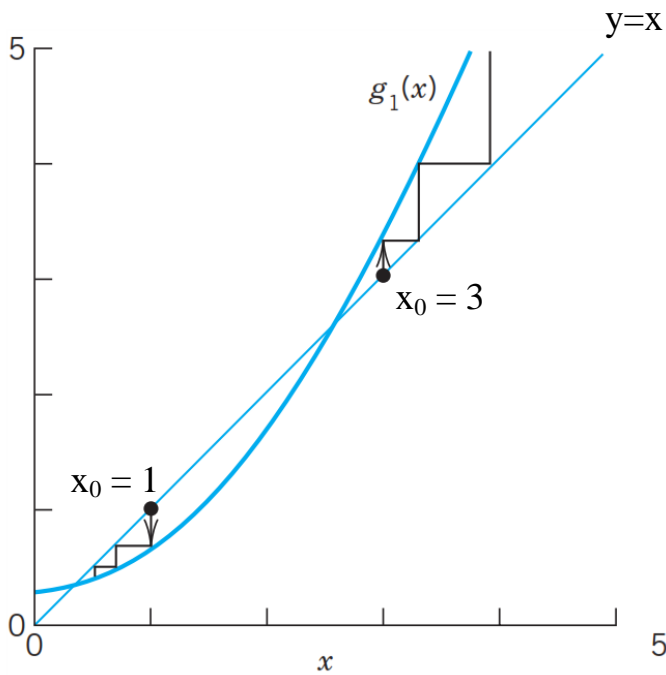
$$x = g_2(x) = 3 - \frac{1}{x}, \quad \Rightarrow \quad x_{n+1} = 3 - \frac{1}{x_n}$$

If Choose $x_0 = 1$

N	x_n	$g_1(x_n)$	$ \epsilon_a $
0	1	2	50%
1	2	2.5	20%
2	2.5	2.6	4%
3	2.6	2.615	0.57%
4	2.615	2.617	0.07%
Convergence			

If Choose $x_0 = 3$

N	x_n	$g_1(x_n)$	$ \epsilon_a $
0	3	2.667	12.5%
1	2.667	2.625	1.6%
2	2.625	2.619	0.22%
3	2.619	2.618	0.04%
4	2.618	2.618	0%
Convergence			



*** Convergence of fixed point method.**

Then if $|g'(x)| < 1$ The iteration process converge for any x_0

For above example

$$g_1(x) = \frac{1}{3}(x^2 + 1) \Rightarrow g'(x) = \frac{2}{3}x$$

$$\therefore |g'(x)| < 1 \Rightarrow \left| \frac{2}{3}x \right| < 1 \rightarrow x < \frac{3}{2} \text{ to converges}$$

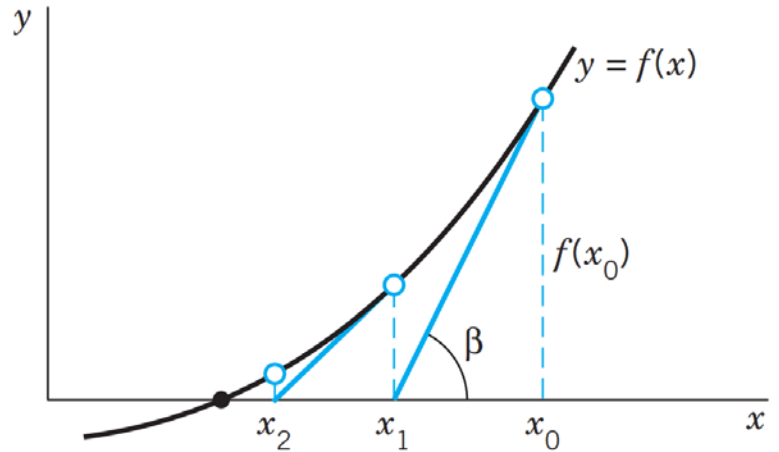
1.3. Newton - Raphson Method

The Newton method is another iteration method for solving equation $f(x) = 0$, where $f(x)$ is assumed to have a continuous derivative $f'(x)$. The method commonly used because of its simplicity and great speed.

$$\tan \beta = f'(x) = \frac{f(x_0)}{x_0 - x_1}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Ex1. Solve the equation $f(x) = x^2 - 3x + 1 = 0$ using Newton Raphson method.

$$f(x) = x^2 - 3x + 1 \Rightarrow f'(x) = 2x - 3$$

$$\therefore x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3x_n + 1}{2x_n - 3} = \frac{2x_n^2 - 3x_n - x_n^2 + 3x_n - 1}{2x_n - 3}$$

$$\therefore x_{n+1} = \frac{x_n^2 - 1}{2x_n - 3}$$

If Choose $x_0 = 1$

N	x_n	x_{n+1}	$ \epsilon_a $
0	1	0	-
1	0	0.333	100%
2	0.333	0.381	12%
3	0.381	0.381	0%
Convergence			

If Choose $x_0 = 3$

N	x_n	x_{n+1}	$ \epsilon_a $
0	3	2.667	12.5%
1	2.667	2.619	1.8%
2	2.619	2.618	0.04%
3	2.618	2.618	0%
Convergence			

H.W. Write a computer program to solve the following equation using the above three methods: Take $x_l = 0$ and $x_u = 3$ For Bisection method.

$$e^{-x} - \frac{x}{N} = 0$$

where N is your number

2. Solution of linear simulation equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \quad (2)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n = b_3 \quad (3)$$

$$\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m \quad (m)$$

Where:

m = No. of equations

n = No. of variables (unknowns)

- i. If **m=n**, there is one exact solution.
- ii. If **m<n**, there is infinite number of solutions.
- iii. If **m>n**, there may be an approximate solution.

For n = m only

2.1 Gauss-Seidel Iteration Method

If take n =3 No. of equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad \Rightarrow \quad x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad \Rightarrow \quad x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3}{a_{22}}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad \Rightarrow \quad x_3 = \frac{b_3 - a_{31}x_1 - a_{32}x_2}{a_{33}}$$

Algorithm for the iteration method

1. Put $x_i = \left(b_i - \sum_{\substack{j=1 \\ j \neq i}}^n a_{ij}x_j \right) / a_{ii}$.

2. Assume values for x_i , $i = 2, 3, \dots, n$

3. Substitute in the equation to find new set of x_i and continue till we get convergence

$$|x_i^{k+1} - x_i^k| < \epsilon \quad \text{where } \epsilon \text{ rate of convergence (very small)}$$

Note: To get convergence, the following condition is preferable

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^n |a_{ij}|$$

Ex. Solve the set of equations

$$x_1 + 4x_2 - x_3 = 6 \quad \dots\dots\dots (1)$$

$$2x_1 - x_2 + x_3 = 3 \quad \dots\dots\dots (2)$$

$$-x_1 + 2x_2 + 4x_3 = 15 \quad \dots\dots\dots (3)$$

Swap eqs. (1) and (2)

$$2x_1 - x_2 + x_3 = 3 \quad \dots\dots\dots (1)$$

$$x_1 + 4x_2 - x_3 = 6 \quad \dots\dots\dots (2)$$

$$-x_1 + 2x_2 + 4x_3 = 15 \quad \dots\dots\dots (3)$$

n Eqs.	0	1	2	-	-	-
$x_1 = \frac{3 + x_2 - x_3}{2}$	-	1.5	0.25	1	1
$x_2 = \frac{6 - x_1 + x_3}{4}$	0	1.125	2.34	2	2
$x_3 = \frac{15 + x_1 - 2x_2}{4}$	0	3.625	2.64	3	3

Continue until the last two column are identical.

*Convergence at $Max |x_i^{k+1} - x_i^k| \leq \epsilon$

$Max |error| \leq \epsilon$ Stop iteration

H.W. Write computer program to solve a set of equation by using Gauss-Seidel Iteration.

2.2 Gauss Elimination Method

Forward Elimination and Back substitution

For $n = m = 3$. The equations in matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$Ax = b$ A : Coefficient matrix

Step (1) Eliminate coefficient of x_1 from equ. (2) & (3)

multiply equ.(1) by $\frac{-a_{21}}{a_{11}}$ and add to equ. (2)

multiply equ.(1) by $\frac{-a_{31}}{a_{11}}$ and add to equ. (3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

Step (2) Eliminate coefficient of x_2 from equ. (3)

multiply equ.(2) by $\frac{-a'_{32}}{a'_{22}}$ and add to equ. (3)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a''_{22} & a''_{23} \\ 0 & 0 & a'_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b''_2 \\ b''_3 \end{bmatrix}$$

Step (3) Back substitution

$$x_3 = \frac{b''_3}{a'_{33}}$$

$$x_2 = \frac{b''_2 - a''_{23}x_3}{a''_{22}}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

Ex. Solve the system by using Gauss elimination method.

$$\begin{aligned} 8x_2 + 2x_3 &= -7 & \dots\dots\dots (1) \\ 3x_1 + 5x_2 + 2x_3 &= 8 & \dots\dots\dots (2) \\ 6x_1 + 2x_2 + 8x_3 &= 26 & \dots\dots\dots (3) \end{aligned}$$

Note.: **Pivoting**, select the equation with maximum x_1 coefficient to eliminate the coefficient in the other equations. (swap eqs. (1), (2) & (3))

$$\begin{aligned} 6x_1 + 2x_2 + 8x_3 &= 26 & \dots\dots\dots (1) \\ 3x_1 + 5x_2 + 2x_3 &= 8 & \dots\dots\dots (2) \\ 8x_2 + 2x_3 &= -7 & \dots\dots\dots (3) \end{aligned}$$

Step (1) Eliminate x_1 from eq.(2), by multiply $\frac{-3}{6} * (1) + eq.(2)$

$$\begin{bmatrix} 6 & 2 & 8 \\ 3 & 5 & 2 \\ 0 & 8 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 26 \\ 8 \\ -7 \end{bmatrix}$$

Can be rewriting the Augmented matrix \tilde{A}

$$\tilde{A} = [A|b] = \begin{array}{ccc|c} \overset{-3/6 * (1)}{\curvearrowright} \mathbf{6} & 2 & 8 & 26 \\ \rightarrow 3 & 5 & 2 & 8 \\ 0 & 8 & 2 & -7 \end{array} \quad \text{Pivot (6)}$$

$$\begin{array}{ccc|c} \mathbf{6} & 2 & 8 & 26 \\ \curvearrowleft \curvearrowright 0 & 4 & -2 & -5 \\ 0 & 8 & 2 & -7 \end{array} \quad \text{Swap (2) \& (3) Because } 8 > 4$$

Step (2) Elimination x_2 from eq.(3) by multiply $\frac{-4}{8} * (2) + eq.(3)$

$$\text{Pivot (8)} \quad \begin{array}{ccc|c} \mathbf{6} & 2 & 8 & 26 \\ \curvearrowleft \curvearrowright 0 & \mathbf{8} & 2 & -7 \\ \rightarrow 0 & 4 & -2 & -5 \end{array} \Rightarrow \begin{array}{ccc|c} \mathbf{6} & 2 & 8 & 26 \\ 0 & 8 & 2 & -7 \\ 0 & 0 & -3 & -3/2 \end{array}$$

Rewriting in algebraic form

$$6x_1 + 2x_2 + 8x_3 = 26 \quad \dots\dots\dots (1)$$

$$8x_2 + 2x_3 = -7 \quad \dots\dots\dots (2)$$

$$-3x_3 = -3/2 \quad \dots\dots\dots (3)$$

Step (3) Back substitution,

$$x_3 = \frac{1}{2}$$

$$x_2 = \frac{1}{8}(-7 - 2x_3) = -1$$

$$x_1 = \frac{1}{6}(26 - 2x_2 - 8x_3) = 4$$

Note: From smaller coefficient of equation can be multiply with factor and after the solution divided it's on the same factor.

$$1000 * (0.001x_1 + 0.02x_2 + 0.0018x_3 = 10^{-3}) \Rightarrow x_1 + 20x_2 + 1.8x_3 = 1$$

2.3 Gauss - Jordan Elimination Method

"Forward & Backward elimination"

Note: Pivoting, selecting the equation with maximum coefficient to eliminate the coefficient in the other equation.

