ORDINARY DIFFERENTIAL EQUATIONS FOR ENGINEERS

— THE LECTURE NOTES FOR MATH-263 (2011)

ORDINARY DIFFERENTIAL EQUATIONS FOR ENGINEERS

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Contents

Chapter 1

INTRODUCTION

1. Definitions and Basic Concepts

1.1 Ordinary Differential Equation (ODE)

An equation involving the derivatives of an unknown function y of a single variable x over an interval $x \in (I)$. More clearly and precisely speaking, a well defined ODE must the following features:

- It can be written in the form:

$$F[x, y, y', y'', \cdots, y^n] = 0; \tag{1.1}$$

  where the mathematical expression on the right hand side contains (1). variable x, (2). function y of x, and (3). some derivatives of y with respect to x;

- The values of variables x, y must be specified in a certain number field, such as $\mathcal{N}$, $\mathcal{R}$, or $\mathcal{C}$;

- The variation region of variable x of Eq. must be specified, such as $x \in (I) = (a, b)$.

1.2 Solution

Any function $y = f(x)$ which satisfies this equation over the interval (I) is called a solution of the ODE. More clearly speaking, function $\phi(x)$ is called a solution of the give EQ. (1.1), if the following requirements are satisfies:

- The function $\phi(x)$ is defined in the region $x \in (I)$;

- The function $\phi(x)$ is differentiable, hence, $\{\phi'(x), \cdots, \phi^{(n)}(x)\}$ all exit, in the region $x \in (I)$;

- With the replacements of the variables $y, y', \cdots, y^{(n)}$ in 1.1 by the functions $\phi(x), \phi'(x), \cdots, \phi^{(n)}(x)$, the EQ. (1.1) becomes an identity over $x \in (I)$. In other words, the right hand side of Eq. (1.1) becomes to zero for all $x \in (I)$.

For example, one can verify that $y = e^{2x}$ is a solution of the ODE

$$y' = 2y,\ x \in (-\infty, \infty),$$

and $y = \sin(x^2)$ is a solution of the ODE

$$xy'' - y' + 4x^3 y = 0,\ x \in (-\infty, \infty).$$

1.3 Order n of the DE

An ODE is said to be order n, if $y^{(n)}$ is the highest order derivative occurring in the equation. The simplest first order ODE is $y' = g(x)$.

Note that the expression F on the right hand side of an n-th order ODE: $F[x, y, y', \ldots, y^{(n)}] = 0$ can be considered as a function of $n + 2$ variables $(x, u_0, u_1, \ldots, u_n)$. Namely, one may write

$$F(x, u_0, u_1, \cdots, u_n) = 0.$$

Thus, the equations

$$xy'' + y = x^3, \quad y' + y^2 = 0, \quad y''' + 2y' + y = 0$$

which are examples of ODE's of second order, first order and third order respectively, can be in the forms:

$$F(x, u_0, u_1, u_2) = xu_2 + u_0 - x^3,$$

$$F(x, u_0, u_1) = u_1 + u_0^2,$$

$$F(x, u_0, u_1, u_2, u_3) = u_3 + 2u_1 + u_0.$$

respectively.

1.4 Linear Equation:

If the function F is linear in the variables $u_0, u_1, \ldots, u_n$, which means every term in F is proportional to $u_0, u_1, \ldots, u_n$, the ODE is said to be **linear**. If, in addition, F is homogeneous then the ODE is said to be homogeneous. The first of the above examples above is linear are linear, the second is non-linear and the third is linear and homogeneous. The general n-th order linear ODE can be written

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx}$$
$$+ a_0(x)y = b(x).$$

1.5 Homogeneous Linear Equation:

The linear DE is homogeneous, if and only if $b(x) \equiv 0$. Linear homogeneous equations have the important property that linear combinations of solutions are also solutions. In other words, if $y_1, y_2, \ldots, y_m$ are solutions and $c_1, c_2, \ldots, c_m$ are constants then

$$c_1 y_1 + c_2 y_2 + \cdots + c_m y_m$$

is also a solution.

1.6 Partial Differential Equation (PDE)

An equation involving the partial derivatives of a function of more than one variable is called PED. The concepts of linearity and homogeneity can be extended to PDE's. The general second order linear PDE in two variables x, y is

$$a(x,y)\tfrac{\partial^2 u}{\partial x^2} + b(x,y)\tfrac{\partial^2 u}{\partial x \partial y} + c(x,y)\tfrac{\partial^2 u}{\partial y^2} + d(x,y)\tfrac{\partial u}{\partial x}$$
$$+e(x,y)\tfrac{\partial u}{\partial y} + f(x,y)u = g(x,y).$$

Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a linear, homogeneous PDE of order 2. The functions $u = \log(x^2+y^2)$, $u = xy$, $u = x^2 - y^2$ are examples of solutions of Laplace's equation. We will not study PDE's systematically in this course.

1.7 General Solution of a Linear Differential Equation

It represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval (I).

For example, given the differential equation

$$y' = 3x^2.$$

Its general solution is

$$y = x^3 + C$$

where C is an arbitrary constant. To select a specific solution, one needs to determine the constant C with some additional conditions. For instance, the constant C can be determined by the value of y at $x = 0$. This condition is called the **initial condition**, which completely determines the solution. More generally, it will be shown in the future that given a, b there is a unique solution y of the differential equation with the initial condition $y(a) = b$. Geometrically, this means that the one-parameter family of curves $y = x^2 + C$ do not intersect one another and they fill up the plane $\mathcal{R}^2$.

1.8 A System of ODE's

The **normal form** of system of ODE's is

$$\begin{aligned} y_1' &= G_1(x, y_1, y_2, \ldots, y_n) \\ y_2' &= G_2(x, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= G_n(x, y_1, y_2, \ldots, y_n). \end{aligned}$$

The number of the equations is called the order of such system. An n-th order ODE of the form $y^{(n)} = G(x, y, y', \ldots, y^{n-1})$ can be transformed in the form of the n-th order system of DE's. If we introduce dependant variables

$$y_1 = y, y_2 = y', \ldots, y_n = y^{n-1}$$

we obtain the equivalent system of first order equations

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_n' &= G(x, y_1, y_2, \ldots, y_n). \end{aligned}$$

For example, the ODE $y'' = y$ is equivalent to the system

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_1. \end{aligned}$$

In this way the study of n-th order equations can be reduced to the study of systems of n first order equations, or say,n-th order of system of ODE's. Systems of equations arise in the study of the motion of particles. For example, if $P(x, y)$ is the position of a particle of mass m at time t, moving in a plane under the action of the force field $(f(x, y), g(x, y)$, we have

$$\begin{aligned} m\frac{d^2x}{dt^2} &= f(x, y), \\ m\frac{d^2y}{dt^2} &= g(x, y). \end{aligned}$$

This is a system of two second order Eq's, it can be easily transformed into a normal form of 4-th order system of ODE's, by introducing the new unknown functions: $x_1 = x, x_2 = x'$ and $y_1 = y, y_2 = y'$.

The general first order ODE in normal form is

$$y' = F(x, y).$$

If F and $\frac{\partial F}{\partial y}$ are continuous one can show that, given a, b, there is a unique solution with $y(a) = b$. Describing this solution is not an easy task and there are a variety of ways to do this. The dependence of the solution on initial conditions is also an important question as the initial values may be only known approximately.

The non-linear ODE $yy' = 4x$ is not in normal form but can be brought to normal form

$$y' = \frac{4x}{y}.$$

by dividing both sides by y.

2. The Approaches of Finding Solutions of ODE

2.1 Analytical Approaches

- Analytical solution methods: finding the exact form of solutions;
- Geometrical methods: finding the qualitative behavior of solutions;
- Asymptotic methods: finding the asymptotic form of the solution, which gives good approximation of the exact solution.

2.2 Numerical Approaches

- Numerical algorithms — numerical methods;
- Symbolic manipulators — Maple, MATHEMATICA, MacSyma.

This course mainly discuss the analytical approaches and mainly on analytical solution methods.

Chapter 2

FIRST ORDER DIFFERENTIAL EQUATIONS

In this chapter we are going to treat linear and separable first order ODE's.

1. Linear Equation

The general first order ODE has the form $F(x, y, y') = 0$ where $y = y(x)$. If it is linear it can be written in the form

$$a_0(x)y' + a_1(x)y = b(x)$$

where $a_0(x),\ a_(x),\ b(x)$ are continuous functions of x on some interval (I).

To bring it to normal form $y' = f(x, y)$ we have to divide both sides of the equation by $a_0(x)$. This is possible only for those x where $a_0(x) \neq 0$. After possibly shrinking (I) we assume that $a_0(x) \neq 0$ on (I). So our equation has the form (standard form)

$$y' + p(x)y = q(x)$$

with

$$p(x) = a_1(x)/a_0(x), \quad q(x) = b(x)/a_0(x),$$

both continuous on (I). Solving for y' we get the normal form for a linear first order ODE, namely

$$y' = q(x) - p(x)y.$$

1.1 Linear homogeneous equation

Let us first consider the simple case: $q(x) = 0$, namely,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = 0. \tag{2.1}$$

To find the solutions, we proceed in the following three steps:

- Assume that the solution exists and in the form $y = y(x)$;
- Find the necessary form of the function $y(x)$. In doing so, by the definition of the solution, one substitute the function $y(x)$ in the Eq., and try to transform the EQ. in such a way that its LHS of EQ. is a complete differentiation $\frac{\mathrm{d}}{\mathrm{d}x}[.....]$, while its RHS is a known function.
- Verify $y(x)$ is indeed a solution.

In the step #2, with the chain law of derivative, one may transform Eq. (2.1) into the following form:

$$\frac{y'(x)}{y} = \frac{\mathrm{d}}{\mathrm{d}x} \ln |y(x)| = -p(x).$$

Its LHS is now changed a complete differentiation with respect to x, while its RHS is a known function. By integrating both sides, we derive

$$\ln |y(x)| = -\int p(x)\mathrm{d}x + C,$$

or

$$y = \pm C_1 \mathrm{e}^{-\int p(x)\mathrm{d}x},$$

where C, as well as $C_1 = \mathrm{e}^C > 0$, is arbitrary constant. As $C_=0$, $y(x) = 0$ is a trivial solution, we derive the necessary form of solution:

$$y = A\mathrm{e}^{-\int p(x)\mathrm{d}x}. \tag{2.2}$$

As the final step of derivation, one can verify the function (2.2) is indeed a solution for any value of constant A. In summary, we deduce that (2.2) is the form of solution, that contains all possible solution. Hence, one can call (2.2) as the general solution of Eq. (2.1).

1.2 Linear inhomogeneous equation

We now consider the general case:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x).$$

We still proceed in the three steps as we did in the previous subsection. However, now in the step #2 one cannot directly transform the LHS of Eq. in the complete differential form as we did for the case of homogeneous Eq. For this purpose, we multiply the both sides of our differential equation with a factor $\mu(x) \neq 0$. Then our equation is equivalent (has the same solutions) to the equation

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \mu(x)q(x).$$

We wish that with a properly chosen function $\mu(x)$,

$$\mu(x)y'(x) + \mu(x)p(x)y(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y(x)].$$

For this purpose, the function $\mu(x)$ must has the property

$$\mu'(x) = p(x)\mu(x), \tag{2.3}$$

and $\mu(x) \neq 0$ for all x. By solving the linear homogeneous equation (2.3), one obtain

$$\mu(x) = e^{\int p(x)\mathrm{d}x}. \tag{2.4}$$

With this function, which is called an **integrating factor**, our equation is now transformed into the form that we wanted:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mu(x)y(x)] = \mu(x)q(x), \tag{2.5}$$

Integrating both sides, we get

$$\mu(x)y = \int \mu(x)q(x)\mathrm{d}x + C$$

with C an arbitrary constant. Solving for y, we get

$$y = \frac{1}{\mu(x)} \int \mu(x)q(x)\mathrm{d}x + \frac{C}{\mu(x)} = y_P(x) + y_H(x) \tag{2.6}$$

as the general solution for the general linear first order ODE

$$y' + p(x)y = q(x).$$

In solution (2.6):

- the first part, $y_P(x)$: a **particular solution** of the inhomogeneous equation,

- the second part, $y_H(x)$: the **general solution of the associate homogeneous equation.**

Note that for any pair of scalars a, b with a in (I), there is a unique scalar C such that $y(a) = b$. Geometrically, this means that the solution curves $y = \phi(x)$ are a family of non-intersecting curves which fill the region $I \times \mathcal{R}$.

Example 1: $y' + xy = x$. This is a linear first order ODE in standard form with $p(x) = q(x) = x$. The integrating factor is

$$\mu(x) = \mathrm{e}^{\int x\mathrm{d}x} = \mathrm{e}^{x^2/2}.$$

Hence, after multiplying both sides of our differential equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{x^2/2}y) = x\mathrm{e}^{x^2/2}$$

which, after integrating both sides, yields

$$\mathrm{e}^{x^2/2}y = \int x\mathrm{e}^{x^2/2}\mathrm{d}x + C = \mathrm{e}^{x^2/2} + C.$$

Hence the general solution is $y = 1 + C\mathrm{e}^{-x^2/2}$.

- The solution satisfying the initial condition: $y(0) = 1$:

$$y = 1, \quad (C = 0);$$

  and

- the solution satisfying I.C., $y(0) = a$:

$$y = 1 + (a - 1)\mathrm{e}^{-x^2/2}, \quad (C = a - 1).$$

Example 2: $xy' - 2y = x^3 \sin x$,
$(x > 0)$. We bring this linear first order equation to standard form by dividing by x. We get

$$y' - \frac{2}{x}y = x^2 \sin x.$$

The integrating factor is

$$\mu(x) = \mathrm{e}^{\int -2\mathrm{d}x/x} = \mathrm{e}^{-2\ln x} = 1/x^2.$$

After multiplying our DE in standard form by $1/x^2$ and simplifying, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}(y/x^2) = \sin x$$

from which $y/x^2 = -\cos x + C$, and

$$y = -x^2 \cos x + Cx^2. \tag{2.7}$$

Note that (2.7) is a family of solutions to the DE

$$xy' - 2y = x^3 \sin x.$$

To determine a special solution, one needs to impose an IC. For this problem, let us exam the following IC's:

1 For given IC: $y(0) = 0$, from the general solution (2.7) we derive that

$$0 = y(0) = 0 + c * 0 = 0.$$

So that, the IC is satisfied for any constant C. The problem has infinitely many solutions.

2 For given IC: $y(0) = b \neq 0$, from the general solution (2.7) we derive that

$$b = y(0) = 0 + c * 0 = 0.$$

So that, the IC cannot be satisfied with any constant C. The problem has no solution.

3 For given IC: $y(a) = b$ where $a \neq 0$ and b is any number, from the general solution (2.7) we derive that

$$b = y(0) = -a^2 \cos a + c * a^2 = a^2(C - \cos a),$$

so that, $C = \cos a + b/a^2$ is uniquely determined. The problem has a unique solution.

This example displays a complicated situation, when the IC is imposed at $x = 0$. Why does this happen? The simple explanation to such abnormality is that because $x = 0$ is a **singular point**, where $p(0) = \infty$. Such an abnormal situation will be discussed more deeply in the future.

2. Nonlinear Equations (I)

2.1 Separable Equations.

The first order ODE $y' = f(x, y)$ is said to be separable if $f(x, y)$ can be expressed as a product of a function of x times a function of y. The DE then has the form:

$$y' = g(x)h(y), \ x \in (I).$$

We apply the procedure of three steps for the solutions as before. Assume that the solution $y = y(x)$ exists, and $h(y) \neq 0$ as $x \in (I)$. Then by dividing both sides by $h[y(x)]$, it becomes

$$\frac{y'(x)}{h[y(x)]} = g(x). \tag{2.8}$$

Of course this is not valid for those solutions $y = y(x)$ at the points where $h[y(x)] = 0$. Furthermore, we assume that the LHS can be written in the form of complete differentiation: $\frac{\mathrm{d}}{\mathrm{d}x}H[y(x)]$, where $H[y(x)]$ is a composite function of x to be determined. Once we find the function $H(y)$, we may write

$$H[y(x)] = \int \frac{y'(x)}{h[y(x)]}\mathrm{d}x = \int g(x)\mathrm{d}x + C.$$

However, by chain rule we have

$$\frac{\mathrm{d}}{\mathrm{d}x}H[y(x)] = H'(y)y'(x).$$

By comparing the above with the LHS of (3.33), it follows that

$$H'(y) = \frac{1}{h(y)}.$$

Thus, we derive that

$$H(y) = \int \frac{\mathrm{d}y}{h(y)} = \int g(x)\mathrm{d}x + C, \tag{2.9}$$

This gives the implicit form of the solution. It determines the value of y implicitly in terms of x. The function given in (2.9) can be easily verified as indeed a solution. Note that with the assumption $h(y) \neq 0$ at the beginning of the derivation, some solution may be excluded in (2.9). As a matter of fact, one can verify that the Eq. may allow the constant solutions,

$$y = y_*, \tag{2.10}$$

as $h(y_*) = 0$.

Example 1: $y' = \frac{x-5}{y^2}$.

To solve it using the above method we multiply both sides of the equation by y^2 to get

$$y^2y' = (x-5).$$

Integrating both sides we get $y^3/3 = x^2/2 - 5x + C$. Hence,

$$y = \left[3x^2/2 - 15x + C_1\right]^{1/3}.$$

Example 2: $y' = \frac{y-1}{x+3}$ $(x > -3)$. By inspection, $y = 1$ is a solution. Dividing both sides of the given DE by $y - 1$ we get

$$\frac{y'}{y-1} = \frac{1}{x+3}.$$

This will be possible for those x where $y(x) \neq 1$. Integrating both sides we get

$$\int \frac{y'}{y-1} \mathrm{d}x = \int \frac{\mathrm{d}x}{x+3} + C_1,$$

from which we get $\ln|y-1| = \ln(x+3) + C_1$. Thus $|y-1| = e^{C_1}(x+3)$ from which $y - 1 = \pm e^{C_1}(x+3)$. If we let $C = \pm e^{C_1}$, we get

$$y = 1 + C(x+3)$$

. Since $y = 1$ was found to be a solution by inspection the general solution is

$$y = 1 + C(x+3),$$

where C can be any scalar. For any (a, b) with $a \neq -3$, there is only one member of this family which passes through (a, b).

However, it is seen that there is a family of lines passing through $(-3, 1)$, while no solution line passing through $(-3, b)$ with $b \neq 1$). Here, $x = -3$ is a singular point.

Example 3: $y' = \frac{y \cos x}{1+2y^2}$. Transforming in the standard form then integrating both sides we get

$$\int \frac{(1+2y^2)}{y} \mathrm{d}y = \int \cos x \, \mathrm{d}x + C,$$

from which we get a family of the solutions:

$$\ln|y| + y^2 = \sin x + C,$$

where C is an arbitrary constant. However, this is not the general solution of the equation, as it does not contains, for instance, the solution: $y = 0$. With I.C.: y(0)=1, we get $C = 1$, hence, the solution:

$$\ln|y| + y^2 = \sin x + 1.$$

2.2 Logistic Equation

$$y' = ay(b - y),$$

where $a, b > 0$ are fixed constants. This equation arises in the study of the growth of certain populations. Since the right-hand side of the equation is zero for $y = 0$ and $y = b$, the given DE has $y = 0$ and $y = b$ as solutions. More generally, if $y' = f(t, y)$ and $f(t, c) = 0$ for all t in some interval (I), the constant function $y = c$ on (I) is a solution of $y' = f(t, y)$ since $y' = 0$ for a constant function y.

To solve the logistic equation, we write it in the form

$$\frac{y'}{y(b - y)} = a.$$

Integrating both sides with respect to t we get

$$\int \frac{y' \mathrm{d}t}{y(b - y)} = at + C$$

which can, since $y' \mathrm{d}t = \mathrm{d}y$, be written as

$$\int \frac{\mathrm{d}y}{y(b - y)} = at + C.$$

Since, by partial fractions,

$$\frac{1}{y(b - y)} = \frac{1}{b}(\frac{1}{y} + \frac{1}{b - y})$$

we obtain

$$\frac{1}{b}(\ln |y| - \ln |b - y|) = at + C.$$

Multiplying both sides by b and exponentiating both sides to the base e, we get

$$\frac{|y|}{|b - y|} = \mathrm{e}^{bC} \mathrm{e}^{abt} = C_1 \mathrm{e}^{abt},$$

where the arbitrary constant $C_1 = \mathrm{e}^{bC} > 0$ can be determined by the initial condition (IC): $y(0) = y_0$ as

$$C_1 = \frac{|y_0|}{|b - y_0|}.$$

Two cases need to be discussed separately.

Case (I), $y_0 < b$: one has $C_1 = |\frac{y_0}{b-y_0}| = \frac{y_0}{b-y_0} > 0$. So that,

$$\frac{|y|}{|b-y|} = \left(\frac{y_0}{b-y_0}\right) \mathrm{e}^{abt} > 0, \quad (t \in (I)).$$

From the above we derive

$$y/(b-y) = C_1 \mathrm{e}^{abt},$$

$$y = (b-y) C_1 \mathrm{e}^{abt}.$$

This gives

$$y = \frac{bC_1 \mathrm{e}^{abt}}{1 + C_1 \mathrm{e}^{abt}} = \frac{b\left(\frac{y_0}{b-y_0}\right) \mathrm{e}^{abt}}{1 + \left(\frac{y_0}{b-y_0}\right) \mathrm{e}^{abt}}.$$

It shows that if $y_0 = 0$, one has the solution $y(t) = 0$. However, if $0 < y_0 < b$, one has the solution $0 < y(t) < b$, and as $t \to \infty$, $y(t) \to b$.

Case (II), $y_0 > b$: one has $C_1 = |\frac{y_0}{b-y_0}| = -\frac{y_0}{b-y_0} > 0$. So that,

$$\left|\frac{y}{b-y}\right| = \left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt} > 0, \quad (t \in (I)).$$

From the above we derive

$$y/(y-b) = \left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt},$$

$$y = (y-b)\left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt}.$$

This gives

$$y = \frac{b\left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt}}{\left(\frac{y_0}{y_0-b}\right) \mathrm{e}^{abt} - 1}.$$

It shows that if $y_0 > b$, one has the solution $y(t) > b$, and as $t \to \infty$, $y(t) \to b$.

It is derived that

- $y(t) = 0$ is an unstable equilibrium state of the system;
- $y(t) = b$ is a stable equilibrium state of the system.

2.3 Fundamental Existence and Uniqueness Theorem

If the function $f(x, y)$ together with its partial derivative with respect to y are continuous on the rectangle

$$(R) : |x - x_0| \le a, \ |y - y_0| \le b$$

there is a unique solution to the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

defined on the interval $|x - x_0| < h$ where

$$h = \min(a, b/M), \quad M = \max |f(x, y)|, \ (x, y) \in (R).$$

Note that this theorem indicates that a solution may not be defined for all x in the interval $|x - x_0| \le a$. For example, the function

$$y = \frac{bCe^{abx}}{1 + Ce^{abx}}$$

is solution to $y' = ay(b - y)$ but not defined when $1 + Ce^{abx} = 0$ even though $f(x, y) = ay(b - y)$ satisfies the conditions of the theorem for all x, y.

The next example show why the condition on the partial derivative in the above theorem is important sufficient condition.

Consider the differential equation $y' = y^{1/3}$. Again $y = 0$ is a solution. Separating variables and integrating, we get

$$\int \frac{dy}{y^{1/3}} = x + C_1$$

which yields

$$y^{2/3} = 2x/3 + C$$

and hence

$$y = \pm(2x/3 + C)^{3/2}.$$

Taking $C = 0$, we get the solution

$$y = \pm(2x/3)^{3/2}, (x \ge 0)$$

which along with the solution $y = 0$ satisfies $y(0) = 0$. Even more, Taking $C = -(2x_0/3)^{3/2}$, we get the solution:

$$y = \begin{cases} 0 & (0 \le x \le x_0) \\ \pm[2(x - x_0)/3]^{3/2}, & (x \ge x_0) \end{cases}$$

which also satisfies $y(0) = 0$. So the initial value problem

$$y' = y^{1/3}, \quad y(0) = 0$$

does not have a unique solution. The reason this is so is due to the fact that

$$\frac{\partial f}{\partial y}(x, y) = \frac{1}{3y^{2/3}}$$

is not continuous when $y = 0$.

Many differential equations become linear or separable after a change of variable. We now give two examples of this.

2.4 Bernoulli Equation:

$$y' = p(x)y + q(x)y^n \quad (n \neq 1).$$

Note that $y = 0$ is a solution. To solve this equation, we set

$$u = y^\alpha,$$

where α is to be determined. Then, we have

$$u' = \alpha y^{\alpha-1} y',$$

hence, our differential equation becomes

$$u'/\alpha = p(x)u + q(x)y^{\alpha+n-1}. \tag{2.11}$$

Now set

$$\alpha = 1 - n,$$

Thus, (2.11) is reduced to

$$u'/\alpha = p(x)u + q(x), \tag{2.12}$$

which is linear. We know how to solve this for u from which we get solve

$$u = y^{1-n}$$

to get

$$y = u^{1/(1-n)}. \tag{2.13}$$

2.5 Homogeneous Equation:

$$y' = F(y/x).$$

To solve this we let

$$u = y/x,$$

so that

$$y = xu, \quad \text{and} \quad y' = u + xu'.$$

Substituting for y, y' in our DE gives

$$u + xu' = F(u)$$

which is a separable equation. Solving this for u gives y via $y = xu$.