Step (1) Forward elimination the same as in Gauss elimination method.

Step (3) Backward elimination: eliminate all elements above the main diagonal.

Ex. Solve the set of equation using Gauss-Jordan Method

$$2x - y + z = 3 \quad \dots\dots\dots (1)$$

$$x + 4y - z = 6 \quad \dots\dots\dots (2)$$

$$-x + 2y + 4z = 15 \quad \dots\dots\dots (3)$$

Sol.

$$\begin{array}{l} \begin{array}{l} -1/2 * \text{eq}(1) \\ \phantom{-1/2 * \text{eq}(1)} \\ -1/2 * (1) + \text{eq}(3) \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 4 & -1 & 6 \\ -1 & 2 & 4 & 15 \end{array} \right] \Rightarrow \begin{array}{l} \\ \\ -1.5/4.5 * (2) + \text{eq}(3) \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 4.5 & -1.5 & 4.5 \\ 0 & 1.5 & 4.5 & 16.5 \end{array} \right] \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 4.5 & -1.5 & 4.5 \\ 0 & 0 & 5 & 15 \end{array} \right]$$

***Back elimination:**

Step (3) eliminate x_3 from eq.(1) & eq.(2)

multiply eq.(3) by $\frac{-1}{5}$ and add to eq. (1)

multiply eq.(3) by $\frac{-1.5}{5}$ and add to eq. (2)

$$\begin{array}{l} -1/5 * (3) + \text{eq}(1) \\ -1.5/5 * (3) + \text{eq}(2) \end{array} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 4.5 & -1.5 & 4.5 \\ 0 & 0 & 5 & 15 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 4.5 & 0 & 9 \\ 0 & 0 & 5 & 15 \end{array} \right]$$

Step (4) eliminate x_2 from eq.(1) by multiply eq. (2) by $\frac{1}{4.5}$ and add to eq. (1)

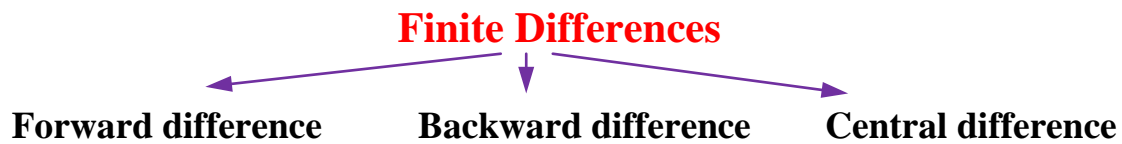
$$1/4.5 * \text{eq}(2) + \text{eq}(1) \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 4.5 & 0 & 9 \\ 0 & 0 & 5 & 15 \end{array} \right] \Rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 4.5 & 0 & 9 \\ 0 & 0 & 5 & 15 \end{array} \right]$$

Step (5)

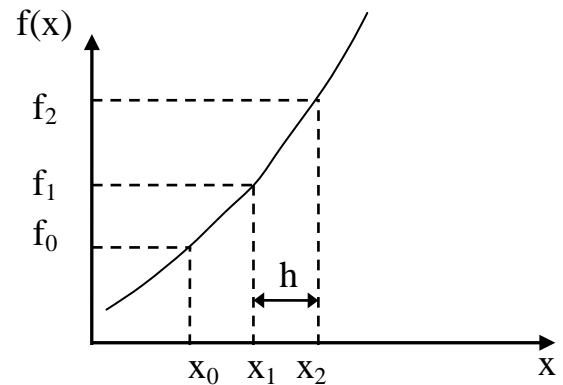
$$x = \frac{2}{2} = 1, \quad y = \frac{9}{4.5} = 2, \quad z = \frac{15}{5} = 3$$

H.W. Write a computer program to solve N-simultaneous linear equation using Gauss elimination and Gauss-Jordan method.

3. Finite Difference, Interpolation and Extrapolation



1. Forward differences:



$$\Delta f_0 = f_1 - f_0$$

$$\Delta f_1 = f_2 - f_1$$

$$\vdots \quad \quad \quad \vdots$$

$$\Delta f_n = f_{n+1} - f_n \quad \text{"1st Forward difference"}$$

$$\Delta^2 f_0 = \Delta(\Delta f_0) = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0)$$

$$= f_2 - 2f_1 + f_0$$

$$\vdots$$

$$\Delta^2 f_n = f_{n+2} - 2f_{n+1} + f_n \quad \text{"2nd Forward difference"}$$

$$\Delta^3 f_n = \Delta(\Delta^2 f_n) = \Delta f_{n+2} - 2\Delta f_{n+1} + \Delta f_n$$

$$= (f_{n+3} - f_{n+2}) - 2(f_{n+2} - f_{n+1}) + (f_{n+1} - f_n)$$

$$\therefore \Delta^3 f_n = f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n$$

Or $\Delta^3 f_n = \Delta^2 f_{n+1} - \Delta^2 f_n$ "3rd Forward difference"

$$\Delta^4 f_n = \Delta^3 f_{n+1} - \Delta^3 f_n$$

$$\Delta^k f_n = \Delta^{k-1} f_{n+1} - \Delta^{k-1} f_n \quad (k = 1, 2, 3, \dots)$$

Ex. Construct the forward difference table for the following data;

x	0	0.5	1	1.5	2.0
f(x)	1.2	1.8	3.1	4.6	7.6

n	x_n	f_n	Δf_n	$\Delta^2 f_n$	$\Delta^3 f_n$	$\Delta^4 f_n$
0	0	1.2	0.6	0.7	-0.5	1.8
1	0.5	1.8	1.3	0.2	1.3	-
2	1.0	3.1	1.5	1.5	-	-
3	1.5	4.6	3.0	-	-	-
4	2.0	7.6	-	-	-	-

2. Backward differences: " ∇ "

$$\nabla f_1 = f_1 - f_0$$

$$\nabla f_n = f_n - f_{n-1} \quad \text{" 1st Backward difference"}$$

$$\nabla^2 f_n = \nabla f_n - \nabla f_{n-1} \quad \text{" 2nd Backward difference"}$$

$$\begin{aligned} &= (f_n - f_{n-1}) - (f_{n-1} - f_{n-2}) \\ &= f_n - 2f_{n-1} + f_{n-2} \end{aligned}$$

$$\nabla^3 f_n = \nabla^2 f_n - \nabla^2 f_{n-1} \quad \text{" 3rd Backward difference"}$$

$$\nabla^k f_n = \nabla^{k-1} f_n - \nabla^{k-1} f_{n-1}$$

Ex. Construct the backward table for the last data.

n	x_n	f_n	∇f_n	$\nabla^2 f_n$	$\nabla^3 f_n$	$\nabla^4 f_n$
0	0	1.2	-	-	-	-
1	0.5	1.8	0.6	-	-	-
2	1.0	3.1	1.3	0.7	-	-
3	1.5	4.6	1.5	0.2	-0.5	-
4	2.0	7.6	3.0	1.5	1.3	1.8

3. Central differences: " δ "

$$\delta f_1 = \frac{f_2 - f_0}{2}$$

Or

$$\delta f_n = \frac{1}{2}(\Delta f_n + \nabla f_n) \quad \text{"1st Central"}$$

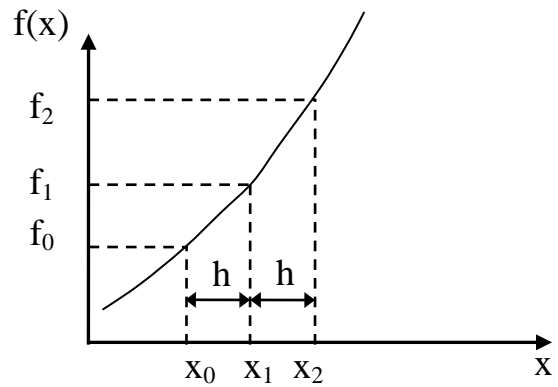
$$= \frac{1}{2}\{(f_{n+1} - f_n) - (f_n - f_{n-1})\}$$

$$\therefore \delta f_n = \frac{1}{2}(f_{n+1} + f_{n-1})$$

$$\delta^2 f_n = \delta(\delta f_n) = \frac{1}{2}(\delta f_{n+1} - \delta f_{n-1})$$

$$\delta^3 f_n = \delta^2(\delta f_n) = \frac{1}{2}(\delta^2 f_{n+1} - \delta^2 f_{n-1})$$

$$\delta^k f_n = \frac{1}{2}(\delta^{k-1} f_{n+1} - \delta^{k-1} f_{n-1})$$



H.W. Construct the central difference table for the data in previous example.

Interpolation and Extrapolation

* Linear interpolation

$$\frac{f_1 - f_0}{x_1 - x_0} = \frac{f - f_0}{x - x_0}$$

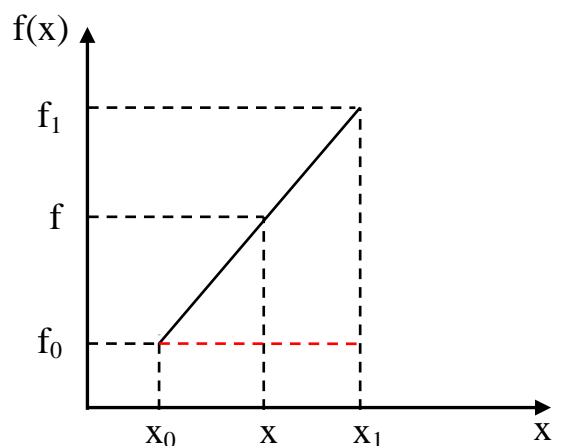
$$(f - f_0)h = (x - x_0)\Delta f_0$$

$$f = f_0 + \frac{x - x_0}{h}\Delta f_0$$

$$r = \frac{x - x_0}{h}$$

$$f = f_0 + r\Delta f_0$$

"linear"



$$h = x_1 - x_0$$

$$= x_{n+1} - x_n \quad \text{"step"}$$

**** Quadratic interpolation**

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2} \Delta^2 f_0$$

*****Newton Forward interpolation formula**

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0 + \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 f + \dots$$

*****Newton Backward interpolation formula**

$$f = f_0 + r\nabla f_0 + \frac{r(r-1)}{2!} \nabla^2 f_0 + \frac{r(r+1)(r+2)}{3!} \nabla^3 f_0 + \frac{r(r+1)(r+2)(r+3)}{4!} \nabla^4 f + \dots$$

Ex. For the same data in last example, find $f(-0.5)$, $f(1.25)$, $f(2.75)$ using linear and quadratic and 4-term Newton formula.

x	0	0.5	1	1.5	2.0
f(x)	1.2	1.8	3.1	4.6	7.6

Sol.

$f(-0.5) = ?$ extrapolation

linear : $f = f_0 + r\Delta f_0$ $r = \frac{x - x_0}{h}$

$x_0 = 0$, $h = 0.5$ "step" $h = x_{n+1} - x_n$ $r = \frac{-0.5}{0.5} = -1$

$f(-0.5) = 1.2 + (-1)(0.6) = 0.6$

Quadratic : $f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2} \Delta^2 f_0$

$f(-0.5) = 1.2 + (-1)(0.6) + \frac{(-1)(-2)}{2} * 0.7 = 1.3$

4-term Newton Forward formula

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0$$

$f(-0.5) = 1.2 + (-1)(0.6) + \frac{(-1)(-2)}{2} (0.7) + \frac{(-1)(-2)(-3)}{6} (-0.5) = 1.8$

$f(1.25) = ?$ Let $x_0 = 1.0$ **Note:** Assume the x_0 nearest to the desired value

linear :
$$r = \frac{x - x_0}{h} = \frac{1.25 - 1}{0.5} = 0.5$$

$$f(1.25) = f_0 + r\Delta f_0 = 3.1 + 0.5 * 1.5 = 3.85$$

Quadratic :

$$f(1.25) = f_0 + r\Delta f_0 + \frac{r(r-1)}{2} \Delta^2 f_0 = 3.1 + 0.5 * 1.5 + \frac{0.5 * (-0.5)}{2} * 1.5 = 3.66$$

4-term Newton formula Let $x_0 = 0.5$ *Because $x_0=1$ not found $\Delta^3 f_0$

$$r = \frac{x - x_0}{h} = \frac{1.25 - 0.5}{0.5} = 1.5$$

$$f = f_0 + r\Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0$$

$$f(1.25) = 1.8 + 1.5 * 1.3 + \frac{1.5 * 0.5}{2} * 0.2 + \frac{1.5 * 0.5 * (-0.5)}{6} * 1.3 = 3.74$$

$f(2.7) = ?$ Let $x_0 = 2.0$ **Note:** Using Backward Table because containing all terms

$$r = \frac{x - x_0}{h} = \frac{2.75 - 2}{0.5} = 1.5$$

linear : $f(2.75) = f_0 + r\nabla f_0 = 7.6 + 1.5 * 3.0 = 12.1$

Quadratic :

$$f(2.75) = f_0 + r\nabla f_0 + \frac{r(r+1)}{2} \nabla^2 f_0 = 7.6 + 1.5 * 3.0 + \frac{1.5 * 2.5}{2} * 1.5 = 14.91$$

4-term Newton formula

$$f = f_0 + r\nabla f_0 + \frac{r(r+1)}{2!} \nabla^2 f_0 + \frac{r(r+1)(r+2)}{3!} \nabla^3 f_0$$

$$f(2.75) = 7.6 + 1.5 * 3.0 + \frac{1.5 * 2.5}{2} * 1.5 + \frac{1.5 * 2.5 * 3.5}{6} * 1.3 = 17.75$$

Note: Newton interpolation formula can be used only when the point are equally spaced ($h = \text{constant}$).

H.W. Write a computer program to interpolate $f(x)$ using Newton forward and backward formula using n-term.

Lagrange Interpolation formula

This method has less accuracy than Newton formula, but it has advantage of it can be used for equally or non-equally spaced data.

$$f(x) = \sum_{k=0}^n N_k(x) f_k$$

$$N_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{(x - x_j)}{(x_k - x_j)} = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{(x - x_j)}{(x_k - x_j)}$$

where, n: number of terms - 1

Ex. if take n = 2

$$f(x) = \sum_{k=0}^2 N_k(x) f_k = N_0(x) f_0 + N_1(x) f_1 + N_2(x) f_2$$

$$N_0 = \prod_{\substack{j=0 \\ j \neq k}}^2 \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)}$$

$$N_1 = \prod_{\substack{j=0 \\ j \neq k}}^2 \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)}$$

$$N_2 = \prod_{\substack{j=0 \\ j \neq k}}^2 \frac{(x - x_j)}{(x_k - x_j)} = \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)}$$

$$\therefore f(x) = \frac{(x - x_1)}{(x_0 - x_1)} \cdot \frac{(x - x_2)}{(x_0 - x_2)} f_0 + \frac{(x - x_0)}{(x_1 - x_0)} \cdot \frac{(x - x_2)}{(x_1 - x_2)} f_1 + \frac{(x - x_0)}{(x_2 - x_0)} \cdot \frac{(x - x_1)}{(x_2 - x_1)} f_2$$

Ex. For the last example, find f(1.25) using 3-term Lagrange interpolation formula.

		x_0	x_1	x_2		
	x	0	0.5	1	1.5	2.0
$x = 1.25$	$f(x)$	1.2	1.8	3.1	4.6	7.6

n = No. of terms - 1 = 3 - 1 = 2

f_0 f_1 f_2

$$f(x) = \sum_{k=0}^2 N_k(x) f_k = N_0(x) f_0 + N_1(x) f_1 + N_2(x) f_2$$

$$N_{k=0} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(1.25 - 1)(1.25 - 1.5)}{(0.5 - 1)(0.5 - 1.5)} = -0.125$$

$$N_{k=1} = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(1.25 - 0.5)(1.25 - 1.5)}{(1 - 0.5)(1 - 1.5)} = 0.75$$

$$N_{k=2} = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(1.25 - 0.5)(1.25 - 1)}{(1.5 - 0.5)(1.5 - 1)} = 0.375$$

$$\therefore f(1.25) = -0.125 * 1.8 + 0.75 * 3.1 + 0.375 * 4.6 = 3.825$$

N.B. Using Lagrange interpolation up to term. (Max n=No. of point - 1)

H.W. Write a general computer program to read points (x_n, f_n) then construct the difference table up $\Delta^{n-1}f$, then estimate $y(x)$ for any x using 4-term Newton and Lagrange interpolation.

4. Numerical Integration and Differentiation

4.1 Integration:

1. Trapezoidal Rule

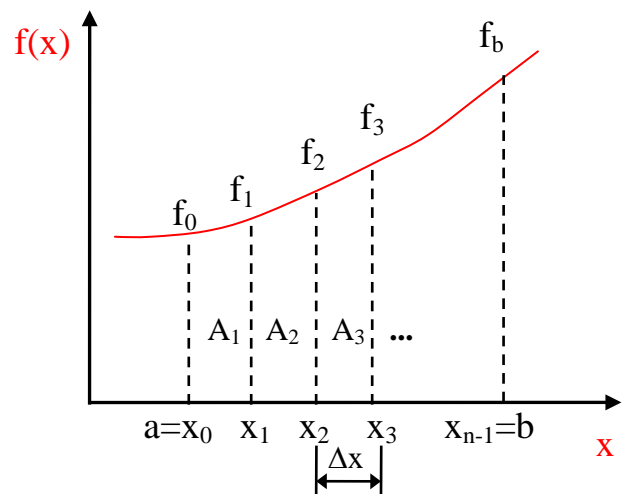
$$\int_a^b f(x) dx$$

n: No. of divisions (strips)

$$\Delta x = \frac{b - a}{n}$$

$$\text{Total area} = \int_a^b f(x) dx = \sum_{i=1}^n A_i$$

$$A_1 = \frac{f_0 + f_1}{2} \cdot \Delta x, \quad A_2 = \frac{f_1 + f_3}{2} \cdot \Delta x, \quad \dots \quad A_n = \dots$$



$$\begin{aligned} x_1 &= x_0 + \Delta x \\ &= a + \Delta x \\ x_2 &= x_1 + \Delta x \\ &\dots \end{aligned}$$

$$\int_a^b f(x) dx = \Delta x \left[\frac{f_a}{2} + f_1 + f_2 + f_3 + \dots + \frac{f_b}{2} \right]$$

2. Simpson Rule

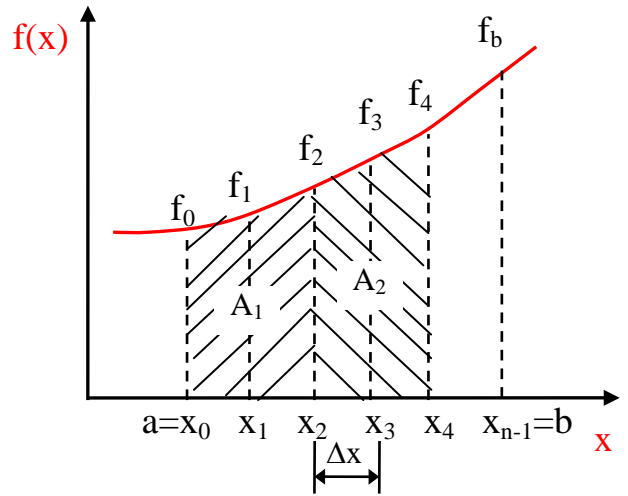
Note: No. of division (strips) should be even

$$A_1 = \frac{\Delta x}{3} (f_0 + 4f_1 + f_2)$$

$$A_2 = \frac{\Delta x}{3} (f_2 + 4f_3 + f_4)$$

⋮

$$A_n = \frac{\Delta x}{3} (f_{n-3} + 4f_{n-2} + f_{n-1})$$



$$\int_a^b f(x) dx = \frac{\Delta x}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 2f_{n-3} + 4f_{n-2} + f_{n-1}]$$

$$f_0 = f_a, f_n = f_b, x_1 = x_0 + \Delta x, x_2 = x_1 + \Delta x \dots$$

Ex. Evaluate $I = \int_0^2 e^{-x^2} dx$ using trapezoidal & Simpson methods. take $h = 0.5$.

Sol. No. of division = $\frac{b-a}{h} = \frac{2-0}{0.5} = 4$

n	x_n	$f_n = e^{-x_n^2}$	Trapezoidal value	Simpson Value
0	0	1		
1	0.5	0.778		
2	1.0	0.367		
3	1.5	0.105		
4	2.0	0.018		
			0.8795	0.88

(1) Trapezoidal $I_t = \Delta x \left[\frac{f_0}{2} + f_1 + f_2 + f_3 + \frac{f_4}{2} \right]$

$$= 0.5 \left[\frac{1}{2} + 0.778 + 0.367 + 0.105 + \frac{0.018}{2} \right] = 0.8795$$

(2) Simpson $I_s = \frac{\Delta x}{3} [f_0 + 4f_1 + 2f_2 + 4f_3 + f_4]$

$$= \frac{0.5}{3} [1 + 4 * 0.778 + 2 * 0.367 + 4 * 0.105 + 0.018] = 0.88$$

H.W Write computer program to evaluate the following integral using trapezoidal, Simpson methods; Take any number of division

$$I = \frac{1}{2\pi} \int_0^{2\pi} x^2 \sin 2x^2 dx$$

4.2 Numerical Differentiation:

$$f'(x) = \frac{\Delta f}{\Delta x} = \frac{\Delta f}{h}$$

$$f'(x_0) = \frac{1}{h} \Delta f_0 = \frac{f_1 - f_0}{h}$$

$$f''(x_0) = \frac{1}{h^2} \Delta^2 f_0, \quad \dots \quad f'''(x_0) = \frac{1}{h^3} \Delta^3 f_0$$

Using Newton interpolation formula

$$f(x) = f_0 + r \Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0 + \dots$$

$$f'(x) = \frac{df}{dx} = \frac{df}{dr} \cdot \frac{dr}{dx}, \quad r = \frac{x - x_0}{h}, \quad \therefore \frac{dr}{dx} = \frac{1}{h}$$

$$\frac{df}{dx} = \frac{1}{h} \cdot \frac{d}{dr} \left[f_0 + r \Delta f_0 + \frac{r(r-1)}{2!} \Delta^2 f_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 f_0 + \dots \right]$$

$$\therefore f'(x) = \frac{1}{h} \left[\Delta f_0 + \frac{2r-1}{2!} \Delta^2 f_0 + \frac{3r^2-6r+2}{3!} \Delta^3 f_0 + \dots \right]$$

** If Using "Three-point formula" find $f'_0, f'_1,$ and f'_2 .

$$\therefore f'(x) = \frac{1}{h} \left[\Delta f_0 + \frac{2r-1}{2!} \Delta^2 f_0 \right] \quad \text{"Three points formula"}$$

(i) At $x = x_0$ $r = 0$

$$f'(x_0) = f'_0 = \frac{1}{h} \left[\Delta f_0 - \frac{1}{2} \Delta^2 f_0 \right] \Rightarrow \therefore f'_0 = \frac{1}{2h} [-3f_0 + 4f_1 - f_2]$$

(ii) At $x = x_1$, $r = \frac{x_1 - x_0}{h} = 1$

$$f'(x_1) = f'_1 = \frac{1}{h} \left[\Delta f_0 + \frac{1}{2} \Delta^2 f_0 \right] \Rightarrow \therefore f'_1 = \frac{1}{2h} [f_2 - f_0]$$