Note that

$$u \equiv a$$

is a solution of

$$xu' = F(u) - u$$

whenever $F(a) = a$ and that this gives

$$y = ax$$

as a solution of

$$y' = f(y/x).$$

Example. $y' = (x - y)/x + y$. This is a homogeneous equation since

$$\frac{x-y}{x+y} = \frac{1 - y/x}{1 + y/x}.$$

Setting $u = y/x$, our DE becomes

$$xu' + u = \frac{1-u}{1+u}$$

so that

$$xu' = \frac{1-u}{1+u} - u = \frac{1 - 2u - u^2}{1+u}.$$

Note that the right-hand side is zero if $u = -1 \pm \sqrt{2}$. Separating variables and integrating with respect to x, we get

$$\int \frac{(1+u)du}{1 - 2u - u^2} = \ln|x| + C_1$$

which in turn gives

$$(-1/2)\ln|1-2u-u^2| = \ln|x| + C_1.$$

Exponentiating, we get

$$\frac{1}{\sqrt{|1-2u-u^2|}} = e^{C_1}|x|.$$

Squaring both sides and taking reciprocals, we get

$$u^2 + 2u - 1 = C/x^2$$

with $C = \pm 1/e^{2C_1}$. This equation can be solved for u using the quadratic formula. If x_0, y_0 are given with

$$x_0 \neq 0, \quad \text{and} \quad u_0 = y_0/x_0 \neq -1$$

there is, by the fundamental, existence and uniqueness theorem, a unique solution with I.C.

$$y(x_0) = y_0.$$

For example, if $x_0 = 1, y_0 = 2$, we have, $u(x_0) = 2$, so, $C = 7$ and hence

$$u^2 + 2u - 1 = 7/x^2$$

Solving for u, we get

$$u = -1 + \sqrt{2 + 7/x^2}$$

where the **positive sign** in the quadratic formula was chosen to make $u = 2, x = 1$ a solution. Hence

$$y = -x + x\sqrt{2 + 7/x^2} = -x + \sqrt{2x^2 + 7}$$

is the solution to the initial value problem

$$y' = \frac{x-y}{x+y}, \quad y(1) = 2$$

for $x > 0$ and one can easily check that it is a solution for all x. Moreover, **using the fundamental uniqueness theorem**, it can be shown that **it is the only solution** defined for all x.

3. Nonlinear Equations (II)— Exact Equation and Integrating Factor

3.1 Exact Equations.

By a region of the (x, y)-plane we mean a connected open subset of the plane. The differential equation

$$M(x, y) + N(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

is said to be exact on a region (R) if there is a function $F(x, y)$ defined on (R) such that

$$\frac{\partial F}{\partial x} = M(x, y); \quad \frac{\partial F}{\partial y} = N(x, y)$$

In this case, if M, N are continuously differentiable on (R) we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{2.14}$$

Conversely, it can be shown that condition (2.14) is also sufficient for the exactness of the given DE on (R) providing that (R) is simply connected, .i.e., has no "holes".

The exact equations are solvable. In fact, suppose $y(x)$ is its solution. Then one can write:

$$\begin{aligned} M\left[x, y(x)\right] + N\left[x, y(x)\right]\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}F\left[x, y(x)\right] = 0. \end{aligned}$$

It follows that

$$F\left[x, y(x)\right] = C,$$

where C is an arbitrary constant. This is an implicit form of the solution $y(x)$. Hence, the function $F(x, y)$, if it is found, will give a family of the solutions of the given DE.

The curves $F(x, y) = C$ are called **integral curves** of the given DE.

Example 1. $2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy^2 + 1 = 0$. Here

$$M = 2xy^2 + 1, \quad N = 2x^2y$$

and $R = \mathcal{R}^2$, the whole (x, y)-plane. The equation is exact on $\mathcal{R}^2$ since $\mathcal{R}^2$ is simply connected and

$$\frac{\partial M}{\partial y} = 4xy = \frac{\partial N}{\partial x}.$$

To find F we have to solve the partial differential equations

$$\frac{\partial F}{\partial x} = 2xy^2 + 1, \quad \frac{\partial F}{\partial y} = 2x^2 y.$$

If we integrate the first equation with respect to x holding y fixed, we get

$$F(x, y) = x^2 y^2 + x + \phi(y).$$

Differentiating this equation with respect to y gives

$$\frac{\partial F}{\partial y} = 2x^2 y + \phi'(y) = 2x^2 y$$

using the second equation. Hence $\phi'(y) = 0$ and $\phi(y)$ is a constant function. The solutions of our DE in implicit form is

$$x^2 y^2 + x = C.$$

Example 2. We have already solved the homogeneous DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x - y}{x + y}.$$

This equation can be written in the form

$$y - x + (x + y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

which is an exact equation. In this case, the solution in implicit form is

$$x(y - x) + y(x + y) = C,$$

i.e.,

$$y^2 + 2xy - x^2 = C.$$

4. Integrating Factors.

If the differential equation $M + Ny' = 0$ is not exact it can sometimes be made exact by multiplying it by a continuously differentiable function $\mu(x, y)$. Such a function is called an *integrating factor*. An integrating factor μ satisfies the PDE:

$$\frac{\partial \mu M}{\partial y} = \frac{\partial \mu N}{\partial x},$$

which can be written in the form

$$\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\mu = N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y}.$$

This equation can be simplified in special cases, two of which we treat next.

- μ is a function of x only. This happens if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = p(x)$$

is a function of x only in which case $\mu' = p(x)\mu$.

- μ is a function of y only. This happens if and only if

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = q(y)$$

is a function of y only in which case $\mu' = -q(y)\mu$.

- $\mu = P(x)Q(y)$. This happens if and only if

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = p(x)N - q(y)M, \tag{2.15}$$

where

$$p(x) = \frac{P'(x)}{P(x)}, \quad q(y) = \frac{Q'(y)}{Q(y)}.$$

If the system really permits the functions $p(x), q(y)$, such that (2.15) hold, then we can derive

$$P(x) = \pm e^{\int p(x)dx}; \qquad Q(y) = \pm e^{\int q(y)dy}.$$

Example 1. $2x^2 + y + (x^2y - x)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 - 2xy}{x^2y - x} = \frac{-2}{x}$$

so that there is an integrating factor $\mu(x)$ which is a function of x only, and satisfies

$$\mu' = -2\mu/x.$$

Hence $\mu = 1/x^2$ is an integrating factor and

$$2 + y/x^2 + (y - 1/x)y' = 0$$

is an exact equation whose general solution is

$$2x - y/x + y^2/2 = C$$

or

$$2x^2 - y + xy^2/2 = Cx.$$

Example 2. $y + (2x - ye^y)y' = 0$. Here

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-1}{y},$$

so that there is an integrating factor which is a function of y only which satisfies

$$\mu' = \mu/y.$$

Hence y is an integrating factor and

$$y^2 + (2xy - y^2e^y)y' = 0$$

is an exact equation with general solution:

$$xy^2 + (-y^2 + 2y - 2)e^y = C.$$

A word of caution is in order here. **The solutions of the exact DE obtained by multiplying by the integrating factor may have solutions which are not solutions of the original DE.** This is due to the fact that μ may be zero and one will have to possibly exclude those solutions where μ vanishes. However, this is not the case for the above Example 2.

Chapter 3

N-TH ORDER DIFFERENTIAL EQUATIONS

1. Introduction

In this lecture we will state and sketch the proof of the fundamental existence and uniqueness theorem for the n-th order DE

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}).$$

The starting point is to convert this DE into a system of first order DE'. Let $y_1 = y, y_2 = y', \dots y^{(n-1)} = y_n$. Then the above DE is equivalent to the system

$$\begin{aligned} \tfrac{dy_1}{dx} &= y_2 \\ \tfrac{dy_2}{dx} &= y_3 \\ &\vdots \\ \tfrac{dy_n}{dx} &= f(x, y_1, y_2, \dots, y_n). \end{aligned} \tag{3.1}$$

More generally let us consider the system

$$\begin{aligned} \tfrac{dy_1}{dx} &= f_1(x, y_1, y_2, \dots, y_n) \\ \tfrac{dy_2}{dx} &= f_2(x, y_1, y_2, \dots, y_n) \\ &\vdots \\ \tfrac{dy_n}{dx} &= f_n(x, y_1, y_2, \dots, y_n). \end{aligned} \tag{3.2}$$

If we let

$$Y = (y_1, y_2, \dots, y_n),$$

$$F(x, Y) = \Big\{ f_1(x, Y), f_2(x, Y), \dots, f_n(x, Y) \Big\}$$

and

$$\frac{dY}{dx} = \left(\frac{dy_1}{dx}, \frac{dy_2}{dx}, \ldots, \frac{dy_n}{dx}\right),$$

the system becomes

$$\frac{dY}{dx} = F(x, Y).$$

2. (*)Fundamental Theorem of Existence and Uniqueness

2.1 Theorem of Existence and Uniqueness (I)

If $f_i(x, y_1, \ldots, y_n)$ and $\frac{\partial f_i}{\partial y_j}$ are continuous on the $n+1$-dimensional box

$$R : |x - x_0| < a,\ |y_i - c_i| < b, (1 \le i \le n)$$

for $1 \le i, j \le n$ with $|f_i(x, y)| \le M$ and

$$\left|\frac{\partial f_i}{\partial y_1}\right| + \left|\frac{\partial f_i}{\partial y_2}\right| + \ldots \left|\frac{\partial f_i}{\partial y_n}\right| < L$$

on R for all i, the initial value problem

$$\frac{dY}{dx} = F(x, Y), \quad Y(x_0) = (c_1, c_2, \ldots, c_n)$$

has a unique solution on the interval:

$$|x - x_0| \le h = \min(a, b/M).$$

The proof is exactly the same as for the proof for $n = 1$. Since $f_i(x, y_1, \ldots, y_n)$, $\frac{\partial f_i}{\partial y_j}$ are continuous in the strip:

$$|x - x_0| \le a,$$

we have an constant L such that

$$|f(x, Y) - f(x, Z)| \le L|Y - Z|.$$

The Picard iterations $Y_k(x)$ defined by

$$Y_0(x) = Y_0 = (c_1, \ldots, c_n),$$

$$Y_{k+1}(x) = Y_0 + \textstyle\int_{x_0}^{x} F(t, Y_k(t))dt,$$

converge to the unique solution Y and

$$|Y(x) - Y_k(x)| \le (M/L)e^{hL}h^{k+1}/(k+1)!.$$

As a corollary of the above theorem we get the following fundamental theorem for n-th order DE's.

2.2 Theorem of Existence and Uniqueness (II)

If $f(x, y_1, \ldots, y_n)$ and $\frac{\partial f}{\partial y_j}$ are continuous on the box

$$R : |x - x_0| \leq a,\ |y_i - c_i| \leq b\ (1 \leq i \leq n)$$

and $|f(x, y_1, \ldots, y_n)| \leq M$ on R, then the initial value problem

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),$$

with I.C's:

$$y^{(i-1)}(x_0) = c_i\ (1 \leq i \leq n)$$

has a unique solution on the interval:

$$|x - x_0| \leq h = \max(a, b/M).$$

2.3 Theorem of Existence and Uniqueness (III)

If $a_0(x), a_1(x), \ldots, a_n(x)$ are continuous on an interval I and $a_0(x) \neq 0$ on I then, for any $x_0 \in I$, that is not an endpoint of I, and any scalars $c_1, c_2, \ldots, c_n$, the initial value problem

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x),$$

with I.C.'s:

$$y^{(i-1)}(x_0) = c_i\ (1 \leq i \leq n)$$

has a unique solution on the interval I.

In this case:

$$f_1 = y_2,\ f_2 = y_3,\ f_n = p_1(x)y_1 + \cdots p_n(x)y_n + q(x)$$

where

$$p_i(x) = a_{n-i}(x)/a_0(x),$$

and

$$q(x) = -b(x)/a_0(x).$$

Hence, $\frac{\partial f}{\partial y_j}$ are continuous

3. Linear Equations

In this chapter, we are only concerned with linear equations.

3.1 Basic Concepts and General Properties

Let us now go to linear equations. The general form is

$$L(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x). \tag{3.3}$$

The function L is called a *differential operator*.

3.1.1 Linearity

The characteristic features of *linear operator* L is that

- With any constants (C_1, C_2),

$$L(C_1y_1 + C_2y_2) = C_1L(y_1) + C_2L(y_2).$$

- With any given functions of x, $p_1(x), p_2(x)$, and the Linear operators,

$$L_1(y) = a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y$$

$$L_2(y) = b_0(x)y^{(n)} + b_1(x)y^{(n-1)} + \cdots + b_n(x)y,$$

the function

$$p_1L_1 + p_2L_2$$

defined by

$$\begin{aligned}(p_1L_1 + p_2L_2)(y) &= p_1(x)L_1(y) + p_2(x)L_2(y)\\ &= \Big[p_(x)a_0(x) + p_2(x)b_0(x)\Big]\, y^{(n)} + \cdots\\ &\quad + [p_1(x)a_n(x) + p_2(x)b_n(x)]\, y\end{aligned}$$

is again a linear differential operator.

- Linear operators in general are subject to the *distributive law*:

$$\begin{aligned}L(L_1 + L_2) &= LL_1 + LL_2,\\ (L_1 + L_2)L &= L_1L + L_2L.\end{aligned}$$

- Linear operators with **constant coefficients** are commutative:

$$L_1L_2 = L_2L_1.$$

Note: In general the linear operators with **non-constant coefficients** are not commutative: Namely,

$$L_1L_2 \neq L_2L_1.$$

For instance, let $L_1 = a(x)\frac{\mathrm{d}}{\mathrm{d}x}$, $L_2 = \frac{\mathrm{d}}{\mathrm{d}x}$.

$$L_1L_2 = a(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} \neq L_2L_1 = \frac{\mathrm{d}}{\mathrm{d}x}\Big[a(x)\frac{\mathrm{d}}{\mathrm{d}x}\Big].$$

3.1.2 Superposition of Solutions

The solutions the linear of equation (3.3) have the following properties:

- For any two solutions y_1, y_2 of (3.3), namely,

$$L(y_1) = b(x), \quad L(y_2) = b(x),$$

the difference $(y_1 - y_2)$ is a solution of the associated homogeneous equation

$$L(y) = 0.$$

- For any pair of of solutions y_1, y_2 of the associated homogenous equation:

$$L(y_1) = 0, \quad L(y_2) = 0,$$

the linear combination

$$(a_1 y_1 + a_2 y_2)$$

of solutions y_1, y_2 is again a solution of the homogenous equation:

$$L(y) = 0.$$

3.1.3 (∗) Kernel of Linear operator $L(y)$

The solution space of $L(y) = 0$ is also called the **kernel** of L and is denoted by $\ker(L)$. It is a subspace of the vector space of real valued functions on some interval I. If y_p is a particular solution of

$$L(y) = b(x),$$

the general solution of

$$L(y) = b(x)$$

is

$$\ker(L) + y_p = \{y + y_p \mid L(y) = 0\}.$$

3.2 New Notations

The differential operator

$$\{L(y) = y'\} \Longrightarrow Dy.$$

The operator

$$L(y) = y'' = D^2 y = D \circ Dy,$$

where $\circ$ denotes composition of functions. More generally, the operator

$$L(y) = y^{(n)} = D^n y.$$

The identity operator I is defined by

$$I(y) = y = D^0 y.$$

By definition $D^0 = I$. The general linear n-th order ODE can therefore be written

$$\Big[a_0(x)D^n + a_1(x)D^{n-1} + \cdots + a_n(x)I\Big](y) = b(x).$$

4. Basic Theory of Linear Differential Equations

In this section we will develop the theory of linear differential equations. The starting point is the fundamental existence theorem for the general n-th order ODE $L(y) = b(x)$, where

$$L(y) = D^n + a_1(x)D^{n-1} + \cdots + a_n(x).$$

We will also assume that $a_0(x) \neq 0, a_1(x), \ldots, a_n(x), b(x)$ are continuous functions on the interval I.

The fundamental theory says that for any $x_0 \in I$, the initial value problem

$$L(y) = b(x)$$

with the initial conditions:

$$y(x_0) = d_1, y'(x_0) = d_2, \ldots, y^{(n-1)}(x_0) = d_n$$

has a unique solution $y = y(x)$ for any $(d_1, d_2, \ldots, d_n) \in \mathcal{R}^n$.

From the above, one may deduce that the general solution of n-th order linear equation contains n arbitrary constants. It can be also deduced that the above solution can be expressed in the form:

$$y(x) = d_1 y_1(x) + \cdots + d_n y_n(x),$$

where the $\{y_i(x)\}, (i = 1, 2, \cdots, n)$ is the set of the solutions for the IVP with IC's:

$$\begin{cases} y^{(i-1)}(x_0) = 1, \ i = 1, 2, \cdots n) \\ y^{(k-1)}(x_0) = 0, \ (k \neq i, 1 \leq k \leq n). \end{cases}$$

In general, if the equation $L(y) = 0$ has a set of n solutions: $\{y_1(x), \cdots y_n(x)\}$ of the equation, such that any solution $y(x)$ of the equation can be expressed in the form:

$$y(x) = c_1 y_1(x) + \cdots + c_n y_n(x),$$

with a proper set of constants $\{c_1, \cdots, c_n\}$, then the solutions $\{y_i(x), i = 1, 2, \cdots, n\}$ is called **a set of fundamental solutions** of the equation.

4.1 Basics of Linear Vector Space

4.1.1 Dimension and Basis of Vector Space, Fundamental Set of Solutions of Eq.

We call the vector space being n-dimensional with the notation by $\dim(V) = n$. This means that there exists a sequence of elements: $y_1, y_2, \ldots, y_n \in V$ such that every $y \in V$ can be uniquely written in the form

$$y = c_1 y_1 + c_2 y_2 + \ldots c_n y_n$$

with $c_1, c_2, \ldots, c_n \in \mathcal{R}$. Such a sequence of elements of a vector space V is called a **basis** for V. In the context of DE's it is also known as a **fundamental set**.

The number of elements in a basis for V is called the dimension of V and is denoted by $\dim(V)$. For instance,

$$e_1 = (1, 0, \ldots, 0), \ e_2 = (0, 1, \ldots, 0), \ldots,$$
$$e_n = (0, 0, \ldots, 1)$$

is the standard basis of geometric vector space $\mathcal{R}^n$.

A set of vectors $v_1, v_2, \cdots, v_n$ in a vector space V is said to **span** or **generate** V if every $v \in V$ can be written in the form

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

with $c_1, c_2, \ldots, c_n \in \mathcal{R}$. Obviously, not any set of n vectors can span the vector space V. It will be seen that $\{v_1, v_2, \cdots, v_n\}$ span the vector space V, if and only if they are linear independent.

4.1.2 Linear Independency

Recall that, as it is learn in Calculus, a set of geometric vectors: $v_1, v_2, \ldots, v_n$ is said to be **linearly independent** if and only if

$$c_1 v_1 + c_2 v_2 + \ldots c_n v_n = 0$$

implies that the scalars $c_1, c_2, \ldots, c_n$ are all zero. The same concept can be introduced for the space of functions. Namely, a set of functions: $v_1(x), v_2(x), \ldots, v_n(x)$ defined on the interval $x \in (I)$ is said to be **linearly independent** if and only if

$$c_1 v_1(x) + c_2 v_2(x) + \ldots c_n v_n(x) = 0$$

for all $x \in (I)$ implies that the scalars $c_1, c_2, \ldots, c_n$ are all zero.

A basis can also be characterized as a linearly independent generating set since the uniqueness of representation is equivalent to linear independence. More precisely,

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = c_1' v_1 + c_2' v_2 + \cdots + c_n' v_n$$

implies

$$c_i = c_i' \qquad \text{for all } i,$$

if and only if $v_1, v_2, \ldots, v_n$ are linearly independent.

To determine the linear independency of given set of functions $\{v_1(x), v_2(x), \cdots, v_n(x)\}$, one may use two methods. To demonstrate the idea, let us take $n = 3$.

Method (I): We pick three points $(x_1, x_2, x_3) \in (I)$. we may derive a system linear equations:

$$c_1 v_1(x_i) + c_2 v_2(x_i) + c_3 v_3(x_i) = 0, \quad (i = 1, 2, 3).$$

If (x_1, x_2, x_3) are chosen properly, such that the determinant of the coefficients of above equations is

$$\Delta = \begin{vmatrix} v_1(x_1) & v_2(x_1) & v_3(x_1) \\ v_1(x_2) & v_2(x_2) & v_3(x_2) \\ v_1(x_3) & v_2(x_3) & v_3(x_3) \end{vmatrix} \neq 0,$$

we derive $c_1 = c_2 = c_3 = 0$. Hence, we may conclude that the set of functions is linear independent.

Example 1. Given a set of functions:

$$\cos(x), \cos(2x), \sin(3x).$$

Show that they are linearly independent. To prove their linear independence, suppose that c_1, c_2, c_3 are scalars such that

$$c_1 \cos(x) + c_2 \cos(2x) + c_3 \sin(3x) = 0 \tag{3.4}$$

for all x. Then setting $x = 0, \pi/2, \pi$, we get

$$\begin{aligned} c_1 + c_2 \qquad &= 0, \\ -c_2 \qquad &= 0, \\ -c_1 + c_2 - c_3 &= 0, \end{aligned}$$

from which $\Delta \neq 0$, hence $c_1 = c_2 = c_3 = 0$. So that, this set of function is (L.I.)

Example 2. Given a set of functions:

$$\{v_1 = x^3, v_2 = |x|^3\}, \; x \in (I) = (-\infty, \infty).$$

Show that they are linearly independent in (I). To show this, one may set two points: $\{x = -1, 1\} \in (I)$. It is then derived that

$$\Delta = \begin{vmatrix} v_1(x_1) & v_2(x_1) \\ v_1(x_2) & v_2(x_2) \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$$

It is done.

A simple example of a set of linearly dependent functions is:

$$\sin^2(x), \cos^2(x), \cos(2x), x \in (I).$$

since we have the identity

$$\cos(2x) = \cos^2(x) - \sin^2(x),$$

which implies that

$$\cos(2x) + \sin^2(x) + (-1)\cos^2(x) = 0,$$

for all $x \in (I)$.

Method (II): We need to assume that the functions $v_i(x)$ are differentiable up to $(n-1)$-th order. From the formula (3.4), by making the derivatives on both sides twice, one may derive:

$$c_1 v_1^{(i)}(x) + c_2 v_2^{(i)}(x) + c_3 v_3^{(i)}(x) = 0, \quad (i = 1, 2), \tag{3.5}$$

for all $x \in (I)$.

We choose **one point** $x_0 \in (I)$, and evaluate the formulas (3.4) and (3.5) at $x = x_0$. Thus, we may derive the following system linear equations:

$$c_1 v_1^{(i)}(x_0) + c_2 v_2^{(i)}(x_0) + c_3 v_3^{(i)}(x_0) = 0, \quad (i = 0, 1, 2).$$

If x_0 is chosen properly, such that the determinant of the coefficients of above equations is

$$\Delta(x_0) = \begin{vmatrix} v_1(x_0) & v_2(x_0) & v_3(x_0) \\ v_1'(x_0) & v_2'(x_0) & v_3'(x_0) \\ v_1''(x_0) & v_2''(x_0) & v_3''(x_0) \end{vmatrix} \neq 0, \tag{3.6}$$

we derive $c_1 = c_2 = c_3 = 0$. Hence, we can conclude that the set of functions is linearly independent.

Example 3. Given the set of functions: $\cos(x), \cos(2x), \cos(3x)$. Show that they are L.I. by th2 method (II). We calculate

$$\Delta(x) = \begin{vmatrix} \cos(x) & \cos(2x) & \cos(3x) \\ -\sin(x) & -2\sin(2x) & -3\sin(3x) \\ -\cos(x) & -4\cos(2x) & -9\cos(3x) \end{vmatrix}$$

and choose $x = \pi/4$. It is found that $\Delta(\pi/4)) = -8$. However, we have $W(0) = 0$. It is seen that one cannot conclude **linear dependency** from the fact that $\Delta(x_0) = 0$ at a point x_0.

Note that the condition (3.6) is just a **sufficient condition** for linear independency, but not a necessary condition. In other words, the condition (3.6) is a **necessary condition** for linear dependency, but not a sufficient condition. Hence, even though $\Delta(x) = 0$ for all $x \in (I)$, one still cannot conclude that the set of functions are linearly dependent.