(iii) At $x = x_2$ $r = \frac{x_2 - x_0}{h} = \frac{2h}{h} = 2$

$$f'_2 = \frac{1}{h} \left[\Delta f_0 + \frac{3}{2} \Delta^2 f_0 \right] \Rightarrow \therefore f'_2 = \frac{1}{2h} [f_0 - 4f_1 + 3f_2]$$

Ex. Consider $f(x) = x^4$ for $x_0 = 0$, $x_1 = 0.2$, $x_3 = 0.4$, $x_4 = 0.6$, and $x_4 = 0.8$. Calculate $f'_0, f'_1, \text{and } f'_2$ using three point formula.

n	x_n	$f_n = x_n^4$
0	0	0
1	0.2	16×10^{-4}
2	0.4	256×10^{-4}
3	0.6	1296×10^{-4}
4	0.8	4096×10^{-4}

$$h = 0.2$$

$$f'_0 = \frac{1}{2h} [-3f_0 + 4f_1 - f_2] =$$

$$f'_1 = \frac{1}{2h} [f_2 - f_0] =$$

$$f'_2 = \frac{1}{2h} [f_0 - 4f_1 + 3f_2] =$$

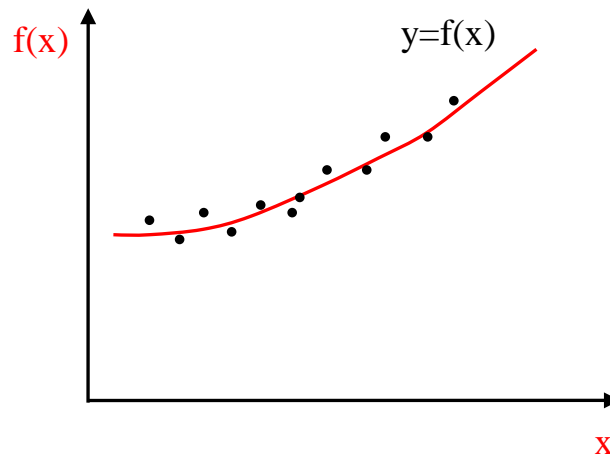
H.W. find $f'_0, f'_1, \text{and } f'_2$ using four-point formula.

N.B. if h not constant using Lagrange interpolation to find differentiation.

$$f(x) = \sum_{k=0}^n N_k(x) f_k, \quad \text{where } N_k(x) = \prod_{\substack{j=0 \\ j \neq k}}^n \frac{(x - x_j)}{(x_k - x_j)}$$

5. Curve Fitting

In **Curve Fitting** we are given **n** points (pairs of numbers) $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ and we want to determine a function $f(x)$ approximately. The type of function (for example, polynomials, exponential functions, sine and cosine functions) may be suggested by the nature of the problem (the underlying physical law, for instance), and in many cases a polynomial of a certain degree will be appropriate.



Using **Least Square Method** to find approximate function

5.1 Linear Fitting

$$y(x) = ax + b \quad a, b : \text{constants}$$

error = exact value - approx. value

$$e_1 = [y_1 - y(x_1)]^2 = [y_1 - (ax_1 + b)]^2 \quad \text{or absolute}$$

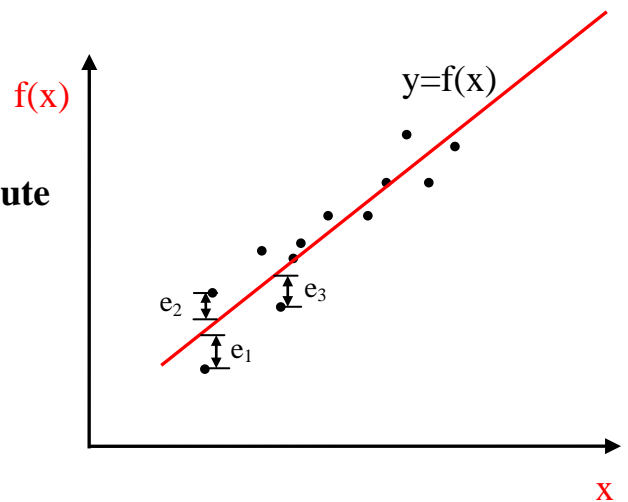
$$e_2 = [y_2 - y(x_2)]^2 = [y_2 - (ax_2 + b)]^2$$

⋮

$$e_n = [y_n - y(x_n)]^2 = [y_n - (ax_n + b)]^2$$

$$\therefore \text{Total error} = E = \frac{1}{2} \sum_{i=1}^n e_i^2 \quad n: \text{No. of points}$$

$$\therefore E = \frac{1}{2} \sum_{i=1}^n [y_i - (ax_i + b)]^2 \quad \text{Should be minimum.}$$



$$\frac{\partial E}{\partial a} = 0, \quad \frac{\partial E}{\partial b} = 0$$

$$\therefore \frac{\partial E}{\partial a} = \frac{1}{n} \sum_{i=1}^n 2 \cdot [y_i - (ax_i + b)] \cdot (-x_i) = 0$$

$$\sum_{i=1}^n [y_i - (ax_i + b)] x_i = 0$$

$$\sum_{i=1}^n y_i x_i = \sum_{i=1}^n (ax_i^2 + bx_i) \quad \text{divided both sides by } n$$

$$\overline{yx} = a\overline{x^2} + b\overline{x} \quad \dots(1)$$

$$\frac{\partial E}{\partial b} = \frac{1}{n} \sum_{i=1}^n 2 \cdot [y_i - (ax_i + b)] \cdot (-1) = 0$$

$$\sum_{i=1}^n y_i = \sum_{i=1}^n (ax_i + b) \quad \text{divided both sides by } n$$

$$\overline{y} = a\overline{x} + b \quad \dots (2)$$

$$\overline{yx} = a\overline{x^2} + b\overline{x} \quad \dots(1)$$

Multiply eq.(2) by \overline{x} and subtract from eq.(1);

$$\therefore a = \frac{\overline{yx} - \overline{y}\overline{x}}{\overline{x^2} - (\overline{x})^2}, \quad \text{and} \quad b = \overline{y} - a\overline{x}$$

Ex. Find the best line representing the following points (0, 1), (1, 2), (2, 4), (3, 8), (4, 10) and calculate the approximate errors.

Sol. $n = 5$

i	x_i	y_i	$y_i x_i$	x_i^2	$y(x_i) = ax_i + b$	e_i	e_i^2
1	0	1	0	0	0.2	0.8	0.64
2	1	2	2	1	2.6	-0.6	0.36
3	2	4	8	4	5.0	-1	1
4	3	8	24	9	7.4	0.6	0.36
5	4	10	40	16	9.8	0.2	0.04
Σ	10	25	74	30	-	-	2.4
Avg.	2	5	14.8	6			

$$a = \frac{\overline{yx} - \overline{y}\overline{x}}{\overline{x^2} - (\overline{x})^2} = \frac{14.8 - (5)(2)}{6 - (2)^2} = 2.4$$

$$b = \overline{y} - a\overline{x} = 5 - (2.4)(2) = 0.2$$

$$\therefore y = 2.4x + 0.2$$

5.2 Polynomial fitting

In General

$$y(x) = a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + \dots + a_0$$

where **m** degree of polynomial

- For example **m = 2**

$$y(x) = a_2 x^2 + a_1 x + a_0$$

$$\frac{\partial E}{\partial a_2} = 0 \quad \Rightarrow \quad \overline{yx^2} = a_2 \overline{x^4} + a_1 \overline{x^3} + a_0 \overline{x^2} \quad \dots(1)$$

$$\frac{\partial E}{\partial a_1} = 0 \quad \Rightarrow \quad \overline{yx} = a_2 \overline{x^3} + a_1 \overline{x^2} + a_0 \overline{x} \quad \dots(2)$$

$$\frac{\partial E}{\partial a_0} = 0 \quad \Rightarrow \quad \overline{y} = a_2 \overline{x^2} + a_1 \overline{x} + a_0 \quad \dots(3)$$

Ex. Find the best second degree curve representing the following points (0, 1), (1, 3), (2, 5), (3, 11), (4, 15).

Sol.

$$y(x) = a_2 x^2 + a_1 x + a_0 \quad ; \quad \mathbf{n = 5}$$

i	x	y	x ²	x ³	x ⁴	yx	yx ²
1	0	1	0	0	0	0	0
2	1	3	1	1	1	3	3
3	2	5	4	8	16	10	20
4	3	11	9	27	81	33	99
5	4	15	16	64	256	60	240
Σ	10	35	30	100	354	106	362
Avg.	2	7	6	20	70.8	21.2	72.4

$$\overline{yx^2} = a_2 \overline{x^4} + a_1 \overline{x^3} + a_0 \overline{x^2} \quad \Rightarrow \quad 72.4 = 70.8a_2 + 20a_1 + 6a_0 \quad \dots(1)$$

$$\overline{yx} = a_2 \overline{x^3} + a_1 \overline{x^2} + a_0 \overline{x} \quad \Rightarrow \quad 21.2 = 20a_2 + 6a_1 + 2a_0 \quad \dots(2)$$

$$\overline{y} = a_2 \overline{x^2} + a_1 \overline{x} + a_0 \quad \Rightarrow \quad 7 = 6a_2 + 2a_1 + a_0 \quad \dots(3)$$

Solve the above linear simultaneous equation using iteration or Gauss method to find a_2 , a_1 , and a_0

5.3 Exponential fitting, $y = ae^{bx}$

Step (1) Take ln for both sides;

$$\ln y = \ln a + bx$$

$$Y = A + bx$$

Step (2) Perform linear fitting to evaluate b, A

$$b = \frac{\overline{Yx} - \overline{Y} \overline{x}}{\overline{x^2} - (\overline{x})^2}, \quad \text{and} \quad A = \overline{Y} - b \overline{x}$$

$$\therefore a = e^A$$

Ex. Find the best exponential function representing the following points (1, 4.7), (2, 9), (3, 25), (4, 80) and calculate the approximate errors.

Sol. $n = 4$

$$y = ae^{bx} \quad \Rightarrow \quad \ln y = \ln a + bx \quad \Rightarrow Y = A + bx$$

$$\text{where } Y = \ln y, \quad A = \ln a$$

i	x	y	Y=lny	Yx	x ²	y=ae ^{bx}	e _i	e _i ²
1	1	4.7	1.548	1.548	1	4.437	0.263	0.07
2	2	9	2.197	4.394	4			
3	3	25	3.219	9.387	9			
4	4	80	4.382	17.53	15			
Σ	10	118.4	11.346	32.859	30			
Avg.	2.5	29.67	2.837	8.215	7.5			

$$b = \frac{\overline{Yx} - \overline{Y} \overline{x}}{\overline{x^2} - (\overline{x})^2} = \frac{8.215 - 2.837 * 2.5}{7.5 - (2.5)^2} = 0.8978$$

$$A = \overline{Y} - b\overline{x} = 2.837 - 0.8978 * 2.5 = 0.5925$$

$$\therefore a = e^A = 1.808$$

$$\therefore y = 1.808e^{0.8978x}$$

5.4 Power fitting

$$y = ax^b$$

Step (1) Take ln for both sides;

$$\ln y = \ln a + b \ln x$$

$$Y = A + bX$$

Step (2) Perform linear fitting to evaluate b, A

$$b = \frac{\overline{YX} - \overline{Y} \overline{X}}{\overline{X^2} - (\overline{X})^2}, \quad \text{and} \quad A = \overline{Y} - b\overline{X}$$

$$\therefore a = e^A$$

H.W. For the same points in the last example, find the best power function? and calculate the approximation error. Ans. $y = 3.614 x^{1.954}$

H.W. Write a computer program to read n-points then ask for the kind of fitting required, then perform the procedure of fitting and evaluate the total error generated.

6. Numerical Solution of ODE

6.1 First order DE.

We know that an ODE of the first order is of the form $F(x, y, y')=0$ and can often be written in the explicit form $y' = f(x, y)$. An **initial value problem** for this equation is of the form;

$$y' = f(x, y), \quad y(x_0) = y_0$$

We shall discuss methods of computing numerical values of the solution. these methods are step-by-step methods, that is, we start from the given $y_0=y(x_0)$ and process stepwise, computing approximate values of the solution of $y(x)$ at the mesh point.

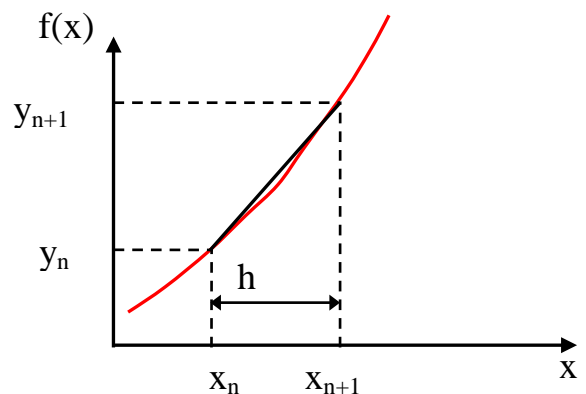
$$x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \dots$$

Ex. $y' - 2y = x \Rightarrow y' = f(x, y) = x + 2y$

(1) Euler Method

Using Taylor series

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \dots$$



For a small value of h , the highest powers h^2, h^3, \dots are very small

$$y_{n+1} = y_n + hy'_n = y_n + hf(x_n, y_n)$$

In the first step we compute

$$y_1 = y_0 + hf(x_0, y_0)$$

In the second step we compute

$$y_2 = y_1 + hf(x_1, y_1)$$

In general;

$$y_{n+1} = y_n + hf(x_n, y_n) \quad n = 0, 1, 2, \dots$$

This is called the **Euler method** or **Euler-Cauchy method**.

Ex. Solve $y' - 2(x + y) = 0$ for x from 0 to 1. given that $y(0) = 1$, use $h = 0.1$.

$$y' = f(x, y) = 2x + 2y$$

Using Euler method $y_{n+1} = y_n + hf(x_n, y_n)$

n	x_n	y_n	hf_n $0.1(2x_n+2y_n)$	y_{n+1}
0	0	1	0.2	1.2
1	0.1	1.2	0.26	1.46
2	0.2	1.46	0.332	1.792
3	0.3			
4	0.4			
5	0.5			
6	0.6			
7	0.7			
8	0.8			
9	0.9			
10	1.0			

(2) Modified Euler Method (Heun's Method)

Euler's method is generally much too inaccurate. For a large and small h the computation becomes prohibitive.

The improved Euler method we compute two values;

Step (1): Predictor

$$K_n = hf(x_n, y_n)$$

$$L_n = hf(x_n + h, y_n + K_n)$$

Step (2): Corrector

$$y_{n+1} = y_n + \frac{1}{2}(K_n + L_n)$$

Hence the **improved Euler method** is a **predictor–corrector method**: In each step we predict a value and then we correct it.

Ex. Solve the least example using modified Euler method.

$$y' = f(x, y) = 2x + 2y, \quad y(0) = 1, \quad h = 0.1$$

$$K_n = hf(x_n, y_n) = 0.1(2x_n + 2y_n)$$

$$L_n = hf(x_n + h, y_n + K_n) = 0.1[2(x_n + 0.1) + 2(y_n + K_n)]$$

$$y_{n+1} = y_n + \frac{1}{2}(K_n + L_n)$$

$$K_0 = 0.1(2 \cdot 0 + 2 \cdot 1) = 0.2$$

$$L_0 = 0.1[2(0 + 0.1) + 2(1 + 0.2)] = 0.26$$

$$y_1 = 1 + \frac{1}{2}(0.2 + 0.26) = 1.23$$

⋮

n	x _n	y _n	K _n	L _n	y _{n+1}
0	0	1	0.2	0.26	1.23
1	0.1	1.23	0.266	0.3392	1.533
2	0.2	1.533			
3	0.3				
4	0.4				
5	0.5				
6	0.6				
7	0.7				
8	0.8				
9	0.9				
10	1.0				

(3) Runge - Kutta Method

A method of great practical importance and much greater accuracy than that of the *improved Euler method* is the *classical Runge–Kutta method of fourth order*, which we call briefly the **Runge–Kutta method**.

The following fourth order Runge-Kutta algorithm, with $x_n = x_0 + nh$.

Step (1): Calculate

$$\begin{aligned}A_n &= hf(x_n, y_n) \\B_n &= hf\left(x_n + \frac{h}{2}, y_n + \frac{A_n}{2}\right) \\C_n &= hf\left(x_n + \frac{h}{2}, y_n + \frac{B_n}{2}\right) \\D_n &= hf(x_n + h, y_n + C_n)\end{aligned}$$

Step (2): Calculate

$$y_{n+1} = y_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$$

Ex. Use the fourth order Runge-Kutta algorithm with a step size $h = 0.2$ to solve initial value problem

$$\frac{dy}{dx} + 2y = \sin 3x \quad \text{with } y(0) = 1$$

in the interval $0 \leq x \leq 1$.

Sol.

$$\therefore y' = f(x, y) = \sin 3x - 2y$$

$$\begin{aligned}A_n &= 0.2 * (\sin 3x_n - 2y_n) \\B_n &= 0.2 \left[\sin 3(x_n + 0.1) - 2\left(y_n + \frac{A_n}{2}\right) \right] \\C_n &= 0.2 \left[\sin 3(x_n + 0.1) - 2\left(y_n + \frac{B_n}{2}\right) \right] \\D_n &= 0.2 [\sin 3(x_n + 0.2) - 2(y_n + C_n)]\end{aligned}$$

$$A_0 = 0.2 * (\sin 3 * 0 - 2 * 1) = -0.4$$

$$B_0 = 0.2 \left[\sin 3(0 + 0.1) - 2 \left(1 + \frac{-0.4}{2} \right) \right] = -0.2609$$

$$C_0 = 0.2 \left[\sin 3(0 + 0.1) - 2 \left(1 + \frac{-0.2609}{2} \right) \right] = -0.28872$$

$$D_0 = 0.2 [\sin 3(0 + 0.2) - 2(1 - 0.28872)] = -0.17158$$

$$\therefore y_1 = 1 + \frac{1}{6} [(-0.4) + 2(-0.2609) + 2(-0.28872) + (-0.17158)] = 0.72153$$

⋮

n	x _n	y _n	A _n	B _n	C _n	D _n	y _{n+1}
0	0	1	-0.4	-0.261	-0.288	-0.171	0.721
1	0.2	0.721	-0.175	-0.097	-0.112	-0.057	0.613
2	0.4	0.613	-0.058	-0.034	-0.038	-0.0348	0.573
3	0.6	0.573					
4	0.8						
5	1.0						

H.W. Write a computer program to solve and first order differential equation using Euler, modified Euler and Runge-Kutta methods.

6.2 Second Order Ordinary Differential Equation:

We know that an ODE of the second order is of the form $F(x, y, y', y'')=0$ and can often be written in the explicit form;

$$y'' = f(x, y, y') \quad \text{with } y(x_0) = y_0 \text{ and } y'(x_0) = y'_0$$

(1) Euler Method

Using Taylor series

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \dots$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n$$

The **Euler method** we compute two steps;

Step (1) $y_{n+1} = y_n + hy'_n + \frac{h^2}{2}f(x_n, y_n, y'_n)$

Step (2) $y'_{n+1} = y'_n + hf(x_n, y_n, y'_n)$

Ex. Solve $y'' - 4y' - 2xy = 0$ given that $y(0) = 1, y'(0) = 1$ for x from 0 to 1 using $h = 0.1$.

Sol.

$$y'' = f(x, y, y') = 2xy + 4y'$$

$$y_{n+1} = y_n + 0.1y'_n + 0.005(2x_n y_n + 4y'_n)$$

$$y'_{n+1} = y'_n + 0.1(2x_n y_n + 4y'_n)$$

Calculate

$$y_1 = 1 + 0.1 \cdot 1 + 0.005(2 \cdot 0 \cdot 1 + 4 \cdot 1) = 1.12$$

$$y'_1 = 1 + 0.1(2 \cdot 0 \cdot 1 + 4 \cdot 1) = 1.4$$

⋮ ⋮

n	x_n	y_n	y'_n	$f(x_n, y_n, y'_n)$	y_{n+1}	y'_{n+1}
0	0	1	1	4	1.12	1.4
1	0.1	1.12	1.4	5.824	1.289	1.982
2	0.2	1.289	1.982			
3	0.3					
4	0.4					
5	0.5					
6	0.6					
7	0.7					

8	0.8					
9	0.9					
10	1.0					

(2) Runge-Kutta-Nyström Method (RKN Method)

RKN methods are direct extensions of RK methods (Runge–Kutta methods) to second-order ODEs $y'' = f(x, y, y')$, as given by Nyström (1925). The best known of these uses the following formulas;

Step (1): Calculate

$$A_n = hf(x_n, y_n, y'_n)$$

$$B_n = hf\left(x_n + \frac{h}{2}, y_n + \frac{K}{2}, y'_n + \frac{A_n}{2}\right) \quad \text{where} \quad K = h\left(y'_n + \frac{A_n}{4}\right)$$

$$C_n = hf\left(x_n + \frac{h}{2}, y_n + \frac{K}{2}, y'_n + \frac{B_n}{2}\right)$$

$$D_n = hf(x_n + h, y_n + L, y'_n + C_n) \quad \text{where} \quad L = h(y'_n + C_n)$$

Step (2): Calculate

$$y_{n+1} = y_n + h\left(y'_n + \frac{1}{6}(A_n + B_n + C_n)\right)$$

Step (3): Calculate

$$y'_{n+1} = y'_n + \frac{1}{6}(A_n + 2B_n + 2C_n + D_n)$$

Ex. Resolve the last example using RKN method, take $h = 0.2$.

$$y'' = f(x, y, y') = 2xy + 4y' \quad , y(0) = y'(0) = 1$$

$$A_n = 0.2(2x_n y_n + 4y'_n)$$

$$B_n = 0.2\left(2\left(x_n + 0.1\right)\left(y_n + \frac{K}{2}\right) + 4\left(y'_n + \frac{A_n}{2}\right)\right) \quad \text{where} \quad K = 0.2\left(y'_n + \frac{A_n}{4}\right)$$

$$C_n = 0.2 \left(2(x_n + 0.1) \left(y_n + \frac{K}{2} \right) + 4 \left(y'_n + \frac{B_n}{2} \right) \right)$$

$$D_n = 0.2 \left(2(x_n + 0.2)(y_n + L) + 4(y'_n + C_n) \right) \quad \text{where} \quad L = 0.2 \left(y'_n + \frac{C_n}{2} \right)$$

$$A_0 = 0.2[2*0*1 + 4*1] = 0.8 \qquad K = 0.2 \left(1 + \frac{0.8}{4} \right) = 0.24$$

$$B_n = 0.2 \left[2(0 + 0.1) \left(1 + \frac{0.24}{2} \right) + 4 \left(1 + \frac{0.8}{2} \right) \right] = 1.165$$

$$C_n = 0.2 \left[2(0 + 0.1) \left(1 + \frac{0.24}{2} \right) + 4 \left(1 + \frac{1.165}{2} \right) \right] = 1.311$$

$$L = 0.2 \left(1 + \frac{1.311}{2} \right) = 0.311$$

$$D_n = 0.2[2(0 + 0.2)(1 + 0.311) + 4(1 + 1.311)] = 1.971$$

$$y_1 = 1 + 0.2 \left(1 + \frac{1}{6}(0.8 + 1.165 + 1.311) \right) = 1.309$$

$$y'_1 = 1 + \frac{1}{6}(0.8 + 2*1.165 + 2*1.311 + 1.971) = 2.287$$

n	x _n	y _n	y' _n	A _n	B _n	C _n	D _n	y _{n+1}	y' _{n+1}
0	0	1	1	0.8	1.165	1.311	1.971	1.309	2.287
1	0.2	1.309	2.287						
2	0.4								
3	0.6								
4	0.8								
5	1.0								

H.W. Write a computer program to solve any second order differential equation using Euler and Runge-Kutta Nyström methods.