To demonstrate this more clearly, let us reconsider the example 2. It is seen that the two functions are differentiable in the interval (I). Hence, the method (II)is applicable. It is derived that

- in the sub-interval $x \geq 0$, one has $v_1(x) = x^3, v_2(x) = x^3$, so that,

$$\Delta(x) = \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix} = 0;$$

- in the sub-interval $x < 0$, one has $v_1(x) = x^3, v_2(x) = -x^3$

$$\Delta(x) = \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix} = 0;$$

Therefore, we have $\Delta(x) = 0$ in the whole interval (I). However, these two function are linearly independent, not linearly dependent. The determinant (3.6) is called the Wronskian of functions $v_1(x), v_2(x), v_3(x)$.

Example 4. Given the set of functions: $\{1, \cos^2 x, \sin^2 x\}$. One can determine that these functions are linearly dependent, as $1 - \cos^2 x - \sin^2 x = 0$ for all $x \in (-\infty, \infty)$. On the other hand, one can shown that as the necessary condition, the wronskian of these functions is zero, for all $x \in (-\infty, \infty)$. Indeed, one can calculate that

$$\Delta(x) = \begin{vmatrix} 1 & \cos^2 x & \sin^2 x \\ 0 & -\sin(2x) & -\sin(2x) \\ 0 & -2\cos(2x) & -\cos(2x) \end{vmatrix} = 0.$$

4.2 Wronskian of n-functions

4.2.1 Definition

If $y_1, y_2, \ldots, y_n$ are n functions which have derivatives up to order $n-1$ then the Wronskian of these functions is the determinant

$$\begin{aligned} W &= W(y_1, y_2, \ldots, y_n) \\ &= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \end{aligned}$$

As is shown for the case $n = 3$, we may conclude that if $W(x_0) \neq 0$ for **some point** x_0, then $y_1, y_2, \ldots, y_n$ are linearly independent.

Note: As we have seen for the case $n = 3$, one cannot conclude linear dependence from the fact that $W(x_0) = 0$ at a point $x = x_0$.

However, it will be seen later that this is not the case if

$$y_1, y_2, \ldots, y_n$$

are solutions of an n-th order linear homogeneous ODE.

4.2.2 Theorem 1

The the Wronskian of n solutions of the n-th order linear ODE

$$L(y) = D^n + a_1(x)D^{n-1} + \cdots + a_n(x) = 0,$$

is subject to the following first order ODE:

$$\frac{dW}{dx} = -a_1(x)W,$$

with solution

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(t)dt}.$$

From the above it follows that the Wronskian of n solutions of the n-th order linear ODE $L(y) = 0$ is either identically zero or vanishes nowhere.

Proof: We prove this theorem for the case of the second order equation only. Let

$$L(y_i) = y_i'' + a_1(x)y_i' + a_2 y_i = 0, \quad (i = 1, 2). \tag{3.7}$$

Note that

$$\begin{aligned}
\frac{\mathrm{d}W}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\begin{vmatrix} y_1, y_2 \\ y_1', y_2' \end{vmatrix} = \begin{vmatrix} y_1', y_2' \\ y_1', y_2' \end{vmatrix} + \begin{vmatrix} y_1, y_2 \\ y_1'', y_2'' \end{vmatrix} \\
&= \begin{vmatrix} y_1, y_2 \\ y_1'', y_2'' \end{vmatrix} \\
&= \begin{vmatrix} y_1, & y_2 \\ -(a_1y_1' + a_2y_1), & -(a_1y_2' + a_2y_2) \end{vmatrix}.
\end{aligned}$$

This yields

$$W'(x) = -a_1(x)W(x), \tag{3.8}$$

so that,

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} a_1(t)dt}.$$

Notes: It is deduced from the above result that the sign of Wronskian $W(x)$ given by a set of solutions of linear homogeneous Eq. is unchangeable in the interval (I). Namely, for any given $x_0 \in (I)$,

- if $W(x_0) = 0$, then $W(x) = 0$ every where in (I);
- if $W(x_0) > 0$, then $W(x) > 0$ every where in (I);
- if $W(x_0) < 0$, then $W(x) < 0$ every where in (I).

In other words, if the sign of Wronskian $W(x)$ given by a set of function $\{y_1, y_2, \cdots, y_n\}$ changes on the interval (I), then this set of functions cannot be a set of solutions for any linear homogeneous Eq.

4.2.3 Theorem 2

If $y_1, y_2, \ldots, y_n$ are solutions of the linear ODE $L(y) = 0$, the following are equivalent:

1 $y_1, y_2, \ldots, y_n$ is a set of fundamental solutions, or a basis for the vector space $V = \ker(L)$;

2 $y_1, y_2, \ldots, y_n$ are linearly independent;

3 $^{(*)}$ $y_1, y_2, \ldots, y_n$ span V;

4 $y_1, y_2, \ldots, y_n$ generate $\ker(L)$;

5 $W(y_1, y_2, \ldots, y_n) \neq 0$ at some point x_0;

6 $W(y_1, y_2, \ldots, y_n)$ is never zero.

Proof. One may first show (2) $\to$ (6). It implies that if $W(y_1, y_2, \ldots, y_n) = 0$ at some point $x = x_0$, $\{y_1, y_2, \ldots, y_n\}$ must be linearly dependent. This can be done based on the theorem of existence and uniqueness of the trivial solution $y = \phi(x) = 0$ for the IVP imposed at $x = x_0$. To prove this, let us consider the solution

$$\phi(x) = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

where $\{c_1, c_2, \cdots, c_n\}$ are a certain set of constants to be determined. We let the values of the function $\phi(x)$ and its derivatives $\phi(i)(x)$ up to

$n-1$-th order evaluated at the point $x = x_0$ all equal to zero. Namely, we set

$$\begin{aligned}
&\phi(x_0) = c_1 y_1(x_0) + \cdots + c_n y_n(x_0) = 0\\
&\phi'(x_0) = c_1 y_1'(x_0) + \cdots + c_n y_n'(x_0) = 0\\
&\quad\vdots\\
&\phi^{(n-1)}(x_0) = c_1 y_1^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0) = 0.
\end{aligned}$$

This yields a system of linear algebraic equations for the constants $\{c_1, c_2, \cdots, c_n\}$. Since the Wronskian $W(x_0) = 0$, the above system would have a set of solution for $\{c_1, c_2, \cdots, c_n\}$, which are not all zero. As a consequence, with this set of constants, the function $\phi(x)$ is the solution of $L(x) = 0$ satisfying the zero initial conditions. However, on the other hand, the system allows the zero solution $y(x) = 0$, which also satisfies the zero initial conditions. From the fundamental theorem of existence and uniqueness, we deduce that these two solution are the same. Namely,

$$\phi(x) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \equiv 0,$$

for all $x \in (I)$. This implied $y_1, y_2, \ldots, y_n$ are linearly dependent.

We now show (2) → (1). We need to prove that a set of linear independent solutions $\{y_1, \cdots, y_n\}$ must be a set of fundamental solutions. It implies that for any given solution $z(x)$, there exists a set of constants,

$$\{c_1, c_2, \cdots, c_n\},$$

such that, the solution $z(x)$ can be expressed as

$$z(x) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

To show this, we denote that

$$\phi(x) = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n \tag{3.9}$$

where

$$\{c_1, c_2, \cdots, c_n\}$$

is a set of constants to be determined. From (2), one can deduce that (5) will be true. Hence, there must be some point x_0, such that the Wronskian $W[y_1, y_2, \ldots, y_n] = W(x_0) \neq 0$. With the formula (3.9), we let

$$\begin{aligned}
&z(x_0) = \alpha_1 = c_1 y_1(x_0) + \cdots + c_n y_n(x_0) = \phi(x_0)\\
&z'(x_0) = \alpha_2 = c_1 y_1'(x_0) + \cdots + c_n y_n'(x_0) = \phi'(x_0)\\
&\vdots\\
&z^{(n-1)}(x_0) = \alpha_n = c_1 y_1^{(n-1)}(x_0) + \cdots + c_n y_n^{(n-1)}(x_0)\\
&\qquad = \phi^{(n-1)}(x_0).
\end{aligned}$$

The above linear system may uniquely determine the constants:

$$\{c_1, c_2, \cdots, c_n\},$$

since its determinant

$$W(x_0) \neq 0.$$

As a consequence, the function $\phi(x)$ is fully determined. On the other hand, the above results also show that the two solutions $z(x)$ and $\phi(x)$ satisfy the same initial conditions at $x = x_0$. From the theorem of uniqueness of solution, it follows that

$$z(x) \equiv \phi(x), \quad \text{for all } x \in (I).$$

QED

4.2.4 The Solutions of $L[y] = 0$ as a Linear Vector Space

Suppose that $v_1(x), v_2(x), \cdots, v_n(x)$ are the linear independent solutions of linear equation $L[y] = 0$. Then any solution of this equation can be written in the form

$$v(x) = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

with $c_1, c_2, \ldots, c_n \in \mathcal{R}$. Especially The zero function $v(x) = 0$ is also solution. The all solutions of linear homogeneous equation form a vector space V with the basis

$$v_1(x), v_2(x), \cdots, v_n(x).$$

The vector space V consists of all possible linear combinations of the vectors:

$$\{v_1, v_2, \ldots, v_n\}.$$

5. Finding the Solutions in terms of the Method with Differential Operators

5.1 Solutions for Equations with Constants Coefficients

In what follows, we shall first focus on the linear equations with constant coefficients:

$$L(y) = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b(x)$$

and present two different approaches to solve them.

5.1.1 Basic Equalities (I).

We may prove the following basic identity of differential operators: for any scalar a,

$$\begin{aligned}(D-a) &= e^{ax}De^{-ax}\\ (D-a)^n &= e^{ax}D^ne^{-ax}\end{aligned} \tag{3.10}$$

where the factors e^{ax}, e^{-ax} are interpreted as linear operators. This identity is just the fact that

$$\frac{\mathrm{d}y}{\mathrm{d}x} - ay = e^{ax}\left(\frac{\mathrm{d}}{\mathrm{d}x}(e^{-ax}y)\right).$$

The formula (3.10) may be extensively used in solving the type of linear equations under discussion.

Note: equalities (3.10) are still applicable, if the parameter $a \in \mathcal{C}$, and operators acting on complex-valued function $y(x)$.

Let us first consider the homogeneous ODE

$$L(y) = P(D)y = 0. \tag{3.11}$$

Here

$$P(D) = (a_0D^{(n)} + a_1D^{(n-1)} + \cdots + a_n)$$

is a n-th order polynomial of operator D, which is associated with $P(D)$, one may write the **characteristic polynomial**:

$$\phi(r) = P(r) = (a_0r^n + a_1r^{n-1} + \cdots + a_n). \tag{3.12}$$

5.1.2 Cases (I)

The characteristic polynomial $\phi(r)$ has n distinct real roots: $r_i \neq r_j, (i \neq j)$. In this case, one may write

$$L(y) = P(D)y = (D-r_1)(D-r_2)\cdots(D-r_n)y = 0.$$

To find solution, one can first solve each factor equations:

$$(D-r_i)y_i = 0, \qquad (i = 1, 2, \cdots, n)$$

separately, which yields a set of independent, so is also fundamental solutions: $\{y_i(x) = \mathrm{e}^{r_ix}\}, (i = 1, 2, \cdots, n)$. The general solution can be written in the form:

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x).$$

5.1.3 Cases (II)

. The characteristic polynomial $\phi(r)$ has multiple roots with m multiplicity.

In this case, one may write

$$L(y) = P(D)y = (D - r_1)^m y = 0.$$

By applying the equalities (3.10), we derive

$$(D - r_1)^m y = (\mathrm{e}^{r_1 x} D^m \mathrm{e}^{-r_1 x})y = 0;$$

so that,

$$D^m(\mathrm{e}^{-r_1 x} y) = 0.$$

This yields

$$(\mathrm{e}^{-r_1 x} y) = a_0 + a_1 x + \cdots + a_{m-1} x^{m-1}),$$

and

$$y(x) = (a_0 + a_1 x + \cdots + a_{n-1} x^{m-1})\mathrm{e}^{r_1 x}.$$

With the above result, one may write

$$\ker\{(D - a)^m\} = \{(a_0 + a_1 x + \cdots + a_{m-1} x^{m-1})e^{ax} \mid a_0, a_1, \ldots, a_{m-1} \in \mathcal{R}\}.$$

5.1.4 Cases (III)

The characteristic polynomial $\phi(r)$ has a multiple complex root with m multiplicity. Note that the complex roots of polynomial $\phi(r)$ with real coefficients must be complex conjugate. So that the complex roots must be paired, as $r_{1,2} = \lambda \pm \mathrm{i}\mu$. For the case of $m = 1$, let r_1, r_2 be simple complex conjugate roots. One may write

$$L(y) = P(D)y = (D - r_1)(D - r_2)y = 0,$$

which yields two sets of the complex solutions:

$$\{\mathrm{e}^{r_1 x}, \mathrm{e}^{r_2 x}\}.$$

We recall the definition of the imaginary number i, from which we have

$$\mathrm{i}^2 = -1; \qquad \mathrm{i}^3 = -\mathrm{i}; \qquad \mathrm{i}^4 = 1; \qquad \mathrm{i}^5 = \mathrm{i}, \cdots.$$

On the other hand, from the Euler formula $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$, where $x \in (\mathcal{R})$, we may determine the complex exponential function of $z = z + \mathrm{i}y$:

$$\mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x (\cos y + \mathrm{i}\sin y). \tag{3.13}$$

Note that the Euler formula may be also derived form the definition of the complex exponential function with the Taylor series:

$$\begin{aligned} e^{ix} = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!} &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n!} \\ &+ i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\ &= \cos x + i \sin x. \end{aligned} \tag{3.14}$$

From the definition (3.13), it can be proved that the formulas

$$D(e^{rx}) = re^{rx}, \qquad D^n(e^{rx}) = r^n e^{rx}$$

are all applicable, when r is a complex number. So that, the basic equalities are now extended to the case with complex number r. Thus, we have the two complex solutions:

$$\begin{aligned} y_1(x) &= e^{r_1 x} = e^{\lambda x}(\cos \mu x + i \sin \mu x), \\ y_2(x) &= e^{r_2 x} = e^{\lambda x}(\cos \mu x - i \sin \mu x) \end{aligned}$$

with a proper combination of these two solutions, one may derive two real solutions:

$$\tilde{y}_1(x) = \frac{1}{2}[y_1(x) + y_2(x)] = e^{\lambda x} \cos \mu x,$$

and

$$\tilde{y}_2(x) = -\frac{1}{2} i [y_1(x) - y_2(x)] = e^{\lambda x} \sin \mu x$$

and the general solution:

$$y(x) = e^{\lambda x}(A \cos \mu x + B \sin \mu x).$$

For the case of multiple complex roots ($m > 1$), one may write

$$L(y) = P(D)y = (D - r_1)^m (D - r_2)^m y = 0.$$

To solve it, one may respectively solve the following two equations:

$$(D - r_1)^m y = 0, \quad (D - r_2)^m y = 0,$$

which yield two sets of the complex solutions:

$$\begin{aligned} &(a_0 + a_1 x + \cdots + a_{m-1} x^{m-1}) e^{r_1 x} \\ &(b_0 + b_1 x + \cdots + b_{m-1} x^{m-1}) e^{r_2 x}, \end{aligned}$$

where $a_i, b_i, (i = 0, \cdots, m-1)$ are arbitrary complex constants in the number field $(\mathcal{C})$.

The general complex solution can be written as

$$y(x) = (a_0 + \cdots + a_{m-1}x^{m-1})e^{\lambda x}(\cos\mu x + \mathrm{i}\sin\mu x), \\ +(b_0 + \cdots + b_{m-1}x^{m-1})e^{\lambda x}(\cos\mu x - \mathrm{i}\sin\mu x),$$

By taking the real part and imaginary part, one derive the general real solution in the form:

$$y(x) = e^{\lambda x}\Big[(\tilde{a}_0 + \tilde{a}_1 x + \cdots + \tilde{a}_{m-1}x^{m-1})\cos\mu x \\ +(\tilde{b}_0 + \tilde{b}_1 x + \cdots + \tilde{b}_{m-1}x^{m-1})\sin\mu x\Big],$$

where $(\tilde{a}_i, \tilde{b}_i), i = 0, \cdots, m-1$ are arbitrary real constants.

5.1.5 Summary

In summary, it can be proved that the following results hold:

- $$\ker\{(D-a)^m\} = \mathrm{span}(e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}).$$

  It means that DE
  $$\{(D-a)^m\}y = 0$$
  has a set of fundamental solutions:
  $$\Big\{e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}\Big\}$$

- $$\ker\{(D-\lambda)^2 + \mu^2)^m\} \\ = \mathrm{span}(e^{\lambda x}f(x), xe^{\lambda x}f(x), \cdots, x^{m-1}e^{\lambda x}f(x)),$$

  where $f(x) = \cos(\mu x)$ or $\sin(\mu x)$

  It means that DE
  $$\{((D-\lambda)^2 + \mu^2)^m\}y = 0$$
  has a set of fundamental solutions:
  $$\Big\{e^{\lambda x}f(x), xe^{\lambda x}f(x), \ldots, x^{m-1}e^{\lambda x}f(x)\Big\},$$

  where $f(x) = \cos(\mu x)$ or $\sin(\mu x)$.

5.1.6 Theorem 1

$$\ker\{P_n(D)Q_m(D)\} = \ker\{P_n(D)\} + \ker\{Q_m(D)\}$$
$$= \left\{ \{y_1\}; \{y_2\} \middle| \begin{array}{l} y_1 \in \ker\{P_n(D)\}, \\ y_2 \in \ker\{Q_m(D)\} \end{array} \right\},$$

provided $P_n(X), Q_m(X)$ are two polynomials with constant coefficients that **have no common roots** .

In other words, assume that DE

$$P_n(D)y = 0,$$

has the set of fundamental solutions:

$$\Big\{p_1(x), p_2(x), \cdots, p_n(x)\}$$

while DE

$$Q_m(D)y = 0,$$

has the set of fundamental solutions:

$$\Big\{q_1(x), q_2(x), \cdots q_m(x)\Big\}.$$

Then DE

$$P_n(D)Q_m(D)y = 0$$

has the set of fundamental solutions:

$$\Big\{p_1(x), \cdots, p_n(x); q_1(x), \cdots q_m(x)\Big\}.$$

Example 1. By using the differential operation method, one can easily solve some inhomogeneous equations. For instance, let us reconsider the example 1. One may write the DE

$$y'' + 2y' + y = x$$

in the operator form as

$$(D^2 + 2D + I)(y) = x.$$

The operator $(D^2 + 2D + I) = \phi(D)$ can be factored as $(D + I)^2$. With (3.10), we derive that

$$(D + I)^2 = (e^{-x}De^x)(e^{-x}De^x) = e^{-x}D^2e^x.$$

Consequently, the DE

$$(D + I)^2(y) = x$$

can be written

$$e^{-x}D^2e^x(y) = x$$

or

$$\frac{d^2}{dx^2}(e^x y) = xe^x$$

which on integrating twice gives

$$e^x y = xe^x - 2e^x + Ax + B,$$

$$y = x - 2 + Axe^{-x} + Be^{-x}.$$

We leave it to the reader to prove that the general solution of DE

$$(D - a)^n y = 0,$$

is

$$\{(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}\}e^{ax}$$

where

$$a_0, a_1, \ldots, a_{n-1} \in \mathcal{R}.$$

Example 2. Now consider the DE

$$y'' - 3y' + 2y = e^x.$$

In operator form this equation is

$$(D^2 - 3D + 2I)(y) = e^x.$$

Since $(D^2 - 3D + 2I) = (D - I)(D - 2I)$, this DE can be written

$$(D - I)(D - 2I)(y) = e^x.$$

Now let

$$z = (D - 2I)(y).$$

Then

$$(D - I)(z) = e^x, \quad (e^x De^{-x})z = e^x.$$

It has the solution

$$z = xe^x + Ae^x.$$

Now $z = (D - 2I)(y)$ is the linear first order DE

$$y' - 2y = xe^x + Ae^x$$

which has the solution

$$y = e^x - xe^x - Ae^x + Be^{2x}.$$

Notice that $-Ae^x + Be^{2x}$ is the general solution of the associated homogeneous DE

$$y'' - 3y' + 2y = 0$$

and that $e^x - xe^x$ is a particular solution of the original DE

$$y'' - 3y' + 2y = e^x.$$

Example 3. Consider the DE

$$y'' + 2y' + 5y = \sin(x)$$

which in operator form is $(D^2 + 2D + 5I)(y) = \sin(x)$. Now

$$D^2 + 2D + 5I = (D + I)^2 + 4I$$

and so the associated homogeneous DE has the general solution

$$Ae^{-x}\cos(2x) + Be^{-x}\sin(2x).$$

All that remains is to find a particular solution of the original DE. We leave it to the reader to show that there is a particular solution of the form $C\cos(x) + D\sin(x)$.

Example 4. Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = 0,$$

with I.C's:

$$y(0) = 1, y'(0) = y''(0) = 0.$$

The DE in operator form is

$$(D^3 - 3D^2 + 7D - 5)(y) = 0.$$

Since

$$\begin{aligned}\phi(r) = r^3 - 3r^2 + 7r - 5 &= (r-1)(r^2 - 2r + 5)\\ &= (r-1)[(r-1)^2 + 4],\end{aligned}$$

we have

$$\begin{aligned} L(y) &= (D^3 - 3D^2 + 7D - 5)(y)\\ &= (D-1)[(D-1)^2 + 4](y)\\ &= [(D-1)^2 + 4](D-1)(y)\\ &= 0. \end{aligned} \tag{3.15}$$

From here, it is seen that the solutions for

$$(D-1)(y) = 0, \tag{3.16}$$

namely,

$$y(x) = c_1 e^x, \tag{3.17}$$

and the solutions for

$$[(D-1)^2 + 4](y) = 0, \tag{3.18}$$

namely,

$$y(x) = c_2 e^x \cos(2x) + c_3 e^x \sin(2x), \tag{3.19}$$

must be the solutions for our equation (3.15). Thus, we derive that the following linear combination

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x), \tag{3.20}$$

must be the solutions for our equation (3.15). In solution (3.20), there are three arbitrary constants (c_1, c_2, c_3). One can prove that this solution is the general solution, which covers all possible solutions of (3.15). For instance, given the I.C.'s:

$$y(0) = 1, y'(0) = 0, y''(0) = 0,$$

from (3.20), we can derive

$$c_1 + c_2 = 1,$$

$$c_1 + c_2 + 2c_3 = 0,$$

$$c_1 - 3c_2 + 4c_3 = 0,$$

and find $c_1 = 5/4, c_2 = -1/4, c_3 = -1/2$.

6. Solutions for Equations with Variable Coefficients

In this lecture we will give a few techniques for solving certain linear differential equations with non-constant coefficients. We will mainly restrict our attention to second order equations. However, the techniques can be extended to higher order equations. The general second order linear DE is

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x).$$

This equation is called a non-constant coefficient equation if at least one of the functions p_i is not a constant function.

6.1 Euler Equations

An important example of a non-constant linear DE is Euler's equation

$$x^2y'' + axy' + by = 0, \tag{3.21}$$

where a, b are constants.

This equation has singularity at $x = 0$. The fundamental theorem of existence and uniqueness of solution holds in the region $x > 0$ and $x < 0$, respectively. So one must solve the problem in the region $x > 0$, or $x < 0$ separately.

Let $x = \pm \mathrm{e}^t, \quad t = \ln|x|$ and

$$y(x) = y(\pm \mathrm{e}^t) = \tilde{y}(t).$$

Then we derive

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \pm e^t \frac{\mathrm{d}y}{\mathrm{d}x}.$$

In operator form, we have

$$\frac{\mathrm{d}}{\mathrm{d}x} = \pm e^{-t}D, \quad x\frac{\mathrm{d}}{\mathrm{d}x} = D,$$

here we have set $D = \frac{\mathrm{d}}{\mathrm{d}t}$. From the above, we derive

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}x^2} &= (\pm 1)^2 e^{-t}De^{-t}D = e^{-2t}e^tDe^{-t}D \\ &= (\pm 1)^2 e^{-2t}(D-1)D,\end{aligned}$$

so that $x^2y'' = D(D-1)\tilde{y}$. By induction one easily proves that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} = (\pm 1)^n e^{-nt}D(D-1)\cdots(D-n+1)$$

so that,

$$x^n y^{(n)} = D(D-1)\cdots(D-n+1)(\tilde{y}).$$

With the variable t, Euler's equation becomes

$$\begin{aligned}&D(D-1)\tilde{y} + aD\tilde{y} + b\tilde{y} \\ &\quad = \frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} + (a-1)\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + b\tilde{y} = q(e^t) = \tilde{q}(t),\end{aligned}$$

which is a linear constant coefficient DE. Solving this for $\tilde{y}$ as a function of t and then making the change of variable $t = \ln(x)$, we obtain the solution of Euler's equation for y as a function of x.

For the region $x < 0$, we may let $-x = \mathrm{e}^t$, or $|x| = \mathrm{e}^t$. Then the equation

$$x^2y'' + axy' + by = 0, \quad (x < 0)$$

is changed to the same form

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} + (a-1)\frac{\mathrm{d}\tilde{y}}{\mathrm{d}t} + b\tilde{y} = 0. \tag{3.22}$$

Hence, we have the solution $y(x) = \tilde{y}(\ln|x|)$ $(x < 0)$.

The above equation with constant coefficients (3.22) can be solved as with the way demonstrated in the previous sections. Assume:

$$P(D) = (D - r_1)(D - r_2).$$

There are three cases:

Cases (I): real roots $r_1 \neq r_2 \Longrightarrow$ two linear independent solutions:

$$\tilde{y}(t) = \Big\{\mathrm{e}^{r_1 t}; \mathrm{e}^{r_2 t}\Big\}.$$

so that

$$y(x) = \Big\{|x|^{r_1}; |x|^{r_2}\Big\}.$$

Cases (II): real roots $r_1 = r_2 \Longrightarrow$ two linear independent solutions:

$$\tilde{y}(t) = \Big\{\mathrm{e}^{r_1 t}; t\mathrm{e}^{r_1 t}\Big\}.$$

so that

$$y(x) = \Big\{|x|^{r_1}; |x|^{r_1}\ln|x|\Big\}.$$

Cases (III): complex roots $r_{1,2} = \lambda \pm \mu$, $\Longrightarrow$ two linear independent complex solutions:

$$\tilde{y}_{1,2}(t) = \Big\{\mathrm{e}^{r_{1,2}t} = \mathrm{e}^{\lambda t}\mathrm{e}^{\pm\mu t}\Big\};$$

or two linear independent real solutions:

$$\tilde{y}_{1,2}(t) = \mathrm{e}^{\lambda t}\Big\{\cos(\mu t); \sin(\mu t)\Big\}.$$

Returning to variable x, we have

$$y_{1,2}(x) = |x|^{\lambda}\Big\{\cos(\mu\ln|x|); \sin(\mu\ln|x|)\Big\};$$

The above approach, can extend to solve the n-th order Euler equation

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = q(x), \tag{3.23}$$

where $a_1, a_2, \ldots a_n$ are constants. Then we may reduce the original Eq. to the Eq. with constant coefficients:

$$P(D)\{\tilde{y}(t)\} = 0,$$

where

$$\begin{aligned} P(D) = {} & D(D-1)\cdots(D-n+1) \\ & + D(D-1)\cdots(D-n+2)a_1 + \cdots \\ & + D(D-1)a_{n-2} + Da_{n-1} + a_n. \end{aligned}$$

Example 1. Solve $x^2y'' + xy' + y = \ln(x)$, $(x > 0)$.

Making the change of variable $x = e^t$ we obtain

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} + \tilde{y} = t$$

whose general solution is $\tilde{y} = A\cos(t) + B\sin(t) + t$. Hence

$$y = A\cos(\ln(x)) + B\sin(\ln(x)) + \ln(x)$$

is the general solution of the given DE.

Example 2. Solve $x^3y''' + 2x^2y'' + xy' - y = 0, \quad (x > 0)$.

This is a third order Euler equation. Making the change of variable $x = e^t$, we get

$$\begin{aligned} & \Big\{(D(D-1)(D-2) + 2D(D-1) + (D-1)\Big\}(\tilde{y}) \\ & = (D-1)(D^2+1)(\tilde{y}) = 0 \end{aligned}$$

which has the general solution $\tilde{y} = c_1e^t + c_2\sin(t) + c_3\cos(t)$. Hence

$$y = c_1x + c_2\sin(\ln(x)) + c_3\cos(\ln(x))$$

is the general solution of the given DE.

7. Finding the Solutions in terms of the Method with Undetermined Parameters

In what follows, we are going to describe another method different from the method of differential operators.

7.1 Solutions for Equations with Constants Coefficients

To illustrate the idea, as a special case, let us first consider the 2-nd order Linear equation with the constant coefficients:

$$L(y) = ay'' + by' + cy = f(x). \tag{3.24}$$

The associate homogeneous equation is:

$$L(y) = ay'' + by' + cy = 0. \tag{3.25}$$

7.2 Basic Equalities (II)

We first give the following basic identities:

$$D(\mathrm{e}^{rx}) = r\mathrm{e}^{rx};\ D^2(\mathrm{e}^{rx}) = r^2\mathrm{e}^{rx};\ \cdots\ D^n(\mathrm{e}^{rx}) = r^n\mathrm{e}^{rx}. \tag{3.26}$$

To solve this equation, we assume that the solution is in the form $y(x) = \mathrm{e}^{rx}$, where r is a constant to be determined. Due to the properties of the exponential function e^{rx}:

$$y'(x) = ry(x);\ y''(x) = r^2y(x);\ \cdots\ y^{(n)} = r^ny(x), \tag{3.27}$$

we can write

$$L(\mathrm{e}^{rx}) = \phi(r)\mathrm{e}^{rx}. \tag{3.28}$$

for any given (r, x), where

$$\phi(r) = ar^2 + br + c.$$

is called the characteristic polynomial. From (3.28) it is seen that the function e^{rx} satisfies the equation (3.24), namely

$$L(\mathrm{e}^{rx}) = 0,$$

as long as the constant r is the root of the characteristic polynomial, i.e.

$$\phi(r) = 0.$$

In general, the polynomial $\phi(r)$ has two roots (r_1, r_2): One can write

$$\phi(r) = ar^2 + br + c = a(r - r_1)(r - r_2).$$

Accordingly, the equation (3.25) has two solutions:

$$\{y_1(x) = \mathrm{e}^{r_1x};\ y_2(x) = \mathrm{e}^{r_2x}\}.$$

Three cases should be discussed separately.

7.3 Cases (I) ($r_1 > r_2$)

When $b^2 - 4ac > 0$, the polynomial $\phi(r)$ has two distinct real roots $(r_1 \neq r_2)$.

In this case, the two solutions, $y_(x); y_2(x)$ are different. The following linear combination is not only solution, but also the **general solution** of the equation:

$$y(x) = Ay_1(x) + By_2(x), \tag{3.29}$$

where A, B are arbitrary constants. To prove that, we make use of the fundamental theorem which states that if y, z are two solutions such that

$$y(0) = z(0) = y_0$$

and

$$y'(0) = z'(0) = y'_0$$

then $y = z$. Let y be any solution and consider the linear equations in A, B

$$\begin{aligned} Ay_1(0) + By_2(0) &= y(0), \\ Ay'_1(0) + By'_2(0) &= y'(0), \end{aligned} \tag{3.30}$$

or

$$\begin{aligned} A + B \quad &= y_0, \\ Ar_1 + Br_2 &= y'_0. \end{aligned} \tag{3.31}$$

Due to $r_1 \neq r_2$, these conditions leads to the unique solution A, B. With this choice of A, B the solution

$$z = Ay_1 + By_2$$

satisfies

$$z(0) = y(0), \quad z'(0) = y'(0)$$

and hence $y = z$. Thus, (3.37) contains all possible solutions of the equation, so, it is indeed the general solution.

7.4 Cases (II) ($r_1 = r_2$)

When $b^2 - 4ac = 0$, the polynomial $\phi(r)$ has double root: $r_1 = r_2 = \frac{-b}{2a}$. In this case, the solution

$$y_1(x) = y_2(x) = \mathrm{e}^{r_1 x}.$$

Thus, for the general solution, one needs to derive another type of *the second solution*. For this purpose, one may use the **method of reduction of order**.

Let us look for a solution of the form $C(x)e^{r_1 x}$ with the undetermined function $C(x)$. By substituting the equation, we derive that

$$L\Big(C(x)e^{r_1 x}\Big) = C(x)\phi(r_1)e^{r_1 x} \\ +a\Big[C''(x) + 2r_1 C'(x)\Big]e^{r_1 x} \\ +bC'(x)e^{r_1 x} = 0.$$

Noting that

$$\phi(r_1) = 0; \quad 2ar_1 + b = 0,$$

we get

$$C''(x) = 0$$

or

$$C(x) = Ax + B,$$

where A, B are arbitrary constants. Thus, the solution:

$$y(x) = (Ax + B)\mathrm{e}^{r_1 x}, \tag{3.32}$$

is a two parameter family of solutions consisting of the linear combinations of the two solutions:

$$y_1 = e^{r_1 x}, \quad y_2 = xe^{r_1 x}.$$

It is also the general solution of the equation. The proof is similar to that given for the case (I) based on the fundamental theorem of existence and uniqueness.

Another approach to treat this case is as follows: Let us consider the case of equation of order n. Then we have the identity:

$$L[\mathrm{e}^{rx}] = \phi(r)\mathrm{e}^{rx}, \tag{3.33}$$

which is valid for all $x \in I$ and arbitrary r. We suppose that the polynomial $\phi(r)$ has multiple root $r = r_1$ with multiplicity m. Hence, we have

$$\phi(r) = (r - r_1)^m R(r).$$

It is seen that

$$\phi(r_1) = \phi'(r_1) = \phi''(r_1) = \cdots = \phi^{(m-1)}(r_1) = 0.$$

We now make derivative with respect to r on the both sides of (3.33), it follows that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r}L\left[\mathrm{e}^{rx}\right] = L\left[\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{e}^{rx}\right] = L\left[x\mathrm{e}^{rx}\right] \\ &= \phi'(r)\mathrm{e}^{rx} + x\phi(r)\mathrm{e}^{rx}. \end{aligned} \tag{3.34}$$

Let $r = r_1$ in (3.33) and (3.34), it follows that

$$L\left[\mathrm{e}^{r_1 x}\right] = L\left[x\mathrm{e}^{r_1 x}\right] = 0. \tag{3.35}$$

One may make derivatives on (3.33) up to the order of $m-1$. As a consequence, we obtain the linear independent solutions:

$$\{\mathrm{e}^{r_1 x}, x\mathrm{e}^{r_1 x}, x^2\mathrm{e}^{r_1 x}, \cdots, x^{m-1}\mathrm{e}^{r_1 x}\}.$$

Example 1. Consider the linear DE $y'' + 2y' + y = x$. Here

$$L(y) = y'' + 2y' + y.$$

- A particular solution of the DE $L(y) = x$ is

$$y_p = x - 2.$$

- The associated homogeneous equation is

$$y'' + 2y' + y = 0.$$

  The characteristic polynomial

$$\phi(r) = r^2 + 2r + 1 = (r+1)^2$$

  has double roots $r_1 = r_2 = -1$.

Thus the general solution of the DE

$$y'' + 2y' + y = x$$

is

$$y = Axe^{-x} + Be^{-x} + x - 2.$$

This equation can be solved quite simply without the use of the fundamental theorem if we make essential use of operators.

7.5 Cases (III) ($r_{1,2} = \lambda \pm \mathrm{i}\mu$)

When $b^2 - 4ac < 0$, the polynomial $\phi(r)$ has two conjugate complex roots $r_{1,2} = \lambda \pm \mathrm{i}\mu$. Noting the basic equalities (II) can be extended to the case with complex number r, we may write the two complex solutions:

$$y_1(x) = \mathrm{e}^{r_1 x} = \mathrm{e}^{\lambda x}(\cos\mu x + \mathrm{i}\sin\mu x),$$
$$y_2(x) = \mathrm{e}^{r_2 x} = \mathrm{e}^{\lambda x}(\cos\mu x - \mathrm{i}\sin\mu x)$$

with a proper combination of these two solutions, one may derive two real solutions:

$$\tilde{y}_1(x) = \frac{1}{2}[y_1(x) + y_2(x)] = \mathrm{e}^{\lambda x}\cos\mu x,$$

and

$$\tilde{y}_2(x) = -\frac{1}{2}\mathrm{i}[y_1(x) - y_2(x)] = \mathrm{e}^{\lambda x}\sin\mu x$$

and the general solution:

$$y(x) = \mathrm{e}^{\lambda x}(A\cos\mu x + B\sin\mu x).$$

7.6 Solutions for Euler Equations

The idea of the undetermined constant method can be applied to solve the Euler equations

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = q(x), \tag{3.36}$$

defined earlier. In doing so, we need to assume that the solutions of Euler Eq. has the form: $y(x) = x^r$ and apply the following properties of the power function.

7.7 Basic Equalities (III)

$$\begin{cases} xy' = rx^r, \\ x^2 y'' = r(r-1)x^r, \\ x^3 y''' = r(r-1)(r-2)x^r, \\ \vdots, \end{cases}$$

By substituting the above into equation (3.36), we derive

$$L[y(x)] = \phi(r)x^r = 0, \quad \text{for all } x \in (I),$$

where

$$\phi(r) = P(r)$$

is called **the characteristic polynomial**. It is derived that if $r = r_0$ is a root of $\phi(r)$, such that $\phi(r_0) = 0$, then $y(x) = x^{r_0}$ must be a solution of the Euler equation.

To demonstrate better, let us consider the special case ($n = 2$):

$$L[y] = x^2y'' + axy' + by = 0,$$

where a, b are constants. In this case, we have the characteristic polynomial $\phi(r) = r(r-1) + ar + b = r^2 + (a-1)r + b$, which in general has two roots $r = (r_1, r_2)$. Accordingly, the equation has two solutions:

$$\{y_1(x) = x^{r_1};\ y_2(x) = x^{r_2}\}.$$

Similar to the problem for the equations with constant coefficients, there are three cases to be discussed separately.

7.8 Cases (I) ($r_1 \neq r_2$)

When $(a-1)^2 - 4b > 0$, the polynomial $\phi(r)$ has two distinct real roots ($r_1 \neq r_2$). In this case, the two solutions, $y_1(x); y_2(x)$ are linear independent. The **general solution** of the equation is:

$$y(x) = Ay_1(x) + By_2(x), \tag{3.37}$$

where A, B are arbitrary constants.

7.9 Cases (II) ($r_1 = r_2$)

When $(a-1)^2 - 4b = 0$, the polynomial $\phi(r)$ has a double root $r_1 = r_2 = \frac{1-a}{2}$. In this case, the solution

$$y_1(x) = y_2(x) = x^{r_1}.$$

Thus, for the general solution, one may derive *the second solution* by using the **method of reduction of order**.

Let us look for a solution of the form $C(x)y_1(x)$ with the undetermined function $C(x)$. By substituting the equation, we derive that

$$\begin{aligned} L\Big[C(x)y_1(x)\Big] &= C(x)\phi(r_1)y_1(x) + C''(x)x^2y_1(x) \\ &+C'(x)\Big[2x^2y_1'(x) + axy_1(x)\Big] = 0. \end{aligned}$$

Noting that

$$\phi(r_1) = 0; \quad 2r_1 + a = 1,$$

we get

$$xC''(x) + C'(x) = 0$$

or $C'(x) = \frac{A}{x}$, and

$$C(x) = A\ln|x| + B,$$

where A, B are arbitrary constants. Thus, the solution:

$$y(x) = (A\ln|x| + B)x^{\frac{1-a}{2}}. \tag{3.38}$$

is a two parameter family of solutions consisting of the linear combinations of the two solutions:

$$y_1 = x^{r_1}, \quad y_2 = x^{r_1}\ln|x|.$$

7.10 Cases (III) ($r_{1,2} = \lambda \pm i\mu$)

When $(a-1)^2 - 4b < 0$, the polynomial $\phi(r)$ has two conjugate complex roots $r_{1,2} = \lambda \pm i\mu$. Thus, we have the two complex solutions:

$$\begin{aligned} y_1(x) &= x^{r_1} = e^{r_1 \ln|x|} \\ &= e^{\lambda \ln|x|}[\cos(\mu\ln|x|) + i\sin(\mu\ln|x|)] \\ &= |x|^{\lambda}[\cos(\mu\ln|x|) + i\sin(\mu\ln|x|)], \\ y_2(x) &= x^{r_2} = e^{r_2 \ln|x|} \\ &= e^{\lambda \ln|x|}[\cos(\mu\ln|x|) - i\sin(\mu\ln|x|)] \\ &= |x|^{\lambda}[\cos(\mu\ln|x|) - i\sin(\mu\ln|x|)]. \end{aligned}$$

With a proper combination of these two solutions, one may derive two real solutions:

$$\tilde{y}_1(x) = \frac{1}{2}[y_1(x) + y_2(x)] = |x|^{\lambda}\cos(\mu\ln|x|),$$

and

$$\tilde{y}_2(x) = -\frac{1}{2}i[y_1(x) - y_2(x)] = |x|^{\lambda}\sin(\mu\ln|x|).$$

and the general solution:

$$|x|^{\lambda}[A\cos(\mu\ln|x|) + B\sin(\mu\ln|x|)].$$

We obtain the same results as what we derived by using the method of differential operator.

Example 1. Solve $x^2y'' + xy' + y = 0, \ (x > 0)$.

The characteristic polynomial for the equation is

$$\phi(r) = r(r-1) + r + 1 = r^2 + 1.$$

whose roots are $r_{1,2} = \pm\mathrm{i}$. The general solution of DE is

$$y = A\cos(\ln x) + B\sin(\ln x).$$

Example 2. Solve $x^3y''' + 2x^2y'' + xy' - y = 0, \quad (x > 0)$.

This is a third order Euler equation. Its characteristic polynomial is

$$\begin{aligned}\phi(r) &= r(r-1)(r-2) + 2r(r-1) + r - 1\\ &= (r-1)[r(r-2) + 2r + 1]\\ &= (r-1)(r^2+1).\end{aligned}$$

whose roots are $r_{1,2,3} = (1, \pm\mathrm{i})$. The general solution od DE:

$$y = c_1x + c_2\sin(\ln|x|) + c_3\cos(\ln|x|).$$

Example 1. Solve $x^2y'' + xy' + y = \ln(x)$, $(x > 0)$.

Making the change of variable $x = e^t$ we obtain

$$\frac{\mathrm{d}^2\tilde{y}}{\mathrm{d}t^2} + \tilde{y} = t$$

whose general solution is $\tilde{y} = A\cos(t) + B\sin(t) + t$. Hence

$$y = A\cos(\ln(x)) + B\sin(\ln(x)) + \ln(x)$$

is the general solution of the given DE.

Example 2. Solve $x^3y''' + 2x^2y'' + xy' - y = 0, \quad (x > 0)$.

This is a third order Euler equation. Making the change of variable $x = e^t$, we get

$$\begin{aligned}&\Big\{(D(D-1)(D-2) + 2D(D-1) + (D-1)\Big\}(\tilde{y})\\ &= (D-1)(D^2+1)(\tilde{y}) = 0\end{aligned}$$

which has the general solution $\tilde{y} = c_1e^t + c_2\sin(t) + c_3\cos(t)$. Hence

$$y = c_1x + c_2\sin(\ln(x)) + c_3\cos(\ln(x))$$

is the general solution of the given DE.

7.11 (*) Exact Equations

The DE $p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$ is said to be exact if

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = \frac{\mathrm{d}}{\mathrm{d}x}(A(x)y' + B(x)y).$$

In this case the given DE is reduced to solving the linear DE

$$A(x)y' + B(x)y = \int q(x)dx + C$$

a linear first order DE. The exactness condition can be expressed in operator form as

$$p_0 D^2 + p_1 D + p_2 = D(AD + B).$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}x}(A(x)y' + B(x)y) = A(x)y'' + (A'(x) \\ + B(x))y' + B'(x)y,$$

the exactness condition holds if and only if $A(x), B(x)$ satisfy

$$A(x) = p_0(x), \quad B(x) = p_1(x) - p_0'(x), \quad B'(x) = p_2(x).$$

Since the last condition holds if and only if

$$p_1'(x) - p_0''(x) = p_2(x),$$

we see that the given DE is exact if and only if

$$p_0'' - p_1' + p_2 = 0$$

in which case

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = \\ \tfrac{\mathrm{d}}{\mathrm{d}x}[p_0(x)y' + (p_1(x) - p_0'(x))y].$$

Example 3. Solve the DE $xy'' + xy' + y = x, \quad (x > 0)$.

This is an exact equation since the given DE can be written

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy' + (x-1)y) = x.$$

Integrating both sides, we get

$$xy' + (x-1)y = x^2/2 + A$$

which is a linear DE. The solution of this DE is left as an exercise.

8. Finding a Particular Solution for Inhomogeneous Equation

In this lecture we shall discuss the methods for producing a particular solution of a special kind for the general linear DE.

In what follows we shall use the operator method to find a particular solution of inhomogeneous equations.

8.1 The Annihilator and the Method of Undetermined Constants

We first consider the following type of inhomogeneous Eq.'s:

$$\begin{aligned} L(y) &= P(D)(y) \\ &= (a_0 D^{(n)} + a_1 D^{(n-1)} + \cdots + a_n D^0)y = b(x). \end{aligned} \tag{3.39}$$

Here, we assume that $L = P(D)$ is a polynomial of differential operator; while the inhomogeneous term $b(x)$ has an **annihilator**. The so-called annihilator of function $b(x)$ is defined as a polynomial of deferential operator $Q(D)$, such that $Q(D)(b(x)) = 0$; or say, the inhomogeneous term $b(x)$ is a solution of linear equation $Q(D)\{y(x))\} = 0$.

It can be shown that if the inhomogeneous term $b(x)$ has an annihilator as $Q(D)$, then one can transform the inhomogeneous equation (3.42) to the homogeneous equation. For this purpose, one may apply the differential operator $Q(D)$ to its both sides of (3.42). It follows that

$$\Phi(D)(y) = Q(D)P(D)(y) = 0.$$

The above method is also called the *Annihilator Method*.

Example 1. Solve the initial value problem

$$y''' - 3y'' + 7y' - 5y = x + e^x,$$
$$y(0) = 1, y'(0) = y''(0) = 0.$$

This DE is non-homogeneous. The associated homogeneous equation was solved in the previous lecture. Note that in this example, In the inhomogeneous term $b(x) = x + e^x$ is in the kernel of $Q(D) = D^2(D-1)$. Hence, we have

$$D^2(D-1)^2[(D-1)^2 + 4](y) = 0$$

which yields

$$y = Ax + B + Cxe^x + c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x).$$

This shows that there is a particular solution of the form

$$y_P = Ax + B + Cxe^x \tag{3.40}$$

which is obtained by discarding the terms:

$$y_H(x) = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x)$$

in the solution space of the associated homogeneous DE.

Now the remaining part of problem is to determine the arbitrary constants A, B, C in the form of particular solution (3.40).

In doing so, substituting (3.40) in the original DE we get

$$y''' - 3y'' + 7y' - 5y = 7A - 5B - 5Ax - Ce^x$$

which is equal to $x + e^x$ if and only if

$$7A - 5B = 0, \quad -5A = 1, \quad -C = 1.$$

So that $A = -1/5$, $B = -7/25$, $C = -1$. Hence the general solution is

$$y = c_1 e^x + c_2 e^x \cos(2x) + c_3 e^x \sin(2x) - x/5 - 7/25 - xe^x.$$

To satisfy the initial condition $y(0) = 0, y'(0) = y''(0) = 0$ we must have

$$\begin{aligned} c_1 + c_2 \quad &= 32/25, \\ c_1 + c_2 + 2c_3 &= 6/5, \\ c_1 - 3c_2 + 4c_3 &= 2 \end{aligned} \tag{3.41}$$

which has the solution

$$c_1 = 3/2, c_2 = -11/50, c_3 = -1/25.$$

It is evident that if the function $b(x)$ can not be annihilated by any linear operator $Q(D)$, the annihilator method will not applicable.

8.1.1 The Annihilators for Some Types of Functions

The annihilators $Q(D)$ for some types of functions $F(x)$ are as follows:

- For the functions

$$F(x) = a_0 + a_1 x + \cdots + a_k x^k,$$

we have the annihilator:

$$Q(D) = D^{k+1}.$$

- For the functions

$$F(x) = (a_0 + a_1 x + \cdots + a_k x^k)e^{ax},$$

we have the annihilator:

$$Q(D) = (D - a)^{k+1}.$$

- For the functions

$$F(x) = e^{\lambda x}\Big[a_0 \cos \mu x + b_0 \sin \mu x\Big],$$

we have the annihilator:

$$Q(D) = [(D - \lambda)^2 + \mu^2].$$

- For the functions

$$F(x) = (a_0 + a_1 x + \cdots + a_k x^k)e^{\lambda x} \cos \mu x,$$

or

$$F(x) = (b_0 + b_1 x + \cdots + b_k x^k)e^{\lambda x} \sin \mu x,$$

we have the annihilator:

$$Q(D) = [(D - \lambda)^2 + \mu^2]^{k+1}.$$

Example 1: Given the functions:

$$\begin{array}{ll} (1) & F_1(x) = x\cos(x) \\ (2) & F_2(x) = x^2 e^x \sin(3x) \\ (3) & F_3(x) = 2x\sin(2x) + 3e^{2x}\cos(5x). \end{array}$$

The annihilators are:

$$\begin{array}{ll} (1) & Q_1(D) = (D^2 + 1)^2 \\ (2) & Q_2(D) = [(D - 1)^2 + 9]^3 \\ (3) & Q_3(D) = (D^2 + 4)^2[(D - 2)^2 + 25]. \end{array}$$

8.1.2 Basic Properties of the Annihilators

The annihilator of function $b(x)$ has the following properties:

1 Assume that $Q(D)$ is the annihilator of both $b_1(x)$ and $b_2(x)$. Then $Q(D)$ will be also an annihilator of $b(x) = c_1 b_1(x) + c_2 b_2(x)$.