7. Numerical Solution of Partial Differential Equation

The PDEs serve as models for elliptic, parabolic, and hyperbolic. For example, the Laplace equation is a representative example of an elliptic type of PDE, and so forth.

A PDE is called **quasilinear** if it is linear in the highest derivatives. Hence a second-order quasilinear PDE in two independent variables x, y is of the form;

$$au_{xx} + 2bu_{xy} + cu_{yy} = F(x, y, u, u_x, u_y)$$

Note:

$$u_x = \frac{\partial u}{\partial x}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_y = \frac{\partial u}{\partial y}, u_{yy} = \frac{\partial^2 u}{\partial y^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$$

u is the unknown function. This equation is said to be of;

Elliptic type	if	$ac - b^2 > 0$	(example: Laplace equation)
Parabolic type	if	$ac - b^2 = 0$	(example: heat equation)
Hyperbolic type	if	$ac - b^2 < 0$	(example: wave equation)

7.1 Solution of 2-D Laplace Equation;

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

To obtain methods of numerical solution. we replace the partial derivatives by corresponding **difference quotients**, as follows. By the **Taylor** formula,

$$u(x + h, y) = u(x, y) + hu_x(x, y) + \frac{h^2}{2}u_{xx}(x, y) + \frac{h^3}{6}u_{xxx}(x, y) + \dots \quad (1)$$

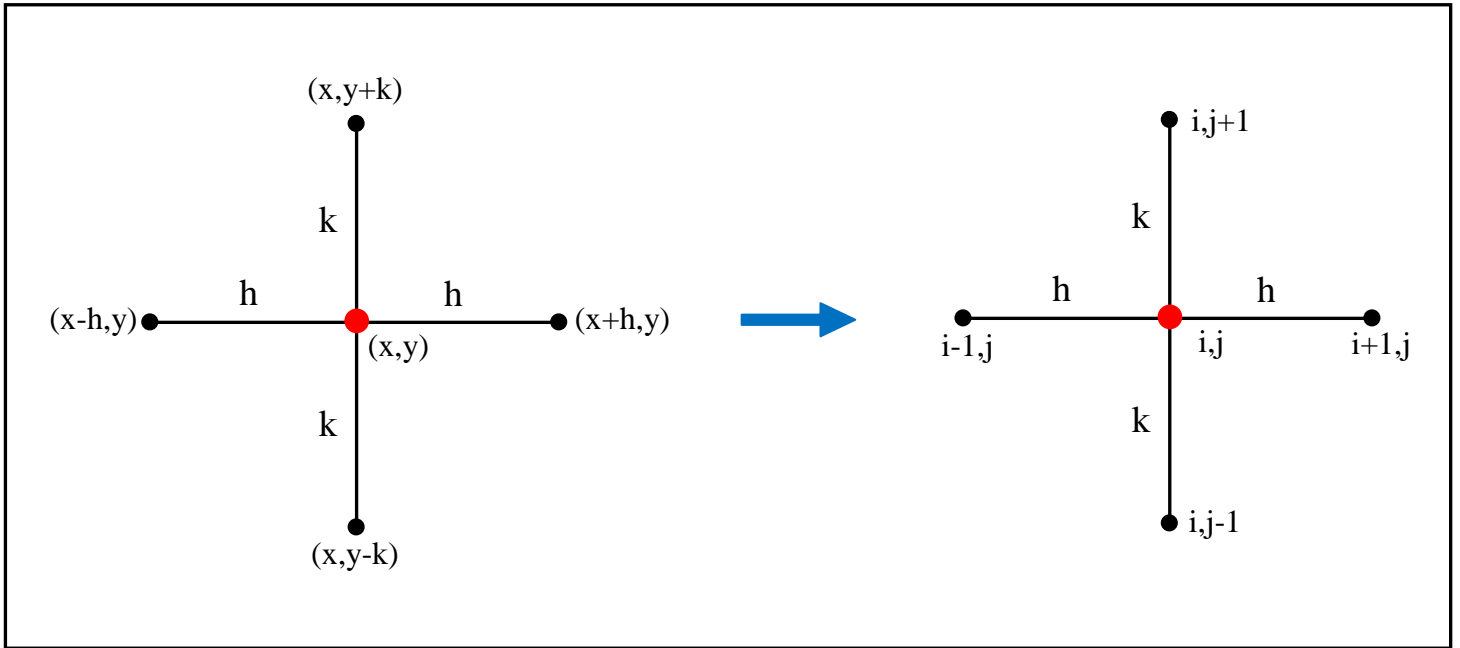
$$u(x - h, y) = u(x, y) - hu_x(x, y) + \frac{h^2}{2}u_{xx}(x, y) - \frac{h^3}{6}u_{xxx}(x, y) + \dots \quad (2)$$

Subtract Eq.(2) from Eq.(1), neglect terms in h^2, h^3, \dots ,

$$u_x(x, y) \approx \frac{1}{2h} [u(x+h, y) - u(x-h, y)] \Rightarrow u_x|_{j,i} = \frac{1}{2h} (u_{i+1,j} - u_{i-1,j})$$

Similarly

$$u_y(x, y) \approx \frac{1}{2k} [u(x, y+k) - u(x, y-k)] \Rightarrow u_y|_{j,i} = \frac{1}{2k} (u_{i,j+1} - u_{i,j-1})$$



To obtain second derivative. Adding Eq.(1) and Eq.(2);

$$u_{xx}(x, y) \approx \frac{1}{h^2} [u(x+h, y) - 2u(x, y) + u(x-h, y)]$$

$$u_{xx}|_{i,j} = \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \quad (4)$$

Similarly

$$u_{yy}(x, y) \approx \frac{1}{k^2} [u(x, y+k) - 2u(x, y) + u(x, y-k)]$$

$$u_{yy}|_{i,j} = \frac{1}{k^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] \quad (5)$$

Substitute Eqs. (4) &(5) into Laplace Equation, get;

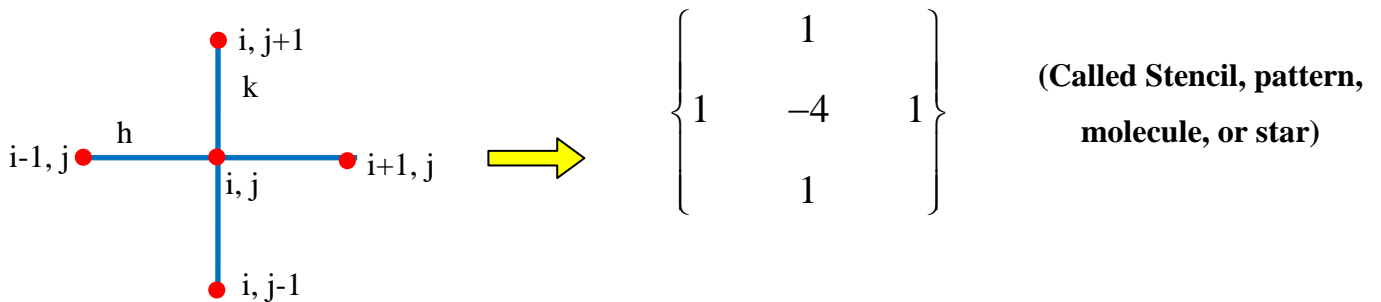
$$\frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + \frac{1}{k^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] = 0$$

* If $h = k$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

$$\therefore u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}$$

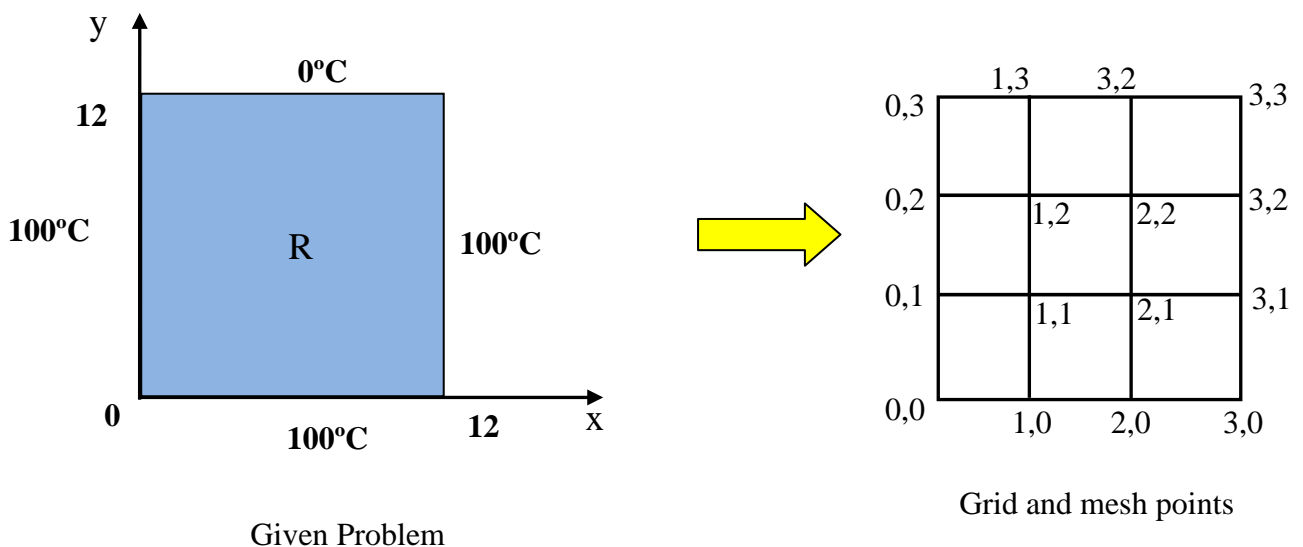
Note:



Can be rewritten the Laplace Equation in stencil as;

$$\left\{ \begin{array}{ccc} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{array} \right\} u = 0$$

Ex. The four sides of a square plate of side 12 cm made of homogeneous material are kept at constant temperature 0 °C and 100 °C as shown figure. Using a grid mesh 4 cm. Find the (steady state) temperature at the mesh point.