2 Assume that $Q(D)$ is the annihilator of both $b(x)$. Then $Q(D) \circ R(D)$ must be also an annihilator of $b(x)$, where $R(D)$ is an arbitrary

polynomial of differential operator. It is seen that a given function will have infinitely many annihilators, if it has an annihilator. It is evident that among these annihilators, there is one annihilator that is the differential operator polynomial of smallest degree.

3 Assume that $b(x) = b_1(x) + b_2(x)$. let $Q_1(D)$ be the annihilator of $b_1(x)$, while $Q_2(D)$ be the annihilator of $b_2(x)$. Then $Q(D) = Q_1(D) \circ Q_2(D)$ is the annihilator of $b(x)$.

Proof: Proofs of these properties are very simple. For instance, to prove the property 3, we may derive that since

$$Q_1(D)[b_1(x)] = Q_2(D)[b_2(x)] = 0,$$

it follows that

$$Q_1(D) \circ Q_2(D)[b_1(x) + b_2(x)] = 0.$$

8.1.3 The Basic Theorems of the Annihilator Method

Theorem 1 Give differential equation:

$$P(D)[y(x)] = b(x), \tag{3.42}$$

and let $Q(D)$ be the annihilator of $b(x)$. Assume that The operators $P(D)$ and $Q(D)$ have **no common root**. Then the Eq. has the following form of particular solution:

$$y_p(x) = \ker\{Q(D)\}$$

Proof: Since

$$Q(D)P(D)[y(x)] = 0, \tag{3.43}$$

and $P(D)$ and $Q(D)$ have no common root, we may first solve the following two Eq.'s:

$$P(D)[y_1(x)] = 0;$$
$$Q(D)[y_2(x)] = 0.$$

and write the general solution of the above Eq.(3.43) as

$$y(x) = \ker\{P(D)\} + \ker\{Q(D)\}.$$

However, it is noticed that $P(D)[y_{\mathrm{H}}(x)] = 0$, or

$$y_{\mathrm{H}} = \ker\{P(D)\}.$$

It is derived that we has the following form of particular solution of Eq. (3.42):

$$y_p(x) = y(x) - y_{\mathrm{H}}(x) = \ker\{Q(D)\}.$$

Theorem 2 Give differential equation:

$$P(D)[y(x)] = b(x) = x^k \mathrm{e}^{ax}, \tag{3.44}$$

Assume that $P(D) = \hat{P}(D)(D-a)^m$, where $\hat{P}(a) \neq 0$ and $m \geq 0$. Then the Eq. (3.44) allows a particular solution in the form:

$$y_p(x) = x^m \ker\{Q(D)\}.$$

where $Q(D) = (D-a)^{k+1}$ is the annihilator of $b(x)$.
Proof: Applying the annihilator on both sides of equation, we get homogeneous equation:

$$P(D)Q(D)y = \hat{P}(D)(D-a)^m(D-a)^{k+1}y = 0.$$

We may write

$$\hat{P}(D)(D-a)^{m+k+1}y = 0. \tag{3.45}$$

Since $\hat{P}(D)$ and $(D-a)^{m+k+1}$ have no common factor, (3.45) has the general solution:

$$\begin{aligned} y(x) = & \Big\{[(c_0 + c_1 x + \cdots c_{m-1}x^{m-1}) \\ & + c_m x^m + \cdots + c_{m+k}x^{m+k}]\mathrm{e}^{ax}\Big\} \\ & + \ker\{\hat{P}(D)\}. \end{aligned}$$

On the other hand, the general solution of the associated homogeneous Eq. can be written as

$$y_{\mathrm{H}}(x) = \ker\{P(D)\} + \Big\{[c_0 + c_1 x + \cdots + c_{m-1}x^{m-1}]\mathrm{e}^{ax}\Big\}.$$

Therefore we derive that a particular solution is in the form:

$$y_p = y(x) - y_{\mathrm{H}}(x) = [c_m x^m + \cdots + c_{m+k}x^{m+k}]\mathrm{e}^{ax},$$

or in the form:

$$y_p = x^m[A_0 + \cdots + A_k x^k]\mathrm{e}^{ax},$$

where

$$[A_0 + \cdots + A_k x^k]\mathrm{e}^{ax} = \ker\{Q(D)\}.$$

The next step is to determine the undermined constants

$$\{A_0, \cdots, A_k\}$$

by substituting y_p to the original inhomogeneous equation:

$$P(D) = x^k e^{ax}.$$

In terms of this theorem, one may directly determine the form of particular solution $y_p(x)$ by skipping the intermediate steps.

Example 2: Find a particular solution for the following equation:

$$\begin{aligned} y'' + 2y' + 2y &= b(t) = b_1(x) + b_2(x) \\ &= 3e^{-t} + 2e^{-t}\cos t + 4e^{-t} t \sin t, \end{aligned} \tag{3.46}$$

Solution: We have

$$P(D)y_H = 0; \qquad P(D) = D^2 + 2D + 2 = [(D+1)^2 + 1].$$

and

$$\begin{aligned} &Q_1(D) = D,\ Q_2(D) = [(D+1)^2 \\ &\Longrightarrow Q(D) = (D+1) \circ [(D+1)^2 + 1]^2. \end{aligned}$$

Then

$$\begin{aligned} &P(D)Q(D)y \\ &\quad = [(D+1)^2 + 1] \cdot (D+1) \cdot [(D+1)^2 + 1]^2 y \\ &\quad = (D+1) \cdot [(D+1)^2 + 1]^3 y = 0. \end{aligned}$$

We solve

$$\begin{aligned} y(t) = Ae^{-t} &+ [b_3 + B_2 t + B_1 t^2]e^{-t}\cos t \\ &+ [d_3 + D_2 t + D_1 t^2]e^{-t}\sin t. \end{aligned}$$

On the other hand, we have

$$y_H = b_3 e^{-t}\cos t + d_3 e^{-t}\sin t.$$

Hence, we have

$$y(t) = y_H + y_P,$$

where

$$\begin{aligned} y_P &= Ae^{-t} + [B_2 t + B_1 t^2]e^{-t}\cos t \\ &\quad + [D_2 t + D_1 t^2]e^{-t}\sin t \\ &= Ae^{-t} + t[B_2 + B_1 t]e^{-t}\cos t \\ &\quad + t[D_2 + D_1 t]e^{-t}\sin t. \end{aligned}$$

However, in terms of the theorem 1-2, since in this case, $m = 1$, we may directly write

$$y_p(x) = \ker\{Q_1(D)\} + t\ker\{Q_2(D)\}.$$

By substituting it into the equation:

$$P(D)y_P = b(t),$$

one may determine the constants: $\{A, B_2, B_1, D_2, D_1\}$.

Example 3: Find a particular solution for the following equation:

$$\begin{aligned} y'' - 4y' + 4y &= b_1(t) + b_2(t) + b_3(t) \\ &= 2t^2 + 4te^{2t} + t\sin 2t. \end{aligned} \tag{3.47}$$

Solution: We have

$$P(D)y_H = 0; \qquad P(D) = D^2 - 4D + 4 = (D-2)^2.$$

and

$$\begin{aligned} &Q_1(D) = D^3,\ Q_2(D) = (D-2)^2,\ Q_3(D) = (D^2+4)^2 \\ &\Longrightarrow \\ &Q(D) = D^3 \circ (D-2)^2 \circ (D^2+4)^2. \end{aligned}$$

Moreover, we may let let $y(t) = y_1(t) + y_2(t) + y_3(t)$ and separate the Eq. (3.46) into three inhomogeneous Eq.' as follows:

$$P(D)[y_1] = b_1(t), \tag{3.48}$$
$$P(D)[y_2] = b_2(t), \tag{3.49}$$
$$P(D)[y_3] = b_3(t). \tag{3.50}$$

Since $P(D)$ has no common root with $Q_1(D)$ and $Q_3(D)$, according to the Theorem 1, we derive that the particular solution:

$$y_{1p}(t) \in \ker\{Q_1(D)\}; \quad y_{3p}(t) \in \ker\{Q_3(D)\}.$$

It follows that

$$\begin{aligned} y_{1p}(t) &= (A_0t^2 + A_1t + A_2); \\ y_{3p}(t) &= [D_0t + D_1)\sin 2t + (E_0t + E_1)\cos 2t. \end{aligned}$$

On the other hand, $P(D)$ has a common root $D = 2$ with $Q_2(D)$. According to the Theorem 2, we have $m = 2, k = 1$, and

$$y_{2p}(t) \in t^2\ker\{Q_2(D)\}.$$

Hence, we have

$$y_{2p}(t) = t^2(B_1 + B_0t)e^{2t}. \tag{3.51}$$

Adding up these three solutions, we obtain

$$y_p(t) = y_{1p}(t) + y_{2p}(t) + y_{3p}(t),$$

or

$$y_p(t) = \ker\{Q_1\} + t^2\ker\{Q_2\} + \ker\{Q_3\}.$$

As the next step, we need to determine the constants $(A_0, A_1, A_2; B_0, B_1; D_0, D_1, D_2)$ by respectively substituting the above solutions y_{1p}, y_{2p}, y_{3p} into the Eq.'s (3.48), then comparing the both sides of Eq.', It is derived that

$$\begin{aligned}
&A_0 = 1/2, A_1 = 1, A_2 = 3/4\\
&B_0 = 2/3, B_1 = 0;\\
&D_0 = 0, D_1 = 1/16\\
&E_0 = 1/8, E_1 = 1/16.
\end{aligned}$$

Note: The Theorem 2 is still applicable, if the parameter $a = (\lambda + \mathrm{i}\mu) \in \mathcal{C}$. In this case it yields a complex solution $y_p(x)$.
Theorem 3 Give differential equation:

$$P(D)y = b(x), \quad x \in \mathcal{R},$$

with with complex inhomogeneous term $b(x) = f(x) + \mathrm{i}g(x)$. Then the complex solution of the above equation can be written as $y(x) = u(x) + \mathrm{i}v(x)$, where $\{u, v\}$ are real solutions subject to the following equations, respectively:

$$P(D)u = f(x), \quad P(D)v = g(x) \quad x \in \mathcal{R}.$$

Thus, to solve the equation:

$$P(D)u = x^k \mathrm{e}^{\lambda x} \cos(\mu x), \text{ or, } P(D)v = x^k \mathrm{e}^{\lambda x} \sin(\mu x),$$

One may first solve the extended complex equation:

$$P(D)y = x^k \mathrm{e}^{ax},$$

where $a = (\lambda + \mathrm{i}\mu)$. This can be done by aid of Theorem 2.

Example 4: Solve

$$y'' - 2y' + 5y = 8\mathrm{e}^x \cos 2x.$$

Let $b(x) = 8\mathrm{e}^{(1+2\mathrm{i})x}$, $y(x) = \Re\{z(x)\}$ and consider

$$z'' - 2z' + 5z = b(x) = 8\mathrm{e}^{(1+2\mathrm{i})x}.$$

Here $P(D) = D^2 - 2D + 5 = (D-1)^2 + 4$, $Q(D) = D - a$. As $a = 1 + 2\mathrm{i}$ and $P(a) = 0$, $m = 1$, we derive

$$z_p = Ax\mathrm{e}^{ax}.$$

To determine A, we calculate:

$$\begin{aligned} z_p' &= A(ax+1)e^{ax}, \\ z_p'' &= Ae^{ax}[a(ax+1)+a]. \end{aligned}$$

By substituting the above into EQ, we derive

$$Ae^{ax}a(ax+2) - 2A(ax+1)e^{ax} + 5xAe^{ax} = 8e^{ax}$$

or

$$A[a(ax+2) - 2(ax+1) + 5x] = 8$$

which leads to

$$A(a^2 - 2a + 5) = 0, \quad 2aA - 2A = 8.$$

Hence, we derive $A = \frac{2}{a-2} = \frac{2}{i} = -2i$, and

$$z_p = -2ie^x(\cos 2x + i \sin 2x).$$

From the Theorem 2, we finally derive

$$y(x) = \Re\{z(x)\} = 2xe^x \sin 2x.$$

8.2 The Method of Variation of Parameters

The method of Variation of parameters is for producing a particular solution of a special kind for the general linear DE in normal form

$$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x)$$

from a fundamental set $\{y_1, y_2, \ldots, y_n\}$ of solutions of the associated homogeneous equation.

In this method we try for a solution of the form

$$y_P = C_1(x)y_1 + C_2(x)y_2 + \cdots + C_n(x)y_n.$$

Then

$$\begin{aligned} y_P' = {} & C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n' \\ & + C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n \end{aligned}$$

and we impose the condition

$$C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n = 0.$$

Then

$$y_P' = C_1(x)y_1' + C_2(x)y_2' + \cdots + C_n(x)y_n'$$

and hence

$$\begin{aligned} y_P'' = &C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'' \\ &+C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n'. \end{aligned}$$

Again we impose the condition

$$C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' = 0$$

so that

$$y_P'' = C_1(x)y_1'' + C_2(x)y_2'' + \cdots + C_n(x)y_n'.$$

We do this for the first $n-1$ derivatives of y, so that

for $1 \le k \le n-1$, we have

$$y_P^{(k)} = C_1(x)y_1^{(k)} + C_2(x)y_2^{(k)} + \cdots C_n(x)y_n^{(k)},$$

with the condition:

$$C_1'(x)y_1^{(k-1)} + C_2'(x)y_2^{(k-1)} + \cdots + C_n'(x)y_n^{(k-1)} = 0.$$

Finally, as $k = n$, we have

$$\begin{aligned} y_P^{(n)} = &C_1(x)y_1^{(n)} + C_2(x)y_2^{(n)} + \cdots C_n(x)y_n^{(n)} \\ &+C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)}. \end{aligned}$$

Now, by substituting $y_P, y_P', \ldots, y_P^{(n)}$ in Eq. $L(y) = b(x)$, we get

$$\begin{aligned} &C_1(x)L(y_1) + C_2(x)L(y_2) + \cdots + C_n(x)L(y_n) \\ &+C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} \\ &= b(x). \end{aligned}$$

However, $L(y_i) = 0$ for $1 \le k \le n$. So that, we have

$$C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x).$$

We thus obtain the system of n linear equations for

$$\left\{C_1'(x), \ldots, C_n'(x)\right\}.$$

$$\begin{aligned} &C_1'(x)y_1 + C_2'(x)y_2 + \cdots + C_n'(x)y_n = 0, \\ &C_1'(x)y_1' + C_2'(x)y_2' + \cdots + C_n'(x)y_n' = 0, \\ &\vdots \\ &C_1'(x)y_1^{(n-1)} + C_2'(x)y_2^{(n-1)} + \cdots + C_n'(x)y_n^{(n-1)} = b(x). \end{aligned}$$

One can solve this linear algebraic system by the Cramer's rule. Suppose that

$$(A)\vec{x} = \vec{b},$$

where

$$\begin{cases} (A) = \left(\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\right) \\ \vec{a}_i = [a_{i1}, a_{i2}, \cdots, a_{in}]^T \\ \vec{x} = [x_1, x_2, \cdots, x_n]^T \\ \vec{b} = [b_1, b_2, \cdots, b_n]^T \end{cases}$$

Then we have the solution:

$$x_j = \frac{\det\left(\vec{a}_1, \cdots, \vec{a}_{j-1}, \vec{b}, \vec{a}_{j+1}, \cdots, \vec{a}_n\right)}{\det\left(\vec{a}_1, \cdots, \vec{a}_{j-1}, \vec{a}_j, \vec{a}_{j+1}, \cdots, \vec{a}_n\right)},$$
$$n = 1, 2, \cdots.$$

For our case, we have

$$\vec{a}_i = [y_i, y_i', \cdots, y_i^{n-1}]^T = \vec{y}_i, \quad (i = 1, 2, \cdots)$$

$$\vec{b} = [0, 0, \cdots, b(x)]^T$$

hence, we solve

$$C_j'(x) = \frac{\det\left(\vec{y}_1, \cdots, \vec{y}_{j-1}, \vec{b}, \vec{y}_{j+1}, \cdots, \vec{y}_n\right)}{\det\left(\vec{y}_1, \cdots, \vec{y}_{j-1}, \vec{y}_j, \vec{y}_{j+1}, \cdots, \vec{y}_n\right)},$$
$$(n = 1, 2, \cdots).$$

Note that we can write

$$\det\left(\vec{y}_1, \cdots, \vec{y}_{j-1}, \vec{y}_j, \vec{y}_{j+1}, \cdots, \vec{y}_n\right) = W(y_1, y_2, \ldots y_n),$$

where $W(y_1, y_2, \ldots y_n)$ is the Wronskian. Moreover, in terms of **the cofactor expansion theory**, we can expand the determinant

$$\det\left(\vec{y}_1, \cdots, \vec{y}_{j-1}, \vec{b}, \vec{y}_{j+1}, \cdots, \vec{y}_n\right)$$

along the column j. Since $\vec{b}$ only has one non-zero element, it follows that

$$\det\left(\vec{y}_1, \cdots, \vec{y}_{j-1}, \vec{b}, \vec{y}_{j+1}, \cdots, \vec{y}_n\right)$$
$$= (-1)^{n+j} b(x) W_i.$$

Here we have use the notation:

$$W_i = W(y_1, \cdots, y_{j-1}, \hat{y}_j, j_{j+1}, \cdots, y_n),$$

where the $\hat{y}_i$ means that y_i is omitted from the Wronskian W.

Thus, we finally obtain we find

$$C_i'(x) = (-1)^{n+i} b(t) \frac{W_i}{W} \mathrm{d}t$$

and, after integration,

$$C_i(x) = \int_{x_0}^{x} (-1)^{n+i} b(t) \frac{W_i}{W} \mathrm{d}t.$$

Note that the particular solution y_P found in this way satisfies

$$y_P(x_0) = y_P'(x_0) = \cdots = y_P^{(n-1)} = 0.$$

The point x_0 is any point in the interval of continuity of the $a_i(x)$ and $b(x)$. Note that y_P is a linear function of the function $b(x)$.

Example 2. Find the general solution of $y'' + y = 1/x$ on $x > 0$.
The general solution of $y'' + y = 0$ is

$$y = c_1 \cos(x) + c_2 \sin(x).$$

Using variation of parameters with

$$y_1 = \cos(x), \quad y_2 = \sin(x), \quad b(x) = 1/x$$

and $x_0 = 1$, we have

$$W = 1, \quad W_1 = \sin(x), \quad W_2 = \cos(x)$$

and we obtain the particular solution

$$y_p = C_1(x)\cos(x) + C_2(x)\sin(x)$$

where

$$C_1(x) = -\int_1^x \frac{\sin(t)}{t} dt, \quad C_2(x) = \int_1^x \frac{\cos(t)}{t} dt.$$

The general solution of $y'' + y = 1/x$ on $x > 0$ is therefore

$$\begin{aligned} y &= c_1 \cos(x) + c_2 \sin(x) \\ &\quad - \left(\int_1^x \tfrac{\sin(t)}{t} dt\right)\cos(x) + \left(\int_1^x \tfrac{\cos(t)}{t} dt\right)\sin(x). \end{aligned}$$

When applicable, the annihilator method is easier as one can see from the DE

$$y'' + y = e^x.$$

With

$$(D-1)e^x = 0.$$

it is immediate that $y_p = e^x/2$ is a particular solution while variation of parameters gives

$$\begin{aligned} y_p = &-\left(\int_0^x e^t \sin(t)dt\right)\cos(x) \\ &+\left(\int_0^x e^t \cos(t)dt\right)\sin(x). \end{aligned} \tag{3.52}$$

The integrals can be evaluated using integration by parts:

$$\begin{aligned} \int_0^x e^t \cos(t)dt &= e^x \cos(x) - 1 + \int_0^x e^t \sin(t)dt \\ &= e^x \cos(x) + e^x \sin(x) - 1 \\ &- \int_0^x e^t \cos(t)dt \end{aligned}$$

which gives

$$\int_0^x e^t \cos(t)dt = \Big[e^x \cos(x) + e^x \sin(x) - 1\Big]/2$$

$$\begin{aligned} \int_0^x e^t \sin(t)dt &= e^x \sin(x) \\ &- \int_0^x e^t \cos(t)dt \\ = &\quad \Big[e^x \sin(x) - e^x \cos(x) + 1\Big]/2 \end{aligned}$$

so that after simplification

$$y_p = e^x/2 - \cos(x)/2 - \sin(x)/2.$$

8.3 Reduction of Order

If y_1 is a non-zero solution of a homogeneous linear n-th order DE $L[y] = 0$, one can always find a new solution of the form

$$y = C(x)y_1$$

for the inhomogeneous DE $L[y] = b(x)$, where $C'(x)$ satisfies a homogeneous linear DE of order $n-1$. Since we can choose $C'(x) \neq 0$, the new solution $y_2 = C(x)y_1$ that we find in this way is not a scalar multiple of y_1.

In particular for $n = 2$, we obtain a fundamental set of solutions y_1, y_2. Let us prove this for the second order DE

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0.$$

If y_1 is a non-zero solution we try for a solution of the form

$$y = C(x)y_1.$$

Substituting $y = C(x)y_1$ in the above we get

$$\begin{aligned} &p_0(x)\Big(C''(x)y_1 + 2C'(x)y_1' + C(x)y_1''\Big) \\ &+p_1(x)\Big(C'(x)y_1 + C(x)y_1'\Big) + p_2(x)C(x)y_1 = 0. \end{aligned}$$

Simplifying, we get

$$p_0y_1C''(x) + (2p_0y_1' + p_1y_1)C'(x) = 0$$

since

$$p_0y_1'' + p_1y_1' + p_2y_1 = 0.$$

This is a linear first order homogeneous DE for $C'(x)$. Note that to solve it we must work on an interval where

$$y_1(x) \neq 0.$$

However, the solution found can always be extended to the places where $y_1(x) = 0$ in a unique way by the fundamental theorem.

The above procedure can also be used to find a particular solution of the non-homogenous DE

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = q(x)$$

from a non-zero solution of

$$p_0(x)y'' + p_1(x)y' + p_2(x)y = 0.$$

Example 4. Solve $y'' + xy' - y = 0$.

Here $y = x$ is a solution so we try for a solution of the form $y = C(x)x$. Substituting in the given DE, we get

$$C''(x)x + 2C'(x) + x(C'(x)x + C(x)) - C(x)x = 0$$

which simplifies to

$$xC''(x) + (x^2 + 2)C'(x) = 0.$$

Solving this linear DE for $C'(x)$, we get

$$C'(x) = Ae^{-x^2/2}/x^2$$

so that

$$C(x) = A\int \frac{dx}{x^2 e^{x^2/2}} + B$$

Hence the general solution of the given DE is

$$y = A_1 x + A_2 x \int \frac{dx}{x^2 e^{x^2/2}}.$$

Example 5. Solve $y'' + xy' - y = x^3 e^x$.

By the previous example, the general solution of the associated homogeneous equation is

$$y_H = A_1 x + A_2 x \int \frac{dx}{x^2 e^{x^2/2}}.$$

Substituting $y_p = xC(x)$ in the given DE we get

$$C''(x) + (x + x/2)C'(x) = x^2 e^x.$$

Solving for $C'(x)$ we obtain

$$\begin{aligned} C'(x) &= \tfrac{1}{x^2 e^{x^2/2}} \left(A_2 + \textstyle\int x^4 \mathrm{e}^{x+x^2/2} \mathrm{d}x\right) \\ &= A_2 \tfrac{1}{x^2 e^{x^2/2}} + H(x), \end{aligned}$$

where

$$H(x) = \frac{1}{x^2 e^{x^2/2}} \int x^4 \mathrm{e}^{x+x^2/2} \mathrm{d}x.$$

This gives

$$C(x) = A_1 + A_2 \int \frac{dx}{x^2 e^{x^2/2}} + \int H(x)\mathrm{d}x,$$

We can therefore take

$$y_p = x\int H(x)\mathrm{d}x,$$

so that the general solution of the given DE is

$$y = A_1 x + A_2 x \int \frac{dx}{x^2 e^{x^2/2}} + y_p(x) = y_H(x) + y_p(x).$$

Chapter 4

LAPLACE TRANSFORMS

1. Introduction

We begin our study of the Laplace Transform with a motivating example: Solve the differential equation

$$y'' + y = f(t) = \begin{cases} 0, & 0 \le t < 10, \\ 1, & 10 \le t < 10 + 2\pi, \\ 0, & 10 + 2\pi \le t. \end{cases}$$

with IC's:

$$y(0) = 0, \quad y'(0) = 0 \tag{4.1}$$

Here, $f(t)$ is **piecewise continuous and any solution would also have y'' piecewise continuous**. To describe the motion of the ball

using techniques previously developed we have to divide the problem into three parts:

- (I) $0 \le t < 10$;
- (II) $10 \le t < 10 + 2\pi$;
- (III) $10 + 2\pi \le t$.