Sol. The equations for 2-D in steady state as;

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

Consider the mesh points;

$$h = k = 4$$

$$4u_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}$$

*** Boundary Condition**

$$u_{1,0} = u_{2,0} = 100 \text{ } ^\circ\text{C}$$

$$u_{1,3} = u_{2,3} = 0 \text{ } ^\circ\text{C}$$

$$u_{0,1} = u_{0,1} = 100 \text{ } ^\circ\text{C}$$

$$u_{3,1} = u_{3,2} = 100 \text{ } ^\circ\text{C}$$

At point (1,1)

$$4u_{1,1} = u_{2,1} + u_{0,1} + u_{1,2} + u_{1,0} \Rightarrow 4u_{1,1} = u_{2,1} + 100 + u_{1,2} + 100$$

$$4u_{1,1} - u_{2,1} - u_{1,2} = 200 \quad (1)$$

At point (2,1)

$$4u_{2,1} = u_{3,1} + u_{1,1} + u_{2,2} + u_{2,0} \Rightarrow 4u_{2,1} = 100 + u_{1,1} + u_{2,2} + 100$$

$$4u_{2,1} - u_{1,1} - u_{2,2} = 200 \quad (2)$$

At point (1,2)

$$4u_{1,2} = u_{2,2} + u_{0,2} + u_{1,3} + u_{1,1} \Rightarrow 4u_{1,2} = u_{2,2} + 100 + 0 + u_{1,1}$$

$$4u_{1,2} - u_{2,2} - u_{1,1} = 100 \quad (3)$$

At point (2,2)

$$4u_{2,2} = u_{3,2} + u_{1,2} + u_{2,3} + u_{2,1} \Rightarrow 4u_{2,2} = 100 + u_{1,2} + 0 + u_{2,1}$$

$$4u_{2,2} - u_{1,2} - u_{2,1} = 100 \quad (4)$$

Solving Eqs. (1), (2), (3), and (4) by Iteration method or Gauss-Jordan method to finding the interior point $u_{1,1}, u_{2,1}, u_{1,2},$ and $u_{2,2}$

Rearrangement above equations

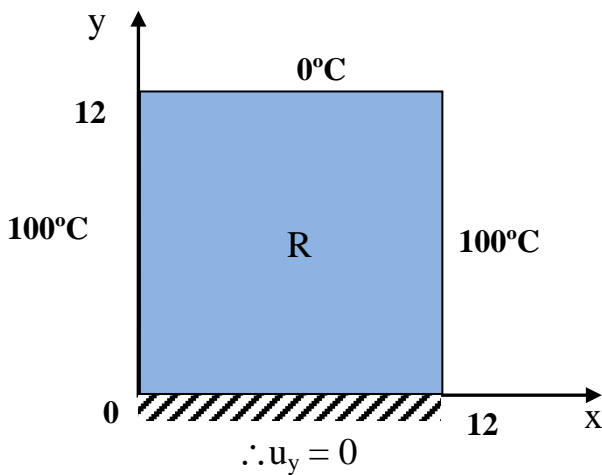
(1) If using Gauss-Jordan elimination method (Practical for small system of equation) as;

$$\begin{array}{rcl}
 -4u_{11} + u_{21} + u_{12} & = & -200 \\
 u_{11} - 4u_{21} + u_{22} & = & -200 \\
 u_{11} - 4u_{12} + u_{22} & = & -100 \\
 u_{21} + u_{12} - 4u_{22} & = & -100
 \end{array}
 \quad \Rightarrow \quad
 \left| \begin{array}{cccc|c}
 -4 & 1 & 1 & 0 & -200 \\
 1 & -4 & 0 & 1 & -200 \\
 1 & 0 & -4 & 1 & -100 \\
 0 & 1 & 1 & -4 & -100
 \end{array} \right|$$

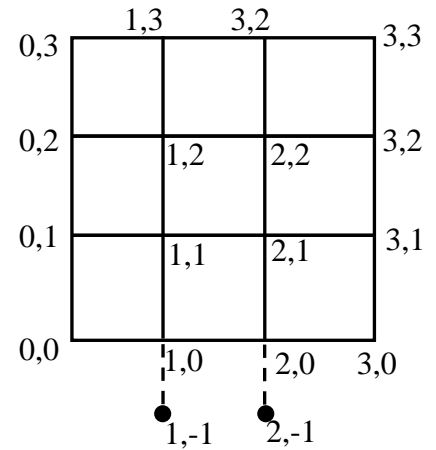
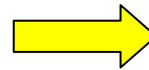
(2) If using Iteration method (Practical for large system of equation) as;

$$\begin{aligned}
 u_{11} &= 0.25u_{21} + 0.25u_{12} + 50 \\
 u_{21} &= 0.25u_{11} + 0.25u_{22} + 50 \\
 u_{12} &= 0.25u_{11} + 0.25u_{22} + 25 \\
 u_{22} &= 0.25u_{21} + 0.25u_{12} + 25
 \end{aligned}$$

Ex. Find the temperature distribution for the following plate.



Given Problem



Grid and mesh points

To find the temperature at isolation side

$$u_y \Big|_{1,0} = \frac{u_{1,1} - u_{1,0}}{2k} = 0 \quad \Rightarrow \quad u_{1,1} = u_{1,0} \quad \dots(a)$$

$$u_y \Big|_{2,0} = \frac{u_{2,1} - u_{2,0}}{2k} = 0 \quad \Rightarrow \quad u_{2,1} = u_{2,0} \quad \dots(b)$$

At point (1,0)

$$4u_{1,0} = u_{2,0} + u_{0,0} + u_{1,1} + u_{1,-1} \Rightarrow 4u_{1,0} = u_{2,0} + 100 + u_{1,1} + u_{1,1}$$

$$4u_{1,0} - u_{2,0} - 2u_{1,1} = 100 \quad (1)$$

At point (2,0)

$$4u_{2,0} = u_{3,0} + u_{1,0} + u_{2,1} + u_{2,-1} \Rightarrow 4u_{2,0} = 100 + u_{1,0} + u_{2,1} + u_{2,-1}$$

$$4u_{2,0} - u_{1,0} - 2u_{2,1} = 100 \quad (2)$$

The other points same the above example

H.W. Complete the other points and solution using Gauss-Seidel Method.

H.W. Write a computer programming to solve 2-D Laplace equation in case of Dirichlet or Neumann Conditions.

7.2 Solution of Parabolic Equation; (1-D Diffuse equaton)

In this section we explain the numerical solution of the prototype of parabolic equation, the one dimensional heat equation;

$$\frac{1}{\alpha} u_t = u_{xx} \quad (1)$$

x : space $0 \leq x \leq L$, t : time $t \geq 0$ α : thermal diffusivity $\left(\frac{m^2}{s} \right)$

The initial condition $u(x, 0) = f(x)$ (given) and boundary condition at $x=0$ and $x =$

L for all $t \geq 0$. Using $\Delta x = h$, $\Delta t = k$

$$u_t \Big|_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} \quad \text{Forward difference} \quad (a)$$

$$u_{xx} \Big|_{i,j} = \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \quad (b)$$

Subs. above Eqs. (a) & (b) in Eq.(1);

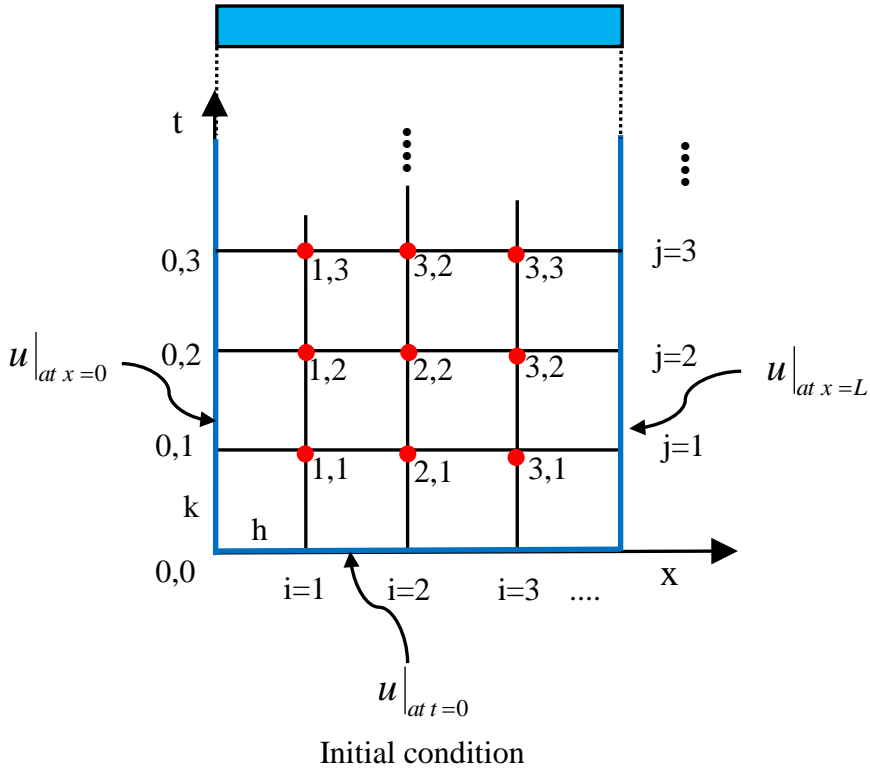
$$\frac{1}{\alpha k} [u_{i,j+1} - u_{i,j}] = \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

$$u_{i,j+1} = \frac{\alpha k}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + u_{i,j}$$

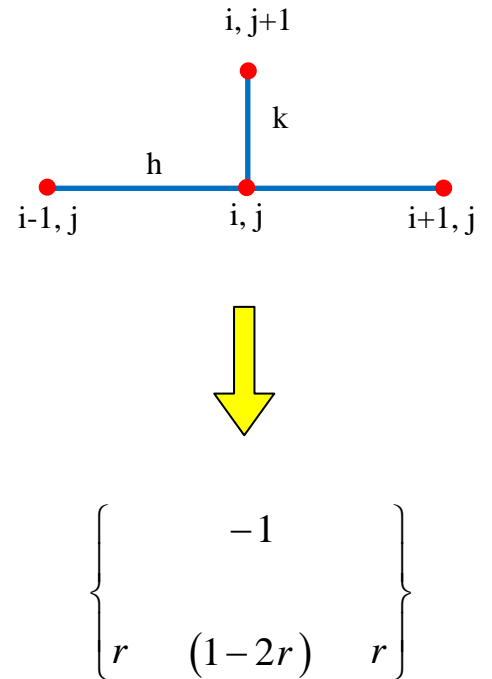
$$u_{i,j+1} = \frac{\alpha k}{h^2} [u_{i+1,j} + u_{i-1,j}] + \left(1 - \frac{2\alpha k}{h^2} \right) u_{i,j}$$

$$u_{i,j+1} = (1 - 2r)u_{i,j} + r[u_{i+1,j} + u_{i-1,j}]$$

where $r = \frac{\alpha k}{h^2}$



Figure; Grid and mesh points



Figure; stencil for Four Points

N. B. For convergence and stability of explicit method is the condition;

$$r \leq \frac{1}{2} \Rightarrow \frac{\alpha k}{h^2} \leq \frac{1}{2}$$

$$\therefore k \leq \frac{h^2}{2\alpha} \quad \text{time step}$$

Ex. A bar is initially at 100 °C, it's ends are attached to a constant temperature body at 0 °C. Find the temperature distribution assuming $\alpha = 0.1$, Take $\Delta x=h=0.25$.

Sol. If $h= 0.25$ and $\alpha = 0.25$

The condition for stability is;

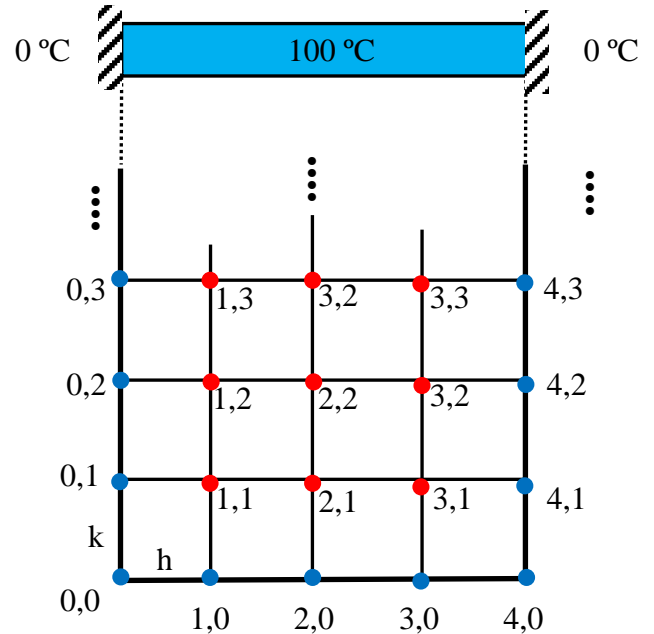
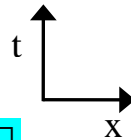
$$\frac{\alpha k}{h^2} \leq \frac{1}{2} \Rightarrow \therefore k \leq \frac{h^2}{2\alpha}$$

$$k \leq \frac{0.25^2}{2*0.1} \rightarrow k \leq 0.3125$$

Take $k = 0.25$

$$r = \frac{\alpha k}{h^2} = \frac{0.1*0.25}{0.25^2} = 0.4$$

$$u_{i,j+1} = (1-2r)u_{i,j} + r[u_{i+1,j} + u_{i-1,j}]$$



Figure; Grid and mesh points

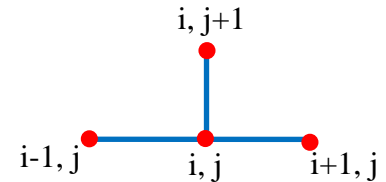
*** Boundary conditions**

At $x = 0$ $u_{0,1} = u_{0,2} = u_{0,3} = \dots = u_{0,j} = 0$

At $x = L$ $u_{4,1} = u_{4,2} = u_{4,3} = \dots = u_{4,j} = 0$

*** Initial conditions At $t = 0$**

$$u_{1,0} = u_{2,0} = u_{3,0} = \dots = u_{i,0} = 100$$



Compute the other points using the above formula

At $t = 0.25$, $j = 1$

$$u_{1,1} = 0.2u_{1,0} + 0.4[u_{2,0} + u_{0,0}] = 0.2 * 100 + 0.4(100 + 0) = 60$$

$$u_{2,1} = 0.2u_{2,0} + 0.4[u_{3,0} + u_{1,0}] = 0.2 * 100 + 0.4(100 + 100) = 100$$

$$u_{3,1} = 0.2u_{3,0} + 0.4[u_{4,0} + u_{2,0}] = 0.2 * 100 + 0.4(0 + 100) = 60$$