(I). The initial value problem determining the motion in part I is

$$y'' + y = 0, \quad y(0) = y'(0) = 0.$$

The solution is

$$y(t) = 0, \; 0 \le t < 10.$$

Taking limits as $t \to 10$ from the left, we find

$$y(10) = y'(10) = 0.$$

(II). The initial value problem determining the motion in part II is

$$y'' + y = 1, \quad y(10) = y'(10) = 0.$$

The solution is

$$y(t) = 1 - \cos(t - 10),\ 10 \le t < 2\pi + 10.$$

Taking limits as $t \to 10 + 2\pi$ from the left, we get

$$y(10 + 2\pi) = y'(10 + 2\pi) = 0.$$

(III). The initial value problem for the last part is

$$y'' + y = 0, \quad y(10 + 2\pi) = y'(10 + 2\pi) = 0$$

which has the solution

$$y(t) = 0,\ 10 + 2\pi \le t < \infty.$$

Putting all this together, we have

$$y(t) = \begin{cases} 0, \quad 0 \le t < 10, \\ 1 - \cos(t - 10), \quad 10 \le t < 10 + 2\pi, \\ 0, \quad 10 + 2\pi \le t. \end{cases}$$

The function $y(t)$ is continuous with continuous derivative

$$y'(t) = \begin{cases} 0, \quad 0 \le t < 10, \\ \sin(t - 10), \quad 10 \le t < 10 + 2\pi, \\ 0, \quad 10 + 2\pi \le t. \end{cases}$$

However the function $y'(t)$ is not differentiable at $t = 10$ and $t = 10+2\pi$. In fact

$$y''(t) = \left\{ \begin{array}{l} 0, \quad 0 \le t < 10, \\ \cos(t - 10), \quad 10 < t < 10 + 2\pi, \\ 0, \quad 10 + 2\pi < t. \end{array} \right\}$$

The left-hand and right-hand limits of $y''(t)$ at $t = 10$ are 0 and 1 respectively. At $t = 10 + 2\pi$ they are 1 and 0.

By a solution we mean any function $y = y(t)$ satisfying the DE for those t not equal to the points of discontinuity of $f(t)$. In this case we have shown that a solution exists with $y(t)$, $y'(t)$ continuous. In the same way, one can show that in general such solutions exist using the fundamental theorem.

We now want to describe is an approach for doing such problems without having to divide the problem into parts. **Such an approach is the Laplace transform.**

2. Laplace Transform

2.1 Definition

2.1.1 Piecewise Continuous Function

Let $f(t)$ be a function defined for $t \geq 0$. The function $f(t)$ is said to be **piecewise continuous**, if

- (1) $f(t)$ converges to a finite limit $f(0+)$ as $t \to 0+$
- (2) for any $c > 0$, the left and right hand limits $f(c-)$, $f(c+)$ of $f(t)$ at c exist and are finite.
- (3) $f(c-) = f(c+) = f(c)$ for every $c > 0$ except possibly a finite set of points or an infinite sequence of points converging to $+\infty$.

Thus the only points of discontinuity of $f(t)$ are jump discontinuities. The function is said to be **normalized** if $f(c) = f(c+)$ for every $c \geq 0$.

2.1.2 Laplace Transform

The Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ of given function $f(x)$ is the function of a new variable s defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt = \lim_{N \to +\infty} \int_0^N e^{-st} f(t)dt, \qquad (4.2)$$

provided that the improper integral exists.

2.2 Existence of Laplace Transform

For the function of exponential order, its Laplace transform exists.

- The function $f(t)$ is said to be of *exponential order*, if there are constants a, M such that
$$|f(t)| \leq Me^{at}$$
for all t.
- *very large of class* of functions is of exponential order. For instance, the solutions of constant coefficient homogeneous DE's are all of exponential order.

- For a function $f(t)$ of **exponential order**, the convergence of the improper integral in (4.2) is guaranteed. In fact, from

$$\begin{aligned}\int_0^N |e^{-st} f(t)| dt &\leq M \int_0^N e^{-(s-a)t} dt \\ &= M \left\{ \frac{1}{s-a} - \frac{e^{-(s-a)N}}{s-a} \right\} \end{aligned} \tag{4.3}$$

which shows that the improper integral converges absolutely when $s > a$. It shows that $|F(s)| < \frac{M}{s-a}$ as $N \to \infty$.

3. Basic Properties and Formulas of Laplace Transform

3.1 Linearity of Laplace Transform

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.$$

3.2 Laplace Transforms for $f(t) = \mathrm{e}^{at}$

The calculation (4.3) shows that

$$\mathcal{L}\{\mathrm{e}^{at}\} = \frac{1}{s-a}$$

for $s > a$. Setting $a = 0$, we get $\mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

The above holds when $f(t)$ is complex valued and $a = \lambda + \mathrm{i}\mu$, $s = \sigma + i\tau$ are complex numbers. The integral exists in this case for $\Re\{s\} > \Re\{a\}$.
For example, this yields

$$\mathcal{L}\{\mathrm{e}^{it}\} = \frac{1}{s-i}, \quad \mathcal{L}\{e^{-it}\} = \frac{1}{s+i}.$$

3.3 Laplace Transforms for $f(t) = \{\sin(bt)$ and $\cos(bt)\}$

Using the linearity property of the Laplace transform

$$\mathcal{L}\{af(t) + bf(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\},$$

by using

$$\begin{cases} \sin(t) = (e^{it} - e^{-it})/2i, \\ \cos(t) = (e^{it} + e^{-it})/2, \end{cases}$$

we find

$$\mathcal{L}\{\sin(bt)\} = \frac{1}{2i}\Big(\frac{1}{s-bi} - \frac{1}{s+bi}\Big) = \frac{b}{s^2+b^2},$$

and

$$\mathcal{L}\{\cos(bt)\} = \frac{1}{2}\Big(\frac{1}{s-bi} + \frac{1}{s+bi}\Big) = \frac{s}{s^2+b^2},$$

for $s > 0$.

3.4 Laplace Transforms for $\{e^{at}f(t); f(bt)\}$

From the definition of the Laplace transform, we derive the following two identities after a change of variable.

$$\mathcal{L}\{e^{at}f(t)\}(s) = \mathcal{L}\{f(t)\}(s-a) = F(s-a),$$
$$\mathcal{L}\{f(bt)\}(s) = \tfrac{1}{b}\mathcal{L}\{f(t)\}(s/b) = \tfrac{1}{b}F(s/b).$$

Using the first of these formulas, we get

$$\mathcal{L}\{e^{at}\sin(bt)\} = \tfrac{b}{(s-a)^2+b^2},$$
$$\mathcal{L}\{e^{at}\cos(bt)\} = \tfrac{s-a}{(s-a)^2+b^2}.$$

3.5 Laplace Transforms for $\{t^n f(t)\}$

From the definition of the Laplace transform,

$$F(s) = \mathcal{L}\{f(t)\}(s) = \int_0^\infty e^{-st}f(t)dt, \tag{4.4}$$

by taking the derivative with respect to s inside the integral, we can derive:

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}(s).$$

For $n = 1$, one can verify the formula directly by differentiating with respect s on the definition of the Laplace transform, whereas the general case follows by induction.

For example, using this formula, we obtain using $f(t) = 1$

$$\mathcal{L}\{t^n\}(s) = (-1)^n \frac{d^n}{ds^n}\frac{1}{s} = \frac{n!}{s^{n+1}}.$$

With $f(t) = \sin(t)$ and $f(t) = \cos(t)$ we get

$$\mathcal{L}\{t\sin(bt)\}(s) = -\frac{d}{ds}\frac{b}{s^2+b^2} = \frac{2bs}{(s^2+b^2)^2},$$

$$\mathcal{L}\{t\cos(bt)\}(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s^2+b^2} = \frac{s^2-b^2}{(s^2+b^2)^2}$$

$$= \frac{1}{s^2+b^2} - \frac{2b^2}{(s^2+b^2)^2}.$$

3.6 Laplace Transforms for $\{f'(t)\}$

The following formula shows how to compute the Laplace transform of $f'(t)$ in terms of the Laplace transform of $f(t)$:

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f(t)\}(s) - f(0).$$

This follows from

$$\mathcal{L}\{f'(t)\}(s) = \int_0^\infty e^{-st} f'(t)dt$$

$$= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt$$

$$= s\int_0^\infty e^{-st} f(t)dt - f(0), \tag{4.5}$$

since $e^{-st}f(t)$ converges to 0 as $t \to +\infty$ in the domain of definition of the Laplace transform of $f(t)$. To ensure that the first integral is defined, we have to assume $f'(t)$ is piecewise continuous. Repeated applications of this formula give

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f(t)\}(s) - s^{n-1}f(0)$$
$$-s^{n-2}f'(0) - \cdots - f^{n-1}(0).$$

The following theorem is important for the application of the Laplace transform to differential equations.

4. Inverse Laplace Transform

4.1 Theorem 1:

If $f(t)$, $g(t)$ are normalized piecewise continuous functions of exponential order then

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$$

implies

$$f = g.$$

4.2 Theorem 2:

The necessary condition for a function $F(s)$ having the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ is

$$\lim_{\Re\{s\}\to\infty} F(s) = 0.$$

4.3 Definition

If $F(s)$ is the Laplace of the normalized piecewise continuous function $f(t)$ of exponential order then $f(t)$ is called the **inverse Laplace transform** of $F(s)$. This is denoted by

$$f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

Note that the inverse Laplace transform is also linear. Using the Laplace transforms we found for $t\sin(bt)$, $t\cos(bt)$ we find

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^2}\right\} = \frac{1}{2b}t\sin(bt),$$

and

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^2}\right\} = \frac{1}{2b^3}\sin(bt) - \frac{1}{2b^2}t\cos(bt).$$

5. Solve IVP of DE's with Laplace Transform Method

In this lecture we will, by using examples, show how to use Laplace transforms in solving differential equations with constant coefficients.

5.1 Example 1

Consider the initial value problem

$$y'' + y' + y = \sin(t), \quad y(0) = 1,\ y'(0) = -1.$$

Step 1

Let $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\begin{aligned}
\mathcal{L}\{y'(t)\} &= sY(s) - y(0) = sY(s) - 1, \\
\mathcal{L}\{y''(t)\} &= s^2Y(s) - sy(0) - y'(0) \\
&= s^2Y(s) - s + 1.
\end{aligned}$$

Taking Laplace transforms of the DE, we get

$$(s^2+s+1)Y(s) - s = \frac{1}{s^2+1}.$$

Step 2

Solving for $Y(s)$, we get

$$Y(s) = \frac{s}{s^2+s+1} + \frac{1}{(s^2+s+1)(s^2+1)}.$$

Step 3

Finding the inverse laplace transform.

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\tfrac{s}{s^2+s+1}\right\} + \mathcal{L}^{-1}\left\{\tfrac{1}{(s^2+s+1)(s^2+1)}\right\}.$$

Since

$$\frac{s}{s^2+s+1} = \frac{s}{(s+1/2)^2+3/4} = \frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2}$$

we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+s+1}\right\} = e^{-t/2}\cos(\sqrt{3}\ t/2) - \frac{1}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\ t/2). \qquad (4.6)$$

Using **partial fractions** we have

$$\frac{1}{(s^2+s+1)(s^2+1)} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2+1}.$$

Multiplying both sides by $(s^2+1)(s^2+s+1)$ and collecting terms, we find

$$1 = (A+C)s^3 + (B+C+D)s^2 + (A+C+D)s + B + D.$$

Equating coefficients, we get $A+C=0$, $B+C+D=0$, $A+C+D=0$, $B+D=1$,

from which we get $A=B=1$, $C=-1$, $D=0$, so that

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+s+1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{3}}e^{-t/2}\sin(\sqrt{3}\,t/2),$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

we obtain

$$y(t) = 2e^{-t/2}\cos(\sqrt{3}\,t/2) - \cos(t).$$

5.2 Example 2

As we have known, a higher order DE can be reduced to a system of DE's. Let us consider the system

$$\begin{aligned}\frac{dx}{dt} &= -2x + y,\\ \frac{dy}{dt} &= x - 2y\end{aligned} \tag{4.7}$$

or

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix};$$

with the initial conditions $x(0) = x_0 = 1$, $y(0) = y_0 = 2$. One may rewrite the system in the form of matrix: $\mathbf{x}'(\mathbf{t}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad \mathbf{x} = (x, y)^T.$$

Step 1

Taking Laplace transforms the system becomes

$$\begin{aligned} sX(s) - 1 &= -2X(s) + Y(s),\\ sY(s) - 2 &= X(s) - 2Y(s), \end{aligned} \tag{4.8}$$

where $X(s) = \mathcal{L}\{x(t)\}$, $Y(s) = \mathcal{L}\{y(t)\}$.

Step 2

Solving for $X(s)$, $Y(s)$. The above linear system of equations can be written in the form:

$$\begin{bmatrix} -2-s & 1 \\ 1 & -2-s \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = -\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

or

$$\begin{aligned} (s+2)X(s) - Y(s) &= x_0 = 1, \\ -X(s) + (s+2)Y(s) &= y_0 = 2. \end{aligned} \tag{4.9}$$

Note that for any given matrix $A(a_{i,j})$, the determinant of the matrix $\Delta(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial; the root of $\Delta(\lambda)$ is called the eigenvalue of the matrix A. In our case, the determinant of the coefficient matrix of the above system of linear equations is just the characteristic polynomial of A:

$$\Delta(s) = -\det(A - sI) = s^2 + 4s + 3 = (s+1)(s+3),$$

whose two roots are the eigenvalues of A: $s_{1,2} = -1, -3$. Using Cramer's rule we get

$$\begin{aligned} X(s) &= \frac{(s+2)x_0 + y_0}{(s-s_1)(s-s_2)} = \frac{(s+4)}{(s+1)(s+3)} \\ Y(s) &= \frac{(s+2)y_0 + x_0}{(s-s_1)(s-s_2)} = \frac{2s+5}{(s+1)(s+3)}. \end{aligned}$$

Step 3

Finding the inverse Laplace transform. By making the partial fraction, we derive

$$\frac{s+4}{(s+1)(s+3)} = \frac{3/2}{s+1} - \frac{1/2}{s+3}, \tag{4.10}$$

$$\frac{2s+5}{(s+1)(s+3)} = \frac{3/2}{s+1} + \frac{1/2}{s+3}, \tag{4.11}$$

we obtain

$$x(t) = \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y(t) = \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

Similarly, one can apply the Laplace method solve more general system of Eq's with $n \times n$ constant coefficients matrix. In the future, it will be demonstrated that there is another approach to solve the system of Eq's based on the eigenvectors of matrix.

It is seen from the above examples that the Laplace transform often reduces the solution of differential equations to a **partial fractions calculation.** If

$$F(s) = P(s)/Q(s)$$

is a ratio of polynomials with the degree of $P(s)$ less than the degree of $Q(s)$ then $F(s)$ can be written as a sum of terms each of which corresponds to an irreducible factor of $Q(s)$. Each factor $Q(s)$ of the form $s-a$ contributes the terms

$$\frac{A_1}{s-a} + \frac{A_1}{(s-a)^2} + \cdots + \frac{A_r}{(s-a)^r}$$

where r is the multiplicity of the factor $s-a$. Each irreducible quadratic factor s^2+as+b contributes the terms

$$\frac{A_1 s + B_1}{s^2+as+b} + \frac{A_2 s + B_2}{(s^2+as+b)^2} + \cdots + \frac{A_r s + B_r}{(s^2+as+b)^r}$$

where r is the degree of multiplicity of the factor s^2+as+b.

5.3 Example 3

Consider the initial value problem

$$y'' + 2ty' - 4y = 9, \quad y(0) = 3,\ y'(0) = 0.$$

Step 1

Let $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\begin{aligned}
\mathcal{L}\{ty'(t)\} &= -\frac{\mathrm{d}}{\mathrm{d}s}[sY(s) - y(0)] = -Y(s) - sY'(s), \\
\mathcal{L}\{y''(t)\} &= s^2Y(s) - sy(0) - y'(0) \\
&= s^2Y(s) - 3s.
\end{aligned}$$

Taking Laplace transforms of the DE, we get

$$-2sY'(s) + (s^2-6)Y(s) - 3s = \frac{9}{s}.$$

or

$$Y'(s) - \frac{(s^2-6)}{2s}Y(s) = -\frac{9}{2s^2} - \frac{3}{2}.$$

Step 2

With the integral factor: $s^3 \mathrm{e}^{-\frac{s^2}{4}}$, the General solution of the above ODE is obtained as

$$Y(s) = A\frac{\mathrm{e}^{\frac{s^2}{4}}}{s^3} + \frac{3}{s} + \frac{21}{s^3}$$

From the definition of Laplace Transform, one may assume that

$$\lim_{s\to\infty} Y(s) = 0.$$

It follows that the arbitrary constant must vanish. $A = 0$. Therefore,

Step 3

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 3 + \frac{21}{2}t^2.$$

6. Further Studies of Laplace Transform

6.1 Step Function

6.1.1 Definition

$$u_c(t) = \begin{cases} 0 & t < c, \\ 1 & t \geq c. \end{cases}$$

6.1.2 Some basic operations with the step function

- Cutting-off head part Generate a new function $F(t)$ by "cutting-off the head part" of another function $f(t)$ at $t = c$:

$$F(t) = f(t)u_c(t) = \begin{cases} 0 & (0 < t < c) \\ f(t) & (t > c). \end{cases}$$

- Cutting-off tail part Generate a new function $F(t)$ by "cutting-off the tail part" of another function $f(t)$ at $t = c$:

$$F(t) = f(t)[1 - u_c(t)] = \begin{cases} f(t) & (0 < t < c) \\ 0 & (t > c). \end{cases}$$

- Cutting-off both head and tail part Generate a new function $F(t)$ by "cutting-off the head part" of another function $f(t)$ at $t = c_1$ and its tail part at $t = c_2 > c_1$:

$$\begin{aligned} F(t) &= f(t)u_{c_1}(t) - f(t)u_{c_2}(t) \\ &= \begin{cases} 0 & (0 < t < c_1) \\ f(t) & (c_1 < t < c_2) \\ 0 & (t > c_2). \end{cases} \end{aligned} \tag{4.12}$$

It is easy to prove that if $0 < t_1 < t_2$, we may write discontinuous function:

$$f(t) = \begin{cases} 0, & (\infty < t < 0) \\ f_1(t), & (0 \leq t < t_1) \\ f_2(t), & (t_1 \leq t < t_2) \\ f_3(t), & (t_2 \leq t < \infty. \end{cases}$$

where $f_k(t), (k = 1, 2, 3)$ are known, and defined on $(-\infty, \infty)$ into the form

$$\begin{aligned} f(t) = & \Big[f_1(t)u_0(t) - f_1(t)u_{t_1}(t)\Big] \\ & +\Big[f_2(t)u_{t_1}(t) - f_2(t)u_{t_2}(t)\Big] \\ & +f_3(t)u_{t_2}(t) \end{aligned}$$

6.1.3 Laplace Transform of Unit Step Function

$$\mathcal{L}\{u_c(t)\} = \frac{\mathrm{e}^{-cs}}{s}.$$

One can derive

$$\mathcal{L}\{u_c(t)f(t-c)\} = \mathrm{e}^{-cs}F(s) = \mathrm{e}^{-cs}\mathcal{L}[f(t)].$$

or

$$\mathcal{L}\{u_c(t)g(t)\} = \mathrm{e}^{-cs}\mathcal{L}[G(t+c)].$$

Accordingly, we have

$$\mathcal{L}^{-1}\left\{\mathrm{e}^{-cs}\mathcal{L}[f(t)]\right\} = u_c(t)f(t-c).$$

Example 1. Determine $\mathcal{L}[f]$, if

$$f(t) = \begin{cases} 0, & (0 \leq t < 1) \\ t-1, & (1 \leq t < 2) \\ 1, & (t \geq 2). \end{cases}$$

Solution: In terms of the unit step function, we can express

$$f(t) = (t-1)u_1(t) - (t-2)u_2(t).$$

Let $g = t$. Then we have

$$f(t) = g(t-1)u_1(t) - g(t-2)u_2(t).$$

It follows that

$$\mathcal{L}[f] = \mathrm{e}^{-s}\mathcal{L}(g) - \mathrm{e}^{2s}\mathcal{L}[g] = \frac{1}{s^2}(\mathrm{e}^{-s} - \mathrm{e}^{-2s}).$$

Example 2. Determine $\mathcal{L}^{-1}\left[\frac{2\mathrm{e}^{-s}}{s^2+4}\right]$.

Solution: We have

$$\mathcal{L}[\sin 2t] = \frac{2}{s^2+4}.$$

Hence,

$$\begin{aligned}\mathcal{L}^{-1}\left[\tfrac{2\mathrm{e}^{-s}}{s^2+4}\right] &= \mathcal{L}^{-1}\Big\{\mathrm{e}^{-s}\mathcal{L}[\sin 2t]\Big\}\\ &= u_1(t)\sin[2(t-1)]\end{aligned}$$

Example 3. Determine $\mathcal{L}^{-1}\left[\frac{(s-4)\mathrm{e}^{-3s}}{s^2-4s+5}\right]$.

Solution: Let

$$\begin{aligned}G(s) &= \tfrac{(s-4)}{s^2-4s+5} = \tfrac{(s-4)}{(s-2)^2+1}\\ &= \left[\tfrac{(s-2)}{(s-2)^2+1} - \tfrac{2}{(s-2)^2+1}\right].\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{L}^{-1}[\mathrm{e}^{-3s}G(s)] &= \mathcal{L}^{-1}\Big\{\mathrm{e}^{-3s}(\mathcal{L}[\mathrm{e}^{2t}\cos t - 2\mathrm{e}^{2t}\sin t])\Big\}\\ &= u_3(t)\mathrm{e}^{2(t-3)}[\cos(t-3) - 2\sin(t-3)]\end{aligned}$$

Example 3'. Find the solution for the IVP:

$$y'' - 4y' + 5 = \mathrm{e}^t u_3(t), \quad y(0) = 0, y'(0) = 1.$$

Solution:

$$\mathcal{L}[y'' - 4y' + 5] = \mathcal{L}[\mathrm{e}^t u_3(t)] = \mathrm{e}^3\mathcal{L}[\mathrm{e}^{t-3}u_3(t)].$$

We have

$$Y(s)(s^2 - 4s + 5) - 1 = \mathrm{e}^3\frac{\mathrm{e}^{-3s}}{s-1}$$

so that

$$\begin{aligned}Y(s) &= \mathrm{e}^3\mathrm{e}^{-3s}\tfrac{1}{(s-1)[(s-2)^2+1]} + \tfrac{1}{[(s-2)^2+1]}\\ &= \mathrm{e}^3\mathrm{e}^{-3s}\left[\tfrac{1}{2}\tfrac{1}{s-1} - \tfrac{1}{2}\tfrac{s-3}{[(s-2)^2+1]}\right] + \tfrac{1}{[(s-2)^2+1]}.\end{aligned}$$

Note that

$$\mathcal{L}^{-1}\Big[\frac{1}{s-1}\Big] = \mathrm{e}^{t}$$

$$\mathcal{L}^{-1}\Big[\frac{s-2}{[(s-2)^2+1]}\Big] = \mathrm{e}^{2t}\cos t$$

$$\mathcal{L}^{-1}\Big[\frac{1}{[(s-2)^2+1]}\Big] = \mathrm{e}^{2t}\sin t.$$

Thus, we have

$$\mathcal{L}^{-1}\Big[\mathrm{e}^{-3s}\frac{1}{s-1}\Big] = \mathrm{e}^{(t-3)}u_3(t)$$

$$\mathcal{L}^{-1}\Big[\mathrm{e}^{-3s}\frac{s-2}{[(s-2)^2+1]}\Big] = \mathrm{e}^{2(t-3)}\cos(t-3)u_3(t)$$

$$\mathcal{L}^{-1}\Big[\mathrm{e}^{-3s}\frac{1}{[(s-2)^2+1]}\Big] = \mathrm{e}^{2(t-3)}\sin(t-3)u_3(t).$$

Finally, we derive

$$\begin{aligned} y(t) &= \mathrm{e}^{2t}\sin t + \tfrac{1}{2}\mathrm{e}^{3}\Big[\mathrm{e}^{t-3} - \mathrm{e}^{2(t-3)}\cos(t-3) \\ &\quad + \mathrm{e}^{2(t-3)}\sin(t-3)\Big]u_3(t) \end{aligned}$$

Example 4. Determine $\mathcal{L}[f]$, if

$$f(t) = \begin{cases} f_1(t), & (0 \le t < t_1) \\ f_2(t), & (t_1 \le t < t_2) \\ f_3(t), & (t_2 \le t < t_3) \\ f_4(t), & (t_3 \le t < \infty. \end{cases}$$

we can express

$$\begin{aligned} f(t) &= f_1(t) - f_1(t)u_{t_1}(t) + f_2(t)u_{t_1}(t) \\ &\quad - f_2(t)u_{t_2}(t) + f_3(t)u_{t_2}(t) \\ &\quad - f_3(t)u_{t_3}(t) + f_4(t)u_{t_3}(t) \\ &= f_1(t) + [f_2(t) - f_1(t)]u_{t_1}(t) \\ &\quad + [f_3(t) - f_2(t)]u_{t_2}(t) \\ &\quad + [f_4(t) - f_3(t)]u_{t_3}(t). \end{aligned}$$

Now we may apply the formula

$$\mathcal{L}\left\{u_c(t)f(t-c)\right\} = \mathrm{e}^{-cs}\mathcal{L}[f(t)](s),$$

or

$$\mathcal{L}\left\{u_c(t)g(t)\right\} = \mathrm{e}^{-cs}\mathcal{L}[g(t+c)](s),$$

Suppose that

$$\mathcal{L}[f_1(t)] = F_1(s)$$

and

$$\mathcal{L}[f_2(t+t_1) - f_1(t+t_1)](s) = G_1(s)$$

$$\mathcal{L}[f_3(t+t_2) - f_2(t+t_2)](s) = G_2(s)$$

$$\mathcal{L}[f_4(t+t_3) - f_3(t+t_3)](s) = G_3(s).$$

Then we can write

$$\mathcal{L}[f(t)](s) = F_1(s) + \mathrm{e}^{-t_1 s}G_1(s) + \mathrm{e}^{-t_2 s}G_2(s) + \mathrm{e}^{-t_3 s}G_3(s)$$

Example 5. Determine $\mathcal{L}[f]$, if

$$f(t) = \begin{cases} 3, & (0 \le t < 2) \\ 1, & (2 \le t < 3) \\ 4t, & (3 \le t < 8) \\ 2t^2, & (8 \le t). \end{cases}$$

we can express

$$f(t) = 3 - 2u_2(t) + (4t-1)u_3(t) + (2t^2 - 4t)u_8(t).$$

Here,

$$g_1(t) = 4t - 1, \quad g_2(t) = 2t^2 - 4t,$$

we have

$$\begin{aligned} g_1(t+3) &= 4t + 11, \\ g_2(t+8) &= 2(t+8)^2 - 4(t+8) \\ &= 2t^2 + 32t + 128 - 4t - 32 \\ &= 2t^2 + 28t + 96. \end{aligned}$$

Hence, we have

$$\mathcal{L}[g_1(t+3)] = G_1(s) = \frac{4}{s^2} + \frac{11}{s}$$

and

$$\mathcal{L}[g_2(t+8)] = G_2(s) = \frac{4}{s^3} + \frac{28}{s^2} + \frac{96}{s}$$

It follows that

$$\mathcal{L}[f(t)](s) = \frac{1}{s} - \frac{2\mathrm{e}^{-2s}}{s} + G_1(s)\mathrm{e}^{-3s} + G_2(s)\mathrm{e}^{-8s}.$$

Example 6. Determine $\mathcal{L}[f]$, if $f(t)$ is a periotic function with period T, and define the 'windowed' version of $f(t)$ as:

$$f_T(t) = \begin{cases} f(t), & (0 \leq t < T) \\ 0, & \text{(Otherwise)}. \end{cases}$$

Then we have

$$\begin{aligned} f(t) = &\, f_T(t) + f_T(t-T)u_T(t) + f_T(t-2T)u_{2T}(t) \\ &+ \cdots + f_T(t-nT)u_{nT}(t) + \cdots \end{aligned}$$

Thus, we have

$$\begin{aligned} \mathcal{L}\left[f(t)\right](s) &= \mathcal{L}\left[f_T\right](s) + \mathcal{L}\left[f_T\right](s)\mathrm{e}^{-sT} \\ &\quad + \mathcal{L}\left[f_T\right](s)\mathrm{e}^{-2sT} + \cdots \\ &\quad + \mathcal{L}\left[f_T\right](s)\mathrm{e}^{-nsT} + \cdots \\ &= \frac{\mathcal{L}\left[f_T\right](s)}{1-\mathrm{e}^{-sT}}, \end{aligned}$$

where

$$\mathcal{L}\left[f_T\right](s) = \int_0^T \mathrm{e}^{-st} f(t)\mathrm{d}t.$$

Example 7. Determine $\mathcal{L}[f]$, if $f(t)$ is the sawtooth wave with period $T = 2$ and

$$f_T(t) = \begin{cases} 2t, & (0 \leq t < 2) \\ 0, & \text{(Otherwise)}. \end{cases}$$

Then we have

$$\mathcal{L}\left[f_T\right](s) = \int_0^T 2t\mathrm{e}^{-st}\mathrm{d}t = \frac{2}{s^2} - 2\frac{(1+2s)\mathrm{e}^{-2s}}{s^2},$$

and

$$\mathcal{L}\left[f(t)\right](s) = \frac{2}{1-\mathrm{e}^{-2s}}\left[\frac{1}{s^2} - \frac{(1+2s)\mathrm{e}^{-2s}}{s^2}\right].$$

Example 8. Solve the IVP with Laplace transform method:

$$y'' + 4y = f(t), \qquad y(0) = y'(0) = 0,$$

where

$$f(t) = \begin{cases} f_T(t), & (n\pi \leq t < (n+1)\pi), \\ & \quad (n = 0, 1, \cdots N). \\ 0, & \text{(Otherwise)}. \end{cases}$$

and

$$f_T(t) = \begin{cases} \sin t, & (0 \le t < \pi) \\ 0, & \text{(Otherwise)}. \end{cases}$$

Solution:

W may write

$$f(t) = f_T(t) + f_T(t-\pi)u_T(t) + f_T(t-2\pi)u_{2\pi}(t) \\ + \cdots + f_T(t-N\pi)u_{N\pi}(t).$$

Thus, we have

$$\mathcal{L}\left[f(t)\right](s) = \mathcal{L}\left[f_T\right](s) + \mathcal{L}\left[f_T\right](s)\mathrm{e}^{-s\pi} \\ +\mathcal{L}\left[f_T\right](s)\mathrm{e}^{-2s\pi} + \cdots + \mathcal{L}\left[f_T\right](s)\mathrm{e}^{-Ns\pi}.$$

Due to

$$\mathcal{L}\left[f_T\right](s) = \int_0^\pi \mathrm{e}^{-st}\sin t \mathrm{d}t = \frac{1+\mathrm{e}^{-s\pi}}{1+s^2},$$

then we have

$$\mathcal{L}\left[f(t)\right](s) = F(s) = \frac{1+\mathrm{e}^{-s\pi}}{1+s^2}\Big[1+\mathrm{e}^{-s\pi}+\mathrm{e}^{-2s\pi} \\ +\cdots+\mathrm{e}^{-Ns\pi}\Big] \\ = \frac{1}{1+s^2}\Big[1+2(\mathrm{e}^{-s\pi}+\mathrm{e}^{-2s\pi}+\cdots+\mathrm{e}^{-Ns\pi}) \\ +\mathrm{e}^{-(N+1)s\pi}\Big].$$

On the other hand, letting

$$\mathcal{L}\left[y(t)\right] = Y(s),$$

from the equation we have

$$Y(s)(s^2+4) = F(s),$$

so that

$$Y(s) = \frac{1}{(1+s^2)(s^2+4)}\Big[1+2(\mathrm{e}^{-s\pi}+\mathrm{e}^{-2s\pi}+\cdots \\ +\mathrm{e}^{-Ns\pi})+\mathrm{e}^{-(N+1)s\pi}\Big].$$

Let

$$H(s) = \frac{1}{(1+s^2)(s^2+4)} = \frac{1}{3}\left[\frac{1}{(1+s^2)} - \frac{1}{(s^2+4)}\right].$$

We have

$$h(t) = \mathcal{L}^{-1}[H(s)] = \left(\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right).$$

Therefore, we may derive

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y(s)] &= \left(\tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t\right) \\
&+2u_{\pi}\left[\tfrac{1}{3}\sin(t-\pi) - \tfrac{1}{6}\sin 2(t-\pi)\right] \\
&+2u_{2\pi}\Big[\tfrac{1}{3}\sin(t-2\pi) \\
&-\tfrac{1}{6}\sin 2(t-2\pi)\Big] + \cdots \\
&+2u_{N\pi}\left[\tfrac{1}{3}\sin(t-N\pi) - \tfrac{1}{6}\sin 2(t-N\pi)\right] \\
&+u_{(N+1)\pi}\Big[\tfrac{1}{3}\sin(t-(N+1)\pi) \\
&-\tfrac{1}{6}\sin 2(t-(N+1)\pi)\Big]
\end{aligned}$$

or

$$\begin{aligned}
y(t) = \mathcal{L}^{-1}[Y(s)] &= \left(\tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t\right) \\
&+2u_{\pi}\left[-\tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t\right] \\
&+2u_{2\pi}\left[\tfrac{1}{3}\sin t - \tfrac{1}{6}\sin 2t\right] + \cdots \\
&+2u_{N\pi}\left[\tfrac{1}{3}(-1)^N\sin t - \tfrac{1}{6}\sin 2t\right] \\
&+u_{(N+1)\pi}\left[\tfrac{1}{3}(-1)^{N+1}\sin t - \tfrac{1}{6}\sin 2t\right].
\end{aligned}$$

6.2 Impulse Function

6.2.1 Definition

Let

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau} & |t| < \tau, \\ 0 & |t| \geq \tau. \end{cases}$$

It follows that

$$I(\tau) = \int_{-\infty}^{\infty} d_\tau(t)\mathrm{d}t = 1.$$

Now, consider the limit,

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t) = \begin{cases} 0 & t \neq 0, \\ \infty & t = 0, \end{cases}$$

which is called the **Dirac δ-function**. Evidently, the Dirac δ-function has the following properties:

1
$$\int_{-\infty}^{\infty} \delta(t)\mathrm{d}t = 1.$$

2
$$\int_A^B \delta(t-c)\mathrm{d}t = \begin{cases} 0 & c \bar{\in} (A,B), \\ 1 & c \in (A,B). \end{cases}$$

3
$$\int_A^B \delta(t-c)f(t)\mathrm{d}t = \begin{cases} 0 & c \bar{\in} (A,B), \\ f(c) & c \in (A,B). \end{cases}$$

6.2.2 Laplace Transform of Impulse Function

$$\mathcal{L}\{\delta(t-c)\} = \begin{cases} \mathrm{e}^{-cs} & c > 0, \\ 1/2 & c = 0 \\ 0 & c < 0. \end{cases}$$

One can derive

$$\mathcal{L}\{\delta(t-c)f(t)\} = \mathrm{e}^{-cs}f(c), \quad (c > 0).$$

Example 1. Solve the IVP:

$$y'' + 4y' + 13y = \delta(t-\pi),$$

with

$$y(0) = 2, \quad y'(0) = 1.$$

Solution: Take the Laplace transform of both sides, and impose IC's. It follows that

$$[s^2Y - 2s - 1] + 4[sY - 2] + 13Y = \mathrm{e}^{-\pi s}$$

We have

$$\begin{aligned} Y &= \tfrac{\mathrm{e}^{-\pi s}+2s+9}{s^2+4s+13} = \tfrac{\mathrm{e}^{-\pi s}+2s+9}{(s+2)^2+9} \\ &= \tfrac{\mathrm{e}^{-\pi s}}{(s+2)^2+9} + \tfrac{2(s+2)}{(s+2)^2+9} + \tfrac{5}{(s+2)^2+9}. \end{aligned}$$

Take the inverse of the Laplace transform on both sides. we derive

$$\begin{aligned} y &= \mathcal{L}^{-1}\left\{\tfrac{\mathrm{e}^{-\pi s}}{3}\mathcal{L}[\mathrm{e}^{-2t}\sin 3t]\right\} \\ &\quad +2[\mathrm{e}^{-2t}\cos 3t] + \tfrac{5}{3}[\mathrm{e}^{-2t}\sin 3t] \\ &= \tfrac{1}{3}u_\pi(t)\mathrm{e}^{-2(t-\pi)}\sin 3(t-\pi) \\ &\quad +2[\mathrm{e}^{-2t}\cos 3t] + \tfrac{5}{3}[\mathrm{e}^{-2t}\sin 3t]. \end{aligned}$$

6.3 Convolution Integral

6.3.1 Theorem

Given

$$\mathcal{L}\{f(t)\} = F(s), \quad \mathcal{L}\{g(t)\} = G(s),$$

one can derive

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\cdot G(s)\} &= f(t) * g(t)\\ &= \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\}\end{aligned}$$

where

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau$$

is called the convolution integral.

Proof:

$$\begin{aligned}\mathcal{L}\{f * g(t)\} &= \int_0^\infty \mathrm{e}^{-st}\left\{\int_0^t f(t-\tau)g(\tau)\mathrm{d}\tau\right\}\mathrm{d}t\\ &= \int_0^\infty \left\{\int_0^t \mathrm{e}^{-st} f(t-\tau)g(\tau)\mathrm{d}\tau\right\}\mathrm{d}t.\end{aligned}$$

However, it can be proven that

$$\begin{aligned}&\textstyle\int_0^\infty \left\{\int_0^t \mathrm{e}^{-st} f(t-\tau)g(\tau)\mathrm{d}\tau\right\}\mathrm{d}t\\ &= \textstyle\int_0^\infty \left\{\int_\tau^\infty \mathrm{e}^{-st} f(t-\tau)g(\tau)\mathrm{d}t\right\}\mathrm{d}\tau.\end{aligned}$$

Now let $u = t - \tau$. We get

$$\begin{aligned}\mathcal{L}\{f * g(t)\} &= \int_0^\infty \left\{\int_0^\infty \mathrm{e}^{-s(u+\tau)} f(u)g(\tau)\mathrm{d}u\right\}\mathrm{d}\tau\\ &= \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}\end{aligned}$$

6.3.2 The properties of convolution integral

- $f(t) * g(t) = g(t) * f(t)$.
- $[cf(t) + dg(t)] * h(t) = c[f(t) * h(t)] + d[g(t) * h(t)]$.
- $1 * g(t) \neq g(t)$
- $0 * g(t) = 0$.

Example 1. Determine $\mathcal{L}^{-1}\left[\frac{G(s)}{(s-1)^2+1}\right]$.

Solution:

$$\mathcal{L}^{-1}\left[\tfrac{G(s)}{(s-1)^2+1}\right] = \mathcal{L}^{-1}[G(s)] * \mathcal{L}^{-1}\left[\tfrac{1}{(s-1)^2+1}\right]$$
$$= \mathrm{e}^{-t}\sin t * g(t).$$

Example 2. Solve the IVP:

$$y'' + \omega^2 y = f(t)$$

with

$$y(0) = \alpha, \quad y'(0) = \beta.$$

Solution: **Step #1**: Taking Laplace transform of both sides, it follows that

$$[s^2 Y - \alpha s - \beta] + \omega^2 Y = F(s).$$

Step #2: We solve

$$Y(s) = \frac{F(s)}{s^2+\omega^2} + \frac{\alpha s}{s^2+\omega^2} + \frac{\beta}{s^2+\omega^2}.$$

Step #3: Taking inverse Laplace transform of both sides, we obtain

$$y(t) = \tfrac{1}{\omega}\int_0^t \sin\omega(t-\tau) f(\tau)\mathrm{d}\tau$$
$$+\alpha\cos\omega t + \tfrac{\beta}{\omega}\sin\omega t.$$

Chapter 5

SERIES SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

1. Series Solutions near an Ordinary Point

1.1 Introduction

A function $f(x)$ of one variable x is said to be **analytic** at a point $x = x_0$ if it has a convergent power series expansion

$$\begin{aligned} f(x) &= \sum_0^\infty a_n(x-x_0)^n \\ &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 \\ &\quad + \cdots + a_n(x-x_0)^n + \cdots \end{aligned}$$

for $|x - x_0| < R$, $R > 0$.

This point $x = x_0$ is also called **ordinary point**. Otherwise, $f(x)$ is said to have a **singularity** at $x = x_0$. The largest such R (possibly $+\infty$) is called the **radius of convergence** of the power series. The series converges for every x with $|x - x_0| < R$ and diverges for every x with $|x - x_0| > R$. There is a formula for $R = \frac{1}{\ell}$, where

$$\ell = \lim_{n\to\infty} |a_n|^{1/n}, \qquad \text{or} \quad \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|},$$

if the latter limit exists. The same is true if $x,\ x_0,\ a_i$ are complex. For example,

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$$

for $|x| < 1$. The radius of convergence of the series is 1. It is also equal to the distance from 0 to the nearest singularity $x = i$ of $1/(x^2+1)$ in the complex plane.

Power series can be integrated and differentiated within the interval (disk) of convergence. More precisely, for $|x - x_0| < R$ we have

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$\int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) \mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} x^n$$

and the resulting power series have R as radius of convergence. If $f(x)$, $g(x)$ are analytic at $x = x_0$ then so is $f(x)g(x)$ and $af + bg$ with any scalars a, b, whose radii of convergence is not smaller than the radii of convergence of the series for $f(x), g(x)$. If $f(x)$ is analytic at $x = x_0$ and $f(x_0) \neq 0$ then $1/f(x_0)$ is analytic at $x = x_0$ with radius of convergence equal to the distance from x_0 to the nearest zero of $f(x)$ in the complex plane.

The following theorem shows that linear DE's with analytic coefficients at x_0 have analytic solutions at x_0 with radius of convergence as big as the smallest of the radii of convergence of the coefficient functions.

1.2 Series Solutions near an Ordinary Point

Theorem 1

If $p_1(x), p_2(x), \ldots, p_n(x), q(x)$ are analytic at $x = x_0$, the solutions of the DE

$$y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = q(x)$$

are analytic with radius of convergence $\geq$ the smallest of the radii of convergence of the coefficient functions

$$p_1(x), p_2(x), \ldots, p_n(x), q(x).$$

The proof of this result follows from the two steps: First find the formal series solutions for the EQ, then prove the formal series solutions are convergent.

Example 1. The coefficients of the DE $y'' + y = 0$ are analytic everywhere, in particular at $x = 0$. Any solution $y = y(x)$ has therefore a series representation

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with infinite radius of convergence. We have

$$y' = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

$$y'' = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n.$$

Therefore, we have

$$y'' + y = \sum_{n=0}^{\infty}((n+1)(n+2)a_{n+2} + a_n)x^n = 0$$

for all x. It follows that

$$(n+1)(n+2)a_{n+2} + a_n = 0$$

for $n \geq 0$. Thus we obtain

$$a_{n+2} = -\frac{a_n}{(n+1)(n+2)}, \quad \text{for } n \geq 0, \tag{5.1}$$

from which we obtain

$$a_2 = -\frac{a_0}{1 \cdot 2}, \; a_3 = -\frac{a_1}{2 \cdot 3}, \; a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{1 \cdot 2 \cdot 3 \cdot 4},$$
$$a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}.$$

(5.1) is called **the recurrence formula.**

By induction one obtains

$$a_{2n} = (-1)^n \frac{a_0}{(2n)!}, \quad a_{2n+1} = (-1)^n \frac{a_1}{(2n+1)!}$$

and hence that

$$\begin{aligned} y &= a_0 \textstyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &= a_0 \cos(x) + a_1 \sin(x). \end{aligned}$$

Example 2. The simplest non-constant DE is $y'' + xy = 0$ which is known as Airy's equation. Its coefficients are analytic everywhere and so the solutions have a series representation

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with infinite radius of convergence. We have

$$\begin{aligned} y'' + xy &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^{n+1}, \\ &= \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n + \sum_{n=1}^{\infty} a_{n-1} x^n, \\ &= 2a_2 + \sum_{n=1}^{\infty}((n+1)(n+2)a_{n+2} + a_{n-1})x^n \\ &= 0 \end{aligned}$$

from which we get **the recurrence formula**:

$$a_2 = 0,$$
$$(n+1)(n+2)a_{n+2} + a_{n-1} = 0$$

for $n \geq 1$. Since $a_2 = 0$ and

$$a_{n+2} = -\frac{a_{n-1}}{(n+1)(n+2)}, \quad \text{for } n \geq 1$$

we have

$$a_3 = -\frac{a_0}{2 \cdot 3}, \ a_4 = -\frac{a_1}{3 \cdot 4}, \ a_5 = 0,$$
$$a_6 = -\frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6},$$
$$a_7 = -\frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}.$$

By induction we get $a_{3n+2} = 0$ for $n \geq 0$ and

$$a_{3n} = (-1)^n \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1) \cdot 3n},$$

$$a_{3n+1} = (-1)^n \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n) \cdot (3n+1)}.$$

Hence $y = a_0 y_1 + a_1 y_2$ with

$$y_1 = 1 - \frac{x^3}{2\cdot 3} + \frac{x^6}{2\cdot 3\cdot 5\cdot 6} - \cdots + (-1)^n \frac{x^{3n}}{2\cdot 3\cdot 5\cdot 6\cdots(3n-1)\cdot 3n} + \cdots,$$

$$y_2 = x - \frac{x^4}{3\cdot 4} + \frac{x^7}{3\cdot 4\cdot 6\cdot 7} - \cdots + (-1)^n \frac{x^{3n+1}}{3\cdot 4\cdot 6\cdot 7\cdots(3n)\cdot(3n+1)} + \cdots.$$

For positive x the solutions of the DE $y'' + xy = 0$ behave like the solutions to a mass-spring system with variable spring constant.

- The solutions oscillate for $x > 0$ with increasing frequency as $|x| \to \infty$.
- For $x < 0$ the solutions are monotone. For example, y_1, y_2 are increasing functions of x for $x \leq 0$.

2. Series Solution near a Regular Singular Point

2.1 Introduction

In this lecture we investigate series solutions for the general linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = b(x),$$

where the functions $a_1, a_2, \ldots, a_n, b$ are analytic at $x = x_0$. If $a_0(x_0) \neq 0$ the point $x = x_0$ is called an **ordinary point** of the DE. In this case, the solutions are analytic at $x = x_0$ since the normalized DE

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = q(x),$$

where

$$p_i(x) = a_i(x)/a_0(x), q(x) = b(x)/a_0(x),$$

has coefficient functions which are analytic at $x = x_0$.

If $a_0(x_0) = 0$, the point $x = x_0$ is said to be a **singular point** for the DE.

If k is the multiplicity of the zero of $a_0(x)$ at $x = x_0$ and the multiplicities of the other coefficient functions at $x = x_0$ is as big then, on cancelling the common factor $(x - x_0)^k$ for $x \neq x_0$, the DE obtained holds even for $x = x_0$ by continuity, has analytic coefficient functions at $x = x_0$ and $x = x_0$ is an ordinary point.

In this case the singularity is said to be **removable**. For example, the DE $xy'' + \sin(x)y' + xy = 0$ has a removable singularity at $x = 0$.

2.2 Series Form of Solutions near a Regular Singular Point

In general, the solution of a linear DE in a neighborhood of a singularity is extremely difficult. However, there is an important special case where this can be done. For simplicity, we treat the case of the general second order homogeneous DE

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \quad (x > x_0),$$

with a singular point at $x = x_0$. We say that $x = x_0$ is a **regular singular point** if the normalized DE

$$y'' + p(x)y' + q(x)y = 0, \quad (x > 0),$$

is such that $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at $x = x_0$. A necessary and sufficient condition for this is that

$$\lim_{x \to x_0}(x - x_0)p(x) = p_0, \quad \lim_{x \to x_0}(x - x_0)^2q(x) = q_0$$

exist and are finite. In this case

$$(x-x_0)p(x) = p_0 + p_1(x-x_0) + \cdots + p_n(x-x_0)^n + \cdots,$$
$$(x-x_0)^2 q(x) = q_0 + q_1(x-x_0) + \cdots + q_n(x-x_0)^n + \cdots.$$

Without loss of generality we can, after possibly a change of variable $x - x_0 = t$, for simplicity, hereafter we assume that the regular singular point is $x_0 = 0$. Then we may re-write the DE in the form:

$$x^2 L[y] = x^2 y'' + x(xp(x))y' + x^2 q(x) y = 0.$$

This DE is an Euler DE if $xp(x) = p_0$, $x^2 q(x) = q_0$. This suggests that we should look for solutions of the form

$$y = x^r \left(\sum_{n=0}^{\infty} a_n x^n \right) = \sum_{n=0}^{\infty} a_n x^{n+r},$$

with $a_0 \neq 0$. Substituting this in the DE gives

$$\begin{aligned} x^2 L[y] &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} \\ &+ \left(\sum_{n=0}^{\infty} p_n x^n \right) \left(\sum_{n=0}^{\infty} (n+r) a_n x^{n+r} \right) \\ &+ \left(\sum_{n=0}^{\infty} q_n x^n \right) \left(\sum_{n=0}^{\infty} a_n x^{n+r} \right) \end{aligned}$$

which, on expansion and simplification, becomes

$$\begin{aligned} x^2 L[y] &= a_0 F(r) x^r + \sum_{n=1}^{\infty} (n+r)(n+r-1) a_n x^{n+r} \\ &+ \sum_{n=1}^{\infty} \Big\{ p_n r a_0 + p_{n-1}(1+r) a_1 + \cdots \\ &\quad + p_1(n-1+r) a_{n-1} + p_0 (n+r) a_n \Big\} x^{n+r} \\ &+ \sum_{n=1}^{\infty} \Big\{ q_n a_0 + q_{n-1} a_1 + \cdots + q_1 a_{n-1} + q_0 a_n \Big\} x^{n+r} \end{aligned}$$

or

$$\begin{aligned} x^2 L[y] &= a_0 F(r) x^r + \sum_{n=1}^{\infty} \Big\{ F(n+r) a_n \\ &\quad + [(n+r-1)p_1 + q_1] a_{n-1} + \cdots \\ &\quad + (r p_n + q_n) a_0 \Big\} x^{n+r}, \end{aligned}$$

where

$$F(r) = r(r-1) + p_0 r + q_0.$$

We call

$$F(r) = r(r-1) + p_0 r + q_0 = 0, \tag{5.2}$$

the **indicial equation**.