At $t = 0.5$, $j = 2$

$$u_{2,1} = 0.2u_{1,1} + 0.4[u_{2,1} + u_{0,1}] = 0.2 * 60 + 0.4(100 + 0) = 52$$

$$u_{2,2} = 0.2u_{2,1} + 0.4[u_{3,1} + u_{1,1}] = 0.2 * 100 + 0.4(60 + 60) = 68$$

$$u_{3,2} = 0.2u_{3,1} + 0.4[u_{4,1} + u_{2,1}] = 0.2 * 60 + 0.4(0 + 100) = 52$$

At t = 0.75, j = 3

H.W. Complete the other points

continue the procedure for until the steady state condition that is not change the value of u at increase the time t.

H.W. Write a computer programming to solve 1-D heat equation.

7.3 Solution of Hyperbolic Equation; (1-D Wave equaton)

In this section we explain the numerical solution of the prototype of hyperbolic equation, the one dimensional wave equation;

$$\frac{1}{c^2} u_{tt} = u_{xx} \quad (1)$$

$$x : space \quad 0 \leq x \leq L \quad , \quad t : time \quad t \geq 0 \quad c : speed \left(\frac{m}{s} \right)$$

Initial displacement $u(x, 0) = f(x)$ (Given)

Initial velocity $u_t(x, 0) = v(x)$ (Given)

boundary condition $u(0, t) = u(L, t) = 0$

Using $\Delta x = h, \quad \Delta t = k$

$$u_{xx} |_{i,j} = \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \quad (a)$$

$$u_{tt} |_{i,j} = \frac{1}{k^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] \quad (b)$$

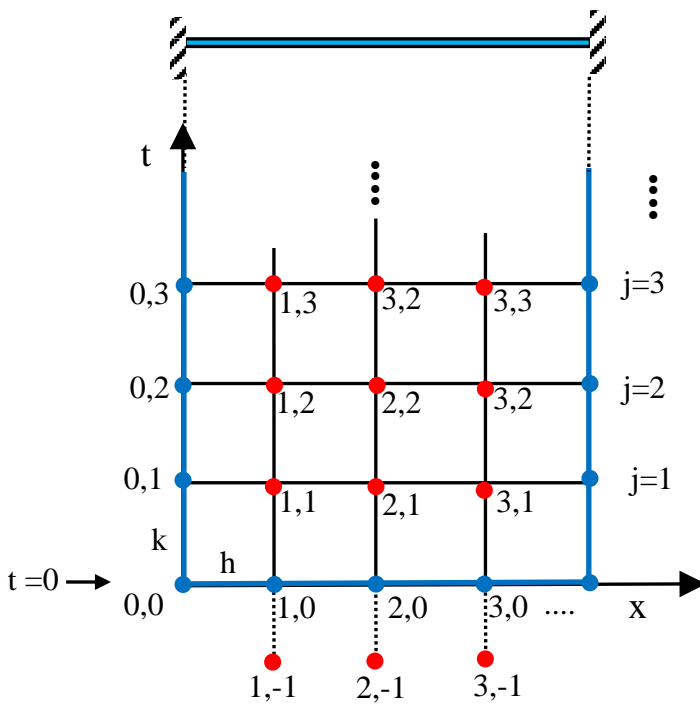
Subs. above Eqs. (a) & (b) in Eq.(1);

$$\frac{1}{c^2 k^2} [u_{i,j+1} - 2u_{i,j} + u_{i,j-1}] = \frac{1}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]$$

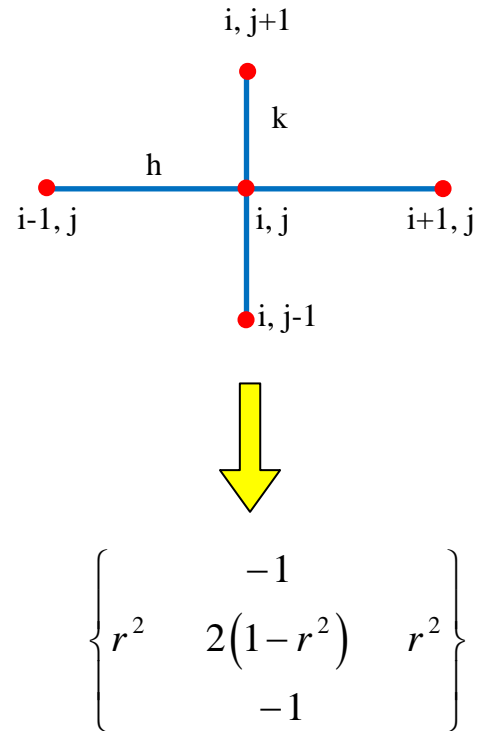
$$u_{i,j+1} = \frac{c^2 k^2}{h^2} [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] + 2u_{i,j} + u_{i,j-1}$$

$$u_{i,j+1} = 2(1-r^2)u_{i,j} + r^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

where $r = \frac{ck}{h}$



Figure; Grid and mesh points



Figure; stencil for five Points

* The condition for convergence and stability of the solution;

$$(1-r^2) \geq 0 \Rightarrow r \leq 1 \Rightarrow \frac{ck}{h} \leq 1 \therefore k \leq \frac{h}{c}$$

At initial velocity $v(x)$ given (At $t = 0, j = 0$)

$$v(x_i) = u_t|_{i,0} = \frac{u_{i,0} - u_{i,-1}}{k}$$

Using Backward difference

$$\therefore u_{i,-1} = u_{i,0} - kv(x_i)$$

Ex. For the string shown, find the motion using $\Delta x = 0.25$, speed $c = 1$ m/s.

$$v(x) = \begin{cases} 2 & 0.25 \leq x \leq 0.75 \\ 0 & \text{otherwise} \end{cases}$$

Sol. $h = \Delta x = 0.25$

$$r \leq 1 \Rightarrow \therefore k \leq \frac{h}{c}$$

$$k \leq 0.25 \Rightarrow \text{take } k = 0.2$$

$$r = \frac{ck}{h} = \frac{1 \cdot 0.2}{0.25} = 0.8$$

$$r^2 = 0.64$$

$$u_{i,j+1} = 2(1-r^2)u_{i,j} + r^2(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

$$u_{i,j+1} = 0.72u_{i,j} + 0.64(u_{i+1,j} + u_{i-1,j}) - u_{i,j-1}$$

* **Boundary conditions;**

$$u_{0,0} = u_{0,1} = u_{0,2} = u_{0,3} = \dots = 0$$

$$u_{4,0} = u_{4,1} = u_{4,2} = u_{4,3} = \dots = 0$$

At $t = 0, j = 0$

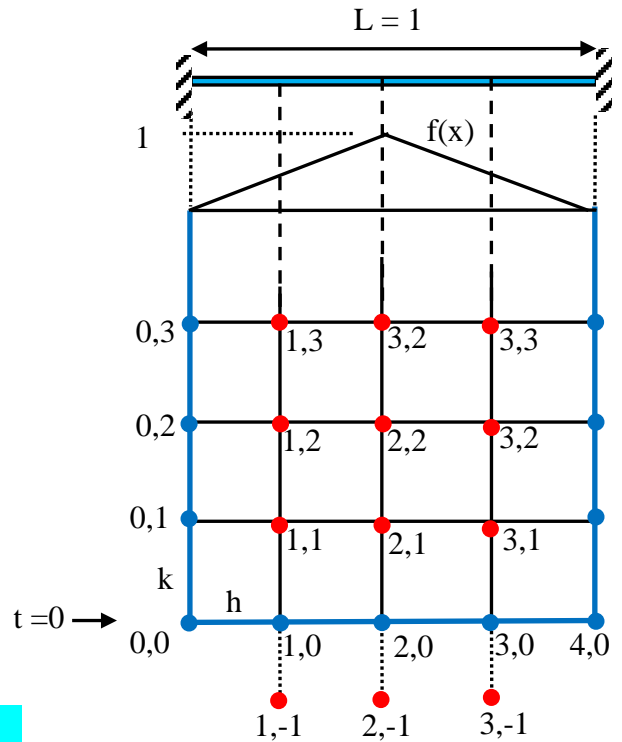
Initial conditions (I.C.)

$$u_{i,0} = f(ih)$$

$$u_{1,0} = f(0.25) = 0.5$$

$$u_{2,0} = f(0.5) = 1$$

$$u_{3,0} = f(0.75) = 0.5$$



Figure; Grid and mesh points

Note: Some time the function $f(x)$ given as; $f(x) = \sin \pi x$

$$u_{i,-1} = u_{i,0} - kv (ih)$$

$$u_{1,-1} = u_{1,0} - kv (0.25) = 0.5 - 0.2 * 2 = 0.1$$

$$u_{2,-1} = u_{2,0} - kv (0.5) = 1 - 0.2 * 2 = 0.6$$

$$u_{3,-1} = u_{3,0} - kv (0.75) = 0.5 - 0.2 * 2 = 0.1$$

$$u_{1,1} = 0.72u_{1,0} + 0.64(u_{2,0} + u_{0,0}) - u_{1,-1} = 0.72 * 0.5 + 0.64(1 + 0) - 0.1 = 0.9$$

$$u_{2,1} = 0.72u_{2,0} + 0.64(u_{3,0} + u_{1,0}) - u_{2,-1} = 0.72 * 1 + 0.64(0.5 + 0.5) - 0.6 = 0.76$$

$$u_{3,1} = 0.72u_{3,0} + 0.64(u_{4,0} + u_{2,0}) - u_{3,-1} = 0.72 * 0.5 + 0.64(0 + 1) - 0.1 = 0.9$$

At t = 0.2, j = 2

$$u_{1,2} = 0.72u_{1,1} + 0.64(u_{2,1} + u_{0,1}) - u_{1,0} = 0.72 * 0.9 + 0.64(0.76 + 0) - 0.5 = 0.6344$$

$$u_{2,2} = 0.72u_{2,1} + 0.64(u_{3,1} + u_{1,1}) - u_{2,0} = 0.72 * 0.76 + 0.64(0.9 + 0.9) - 1 = 0.6992$$

$$u_{3,2} = 0.72u_{3,1} + 0.64(u_{4,1} + u_{2,1}) - u_{3,0} = 0.72 * 0.9 + 0.64(0 + 0.76) - 0.5 = 0.6344$$

At t = 0.4, j = 3

Note: Can be calculate half points and using the symmetry to obtained the others points

H.W. Complete the other points

H.W. Write a computer programming to solution the hyperbolic equation.