Equating coefficients of the series on the right side to zero, namely set

$$\begin{aligned} F(n+r)a_n = & -[(n+r-1)p_1 + q_1]a_{n-1} \\ & - \cdots - (rp_n + q_n)a_0 \end{aligned} \tag{5.3}$$

for $n \geq 1$, results in the recurrence formula:

$$a_n(r) = \frac{-\big[(n+r-1)p_1+q_1\big]a_{n-1}-\cdots-(rp_n+q_n)a_0}{F(n+r)} \tag{5.4}$$

and

$$x^2 L[y(x,r)] = a_0 F(r) x^r. \tag{5.5}$$

It is important to note that **when the coefficients $a_n(r)$ of series solution $y(x)$ are given by the recurrence formula (5.4), the formula (5.5) will be valid for all x and any parameters (r, a_0).** The above recurrence equation (5.4), may be re-written in the form:

$$a_n(r) = R[a_0, a_1(r), \cdots, a_{n-1}(r), r, n]. \tag{5.6}$$

It may be deduced that $a_n(r) = a_0 \hat{a}_n(r)$, and $\hat{a}_n(r)$ can be uniquely determined from (5.6) by setting $\hat{a}_0 = 1$ as follows:

$$\begin{aligned} \hat{a}_n(r) &= R[1, \hat{a}_1(r), \cdots, \hat{a}_{n-1}(r), r, n] \\ &= \frac{-\big[(n+r-1)p_1+q_1\big]\hat{a}_{n-1}-\cdots-(rp_n+q_n)}{F(n+r)} \end{aligned} \tag{5.7}$$

for $n \geq 1$.

The indicial equation (5.2) has two roots: r_1, r_2. Three cases should be discussed separately.

2.2.1 Case (I): The roots $(r_1 - r_2 \neq N)$

Two roots do'nt differ by an integer. In this case,

$$F(r) = (r - r_1)(r - r_2).$$

With $r = r_1$ and $r = r_2$, we obtain the linearly independent solutions

$$\begin{aligned} y_1 &= |x|^{r_1} \left(\sum_{n=0}^{\infty} \hat{a}_n(r_1) x^n \right), \\ y_2 &= |x|^{r_2} \left(\sum_{n=0}^{\infty} \hat{a}_n(r_2) x^n \right). \end{aligned} \tag{5.8}$$

It can be shown that the radius of convergence of the infinite series is the distance to the singularity of the DE nearest to the singularity $x = 0$.

When $r_1 = r_2$ or $r_1 = r_2 + N$, only the first solution $y_1(x)$ corresponding to $r = r_1$

$$y_1 = |x|^{r_1} \left(\sum_{n=0}^{\infty} \hat{a}_n(r_1) x^n \right).$$

can be obtained through the above procedure.

In those cases, a **second linearly independent solution can then be found by the method of reduction of order**.

However, a second solution can be also found with the method shown below.

2.2.2 Case (II): The roots ($r_1 = r_2$)

In this case, we have $F(r) = (r - r_1)^2$. Setting $\hat{a}_0 = 1$, with $\{\hat{a}_n(r), (n \geq 1)\}$ determined by (5.7), from the equality (5.5) we get

$$x^2 y'' + x^2 p(x) y' + x^2 q(x) y = (r - r_1)^2 x^r,$$

which is valid for all values of variables x and r. Differentiating this equation with respect to r, we get

$$\begin{aligned} & x^2 \left(\tfrac{\partial y}{\partial r} \right)'' + x^2 p(x) \left(\tfrac{\partial y}{\partial r} \right)' + x^2 q(x) \tfrac{\partial y}{\partial r} \\ & = 2(r - r_1) x^r + (r - r_1)^2 x^r \ln(x). \end{aligned}$$

Setting $r = r_1$, we find that

$$\begin{aligned} y_2 = \frac{\partial y}{\partial r}(x, r_1) &= x^{r_1} \Big[\sum_{n=0}^{\infty} \hat{a}_n(r_1) x^n \Big] \ln(x) \\ & + x^{r_1} \sum_{n=0}^{\infty} \hat{a}'_n(r_1) x^n \\ &= y_1 \ln(x) + x^{r_1} \sum_{n=0}^{\infty} \hat{a}'_n(r_1) x^n, \end{aligned}$$

where $\hat{a}'_n(r)$ is the derivative of $\hat{a}_n(r)$ with respect to r. This is a second linearly independent solution.

2.2.3 Case (III): The roots ($r_1 - r_2 = N > 0$)

For this case, $r_1 = r_2 + N$. Thus,

$$F(r_2 + N) = 0.$$

Thus, from (5.7) we find $\hat{a}_N(r_2) = \infty$. Consequently, we cannot derive $y_2(x)$ with the root r_2 as in case (I).

To resolve the difficulty, we now set $a_0 = (r - r_2)$,

$$a_n(r) = (r - r_2)\hat{a}_n(r), \quad (n \geq 1),$$

and consider the function

$$y(x, r) = \sum_{n=0}^{\infty} a_n(r) x^{n+r}.$$

Thus, from the equality (5.5) we get

$$x^2 L[y(x, r)] = (r - r_1)(r - r_2)^2 x^r.$$

Differentiating both sides of this equality with respect to r, we get

$$\begin{aligned} x^2 L[\tfrac{\partial}{\partial r} y(x, r)] &= (r - r_1)(r - r_2)^2 x^r \ln(x) \\ &+(r - r_2)\Big[(r - r_2) + 2(r - r_1)\Big] x^r. \end{aligned}$$

Setting $r = r_2$, it is seen that $y_2(x) = \frac{\partial}{\partial r} y(x, r_2)$ is a solution of the given DE. This leads to

$$y_2 = \ln|x| \sum_{n=0}^{\infty} a_n(r_2) x^{n+r_2} + x^{r_2} \sum_{n=0}^{\infty} a_n'(r_2) x^n. \tag{5.9}$$

We are now going to further investigate the above result.

1. <u>The simplification of the first power series term</u>: One may deduce that

- as $n \leq (N - 1)$,

  $$a_n(r_2) = 0,$$

  This directly follows from the definition $a_n(r) = (r - r_2)\hat{a}_n(r)$;

- as $n = N$,

  $$a_N(r_2) = \lim_{r \to r_2} (r - r_2)\hat{a}_N(r) = a < \infty,$$

- as $n \geq N + 1$, or $n = N + k$,

  $$a_n(r_2) = a_{N+k}(r_2) = a\hat{a}_k(r_1).$$

Proof: We re-write the recurrence formula (5.3) as

$$\begin{aligned} a_n(r) &= (r - r_2)R[\hat{a}_0, \cdots, \hat{a}_N, \cdots, \hat{a}_{n-1}, r, n] \\ &= -\tfrac{[(n+r-1)p_1+q_1]\hat{a}_{n-1}(r)}{F(n+r)}(r - r_2) \\ &- \cdots - \tfrac{[(N+r-1)p_{n-N}+q_{n-N}]\hat{a}_N(r)}{F(n+r)}(r - r_2) \\ &- \cdots - \tfrac{[rp_n+q_n]\hat{a}_0(r-r_2)}{F(n+r)}. \end{aligned} \tag{5.10}$$

As $r \to r_2$, it is reduced to

$$\begin{aligned} a_n(r_2) &= -\tfrac{[(n+r_2-1)p_1+q_1]a_{n-1}(r_2)}{F(n+r_2)} \\ &- \cdots - \tfrac{[(N+r_2-1)p_{n-N}+q_{n-N}]a}{F(n+r_2)}. \end{aligned} \tag{5.11}$$

With $n+r_2 = N+k+r_2 = k+r_1$, $n-N = k$, by denoting $a_{N+k}(r_2) = ab_k$, we may re-write the above formula as

$$\begin{aligned} b_k &= -\tfrac{[(k+r_1-1)p_1+q_1]b_{n-1}}{F(k+r_1)} \\ &- \cdots - \tfrac{[(r_1-1)p_k+q_k]b_0}{F(k+r_1)}. \end{aligned} \tag{5.12}$$

Due to $b_0 = 1$, this yields $b_k = \hat{a}_k(r_1)$.
By substituting the above results to the first power series on the right hand side of (5.9), we derive

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(r_2)x^{n+r_2} &= \sum_{N+k=0}^{\infty} a_{N+k}(r_2)x^{N+k+r_2} \\ &= a\sum_{k=0}^{\infty} \hat{a}_k(r_1)x^{k+r_1} \\ &= ay_1(x). \end{aligned}$$

2. <u>The simplification of the second power series term</u>: we now investigate the second power series on the right hand side of (5.9). One may deduce that

- as $n \le (N-1)$,

  $$a'_n(r_2) = \hat{a}_n(r_2)$$

  This directly follows from the definition $a_n(r) = (r - r_2)\hat{a}_n(r)$;

- as $n = N$, though $\hat{a}_N(r_2) = \infty$, we may have

  $$a'_N(r_2) = \lim_{r \to r_2}[(r - r_2)\hat{a}_N(r)]' = \tilde{a}_N < \infty;$$

- as $n \geq N+1$, from the recurrence formula (5.10), we derive

$$\begin{aligned} a'_n(r_2) &= \tilde{a}_n(r_2) = \lim_{r \to r_2}[(r-r_2)\hat{a}_n(r)]' \\ &- \tfrac{[(n+r_2-1)p_1+q_1]\tilde{a}_{n-1}(r_2)}{F(n+r_2)} \\ &- \cdots - \tfrac{[(N+r_2-1)p_{n-N}+q_{n-N}]\tilde{a}_N}{F(n+r_2)} \\ &- \cdots - \tfrac{[r_2 p_n+q_n]\hat{a}_0}{F(n+r_2)}. \end{aligned} \tag{5.13}$$

Here the sequence $\{\tilde{a}_{N+1}, \cdots \tilde{a}_{n-1}\}$ are calculated by setting $r = r_2$ and $\hat{a}_N(r) = \tilde{a}_N$ in the recurrence formula (5.7). Thus, we may write the second power series as

$$|x|^{r_2} \sum_{n=0}^{\infty} \tilde{a}_n(r_2) x^n,$$

where

$$\tilde{a}_n(r_2) = \begin{cases} 1 & n = 0; \\ \hat{a}_n(r_2) & 0 < n < N; \\ \tilde{a}_N & n = N. \end{cases}$$

Whereas for $n > N$,

$$\begin{aligned} \tilde{a}_n = & R[1, \hat{a}_1, \cdots, \hat{a}_{N-1}, \tilde{a}_N, \tilde{a}_{N+1}, \\ & \cdots, \tilde{a}_{n-1}, r_2, n], \end{aligned}$$

which is obtained by setting $r = r_2$ in the recurrence formula (5.7) and replacing the variable $\hat{a}_N(r_2)$ by the number $\tilde{a}_N$.

Thus, we finally derive

$$y_2(x) = a y_1(x) \ln|x| + |x|^{r_2} \left(\sum_{n=0}^{\infty} \tilde{a}_n(r_2) x^n \right). \tag{5.14}$$

Thus, we obtain two linearly independent solution. The above method is due to Frobenius and is called the **Frobenius method**.

2.3 Summary

THEOREM 2.3.1 *(Frobenius Theorem)*
Give the normalized DE

$$y'' + p(x)y' + q(x)y = 0, \quad (x > 0).$$

Suppose say that $x = 0$ *is a* **regular singular point**, *and*

$$xp(x) = p_0 + p_1 x + \cdots + p_n x^n + \cdots,$$

$$x^2q(x) = q_0 + q_1x + \cdots + q_nx^n + \cdots.$$

Then, we have the indicial equation:

$$F(r) = r(r-1) + p_0r + q_0 = 0,$$

which has two roots:r_1, r_2. There three distinct cases:

1 **Case (I): The roots** $(r_1 - r_2 \neq N)$

The system has the linearly independent solutions:

$$y_1 = |x|^{r_1}\Big(1 + \sum_{n=1}^{\infty} \hat{a}_n(r_1)x^n\Big),$$
$$y_2 = |x|^{r_2}\Big(1 + \sum_{n=1}^{\infty} \hat{a}_n(r_2)x^n\Big).$$

2 **Case (II): The roots** $(r_1 = r_2)$ The system has the linearly independent solutions:

$$y_1 = |x|^{r_1}\Big(1 + \sum_{n=1}^{\infty} \hat{a}_n(r_1)x^n\Big),$$
$$y_2 = y_1(x)\ln|x| + |x|^{r_1}\Big(\sum_{n=0}^{\infty} b_nx^n\Big),$$

where $b_n = \hat{a}'_n(r_2)$; $b_0 = 0$, since $\hat{a}_0 = 1$.

3 **Case (III): The roots** $(r_1 = r_2 + N)$ The system has the linearly independent solutions:

$$y_1 = |x|^{r_1}\Big(1 + \sum_{n=1}^{\infty} \hat{a}_n(r_1)x^n\Big),$$
$$y_2 = ay_1(x)\ln|x| + |x|^{r_2}\Big(\sum_{n=0}^{\infty} \tilde{a}_nx^n\Big),$$

where $\tilde{a}_0 = 1$ and $a_0(r) = (r - r_2)$.

Example 1. The DE $2xy'' + y' + 2xy = 0$ has a regular singular point at $x = 0$ since $xp(x) = 1/2$ and $x^2q(x) = x^2$. Noting that

$$p_0 = \frac{1}{2},\ p_1 = p_2 = \cdots = 0,$$

and

$$q_0 = q_1 = 0,\ q_2 = 1,\ q_3 = q_4 = \ldots = 0,$$

we derive the indicial equation:

$$r(r-1)+\frac{1}{2}r=r(r-\frac{1}{2})=0.$$

The roots are $r_1 = 1/2$, $r_2 = 0$ which do not differ by an integer. For the recurrence formula, we have

$$\begin{aligned}(r+1)(r+\tfrac{1}{2})a_1 &= 0,\\ (n+r)(n+r-\tfrac{1}{2})a_n &= -a_{n-2} \quad \text{for } n \geq 2,\end{aligned} \tag{5.15}$$

so that

$$a_n = -2a_{n-2}/(r+n)(2r+2n-1)$$

for $n \geq 2$. Hence $0 = a_1 = a_3 = \cdots a_{2n+1}$ for $n \geq 0$ and

$$\begin{aligned}a_2 &= -\tfrac{2}{(r+2)(2r+3)}a_0,\\ a_4 &= -\tfrac{2}{(r+4)(2r+7)}a_2\\ &= \tfrac{2^2}{(r+2)(r+4)(2r+3)(2r+7)}a_0.\end{aligned}$$

It follows by induction that

$$\begin{aligned}a_{2n} &= (-1)^n\frac{2^n}{(r+2)(r+4)\cdots(r+2n)}\\ &\times\frac{1}{(2r+3)(2r+7)\cdots(2r+4n-1)}a_0.\end{aligned} \tag{5.16}$$

Setting, $r = 1/2,\ 0,\ a_0 = 1$, we get

$$\begin{aligned}y_1 &= \sqrt{x}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(5\cdot 9\cdots(4n+1))n!},\\ y_2 &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(3\cdot 7\cdot\cdots(4n-1))n!}.\end{aligned}$$

The infinite series have an infinite radius of convergence since $x = 0$ is the only singular point of the DE.

Example 2. The DE $xy'' + y' + y = 0$ has a regular singular point at $x = 0$ with $xp(x) = 1$, $x^2q(x) = x$. Noting that

$$p_0 = 1,\, p_1 = p_2 = \cdots = 0,$$

and

$$q_0 = 0,\, q_1 = 1,\, q_2 = q_3 = q_4 = \ldots = 0,$$

we get the indicial equation:

$$r(r-1)+r=r^2=0.$$

This equation has only one root $x=0$. The recursion equation is

$$(n+r)^2 a_n = -a_{n-1}, \quad n \geq 1.$$

The solution with $a_0 = 1$ is

$$a_n(r) = (-1)^n \frac{1}{(r+1)^2(r+2)^2 \cdots (r+n)^2}.$$

setting $r=0$ gives the solution

$$y_1 = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(n!)^2}.$$

Taking the derivative of $a_n(r)$ with respect to r we get, using

$$a_n'(r) = a_n(r) \frac{d}{dr} \ln\left[a_n(r)\right]$$

(logarithmic differentiation),we get

$$a_n'(r) = -\left(\frac{2}{r+1} + \frac{2}{r+2} + \cdots + \frac{2}{r+n}\right) a_n(r)$$

so that

$$a_n'(0) = 2(-1)^{n+1} \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{(n!)^2}.$$

Therefore a second linearly independent solution is

$$y_2 = y_1 \ln(x) + 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}}{(n!)^2} x^n.$$

The above series converge for all x. Any bounded solution of the given DE must be a scalar multiple of y_1.

Example 3.

$$L[y] = x^2 y'' + x y' + (x^2 - 1) y = 0.$$

This DE has a regular singular point at $x = 0$, the indicial equation is

$$F(r) = r(r-1) + r - \nu^2 = r^2 - \nu^2 = (r-1)(r+1)$$

whose roots are

$$r_1 = 1, \quad r_2 = -1.$$

This is the case (III). Let

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n x^n,$$

we may derive

$$L[y(r, x)] = a_0(r^2 - 1)x^r + a_1[(r + 1)^2 - 1]x^{r+1} + \sum_{n=2}^{\infty} \Big\{[(r + n)^2 - 1]a_n + a_{n-2}\Big\} x^{r+n} = 0.$$

The recursion equations are

$$[(1 + r)^2 - 1]a_1 = 0,$$
$$[(n + r)^2 - 1]a_n = -a_{n-2}, \quad \text{for } n \geq 2.$$

The first solution $y_1(x)$ can be obtained as in the last section. Let $r = r_1 = 1$ and $a_0 = 1$, we have

$$a_1 = 0,$$
$$a_n = -\frac{a_{n-2}}{n(n + 2)}, \quad n \geq 2.$$

It is derived that for the odd numbers,

$$a_1 = a_3 = a_5 = \cdots = 0;$$

while for the even numbers,

$$a_2 = \frac{(-1)a_0}{2^2(2)(1)}, \quad a_4 = \frac{(-1)a_2}{2^2(3)(2)}, \ \cdots$$

In general, for $n = 2m$,

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m + 1)!m!}.$$

We obtain the first solution:

$$y_1 = x \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m + 1)!m!}.$$

In practice, one define the Bessel function $J_1(x) = 2y_1(x)$.

The second solution has the form:

$$y_2 = ay_1(x)\ln x + x^{-1}\Big(\sum_{n=0}^{\infty} c_n x^n\Big),$$

Here, as it is known from the Frobenius theorem, $c_0 = 1$.

We now compute

$$\begin{aligned}
y_2'(x) &= ay_1'(x)\ln x + ay_1(\ln x)' + x^{-1}\sum_{n=0}^{\infty} nc_n x^{n-1} \\
&\quad -x^{-2}\Big(\sum_{n=0}^{\infty} c_n x^n\Big), \\
y_2''(x) &= ay_1''(x)\ln x + 2ay_1'(\ln x)' + ay_1(\ln x)'' \\
&\quad +x^{-1}\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x^{-2}\sum_{n=0}^{\infty} nc_n x^{n-1} \\
&\quad +2x^{-3}\sum_{n=0}^{\infty} c_n x^n,
\end{aligned}$$

Then we let

$$L[y_2] = p_0(x)y_2'' + p_1(x)y_2' + p_2(x)y_2 = 0.$$

It follows that

$$\begin{aligned}
L[y_2] &= a\ln x\Big[p_0(x)y_1'' + p_1(x)y_1' + p_2(x)y_1\Big] \\
&\quad +\Big[x^2 ay_1(\ln x)'' + xay_1(\ln x)'\Big] + 2axy_1' \\
&\quad +\sum_{n=0}^{\infty} [(n-1)(n-2)c_n + (n-1)c_n - c_n]x^{n-1} \\
&\quad +\sum_{n=0}^{\infty} c_n x^{n+1}.
\end{aligned}$$

Note that

$$L[y_1] = p_0(x)y_1'' + p_1(x)y_1' + p_2(x)y_1 = 0,$$

and

$$x^2 ay_1(\ln x)'' + xay_1(\ln x)' = 0,$$

we obtain

$$\begin{aligned}
L[y_2] &= 2axy_1' \\
&\quad +\sum_{n=0}^{\infty} [(n-1)(n-2)c_n + (n-1)c_n - c_n]x^{n-1} \\
&\quad +\sum_{n=0}^{\infty} c_n x^{n+1},
\end{aligned}$$

or,

$$\sum_{n=0}^{\infty}[(n-1)(n-2)c_n+(n-1)c_n-c_n]x^{n-1}$$
$$+\sum_{n=0}^{\infty}c_nx^{n+1}=-2ax\sum_{m=0}^{\infty}\frac{(-1)^m(2m+1)x^{2m}}{2^{2m}(m+1)!m!}.$$

It may re-written as

$$-c_1+[0\cdot c_2+c_0]x+\sum_{n=2}^{\infty}[(n^2-1)c_{n+1}-c_{n-1}]x^n$$
$$=-2a\left[x+\sum_{m=1}^{\infty}\frac{(-1)^m(2m+1)x^{2m+1}}{2^{2m}(m+1)!m!}\right].$$

It follows that

$$c_1=0,\ a=-c_0/2,$$

Since $c_0=1$, we have $a=-\frac{1}{2}$. Furthermore, we have

$$c_3=c_5=\cdots=0.$$

Moreover, let $n=2m+1$, we have

$$[(2m+1)^2-1]c_{2m+2}-c_{2m}$$
$$=\frac{(-1)^m(2m+1)x^{2m+1}}{2^{2m}(m+1)!m!},\quad m=1,2,\cdots$$

As $m=1$, we have

$$(3^2-1)c_4+c_2=(-1)3/(2^2\cdot 2!).$$

$$c_2\Rightarrow c_4\Rightarrow c_6\Rightarrow\cdots$$

So that, c_2 is arbitrary constant. In practice, one set $c_2=1/2^2$. Thus, we have

$$c_4=\frac{-1}{2^4\cdot 2}\left[\frac{3}{2}+1\right]=\frac{-1}{2^4\cdot 2!}\left[\left(1+\frac{1}{2}\right)+1\right]$$

We define

$$H_0=1,\ H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n},\quad (n\geq 1).$$

Then, we have

$$c_4=\frac{-1}{2^4\cdot 2!}\left(H_2+H_1\right).$$

In general, one may write

$$c_{2m} = \frac{(-1)^{m+1}}{2^{2m} m!(m-1)!} \left(H_m + H_{m-1}\right).$$

We finally obtain:

$$\begin{aligned} y_2(x) &= -\tfrac{1}{2} y_1(x) \ln x \\ &+ \frac{1}{x} \left[1 + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \left(H_m + H_{m-1}\right) x^{2m}}{2^{2m} m!(m-1)!} \right]. \end{aligned}$$

3. (*)Bessel Equation

In this lecture we study an important class of functions which are defined by the differential equation

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0,$$

where $\nu \geq 0$ is a fixed parameter. This DE is known **Bessel's equation of order** ν.

This equation has $x = 0$ as its only singular point. Moreover, this singular point is a regular singular point since

$$x p(x) = 1, \quad x^2 q(x) = x^2 - \nu^2.$$

Bessel's equation can also be written

$$y'' + \frac{1}{x} y' + (1 - \frac{\nu^2}{x^2}) = 0$$

which for x large is approximately the DE $y'' + y = 0$ so that we can expect the solutions to oscillate for x large. The indicial equation is

$$r(r-1) + r - \nu^2 = r^2 - \nu^2$$

whose roots are

$$r_1 = \nu, \quad r_2 = -\nu.$$

Note that

$$p_0 = 1, \, p_1 = p_2 = \cdots = 0,$$

and

$$q_0 = -\nu^2, \, q_1 = 0, \, q_2 = 1, \, q_3 = q_4 = \ldots = 0,$$

The recursion equations are

$$[(1+r)^2-\nu^2]a_1 = 0,$$
$$[(n+r)^2-\nu^2]a_n = -a_{n-2},$$
$$\text{for } n \geq 2.$$

The general solution of these equations is

$$a_{2n+1} = 0, \quad \text{for } n \geq 0$$

and

$$a_{2n}(r) = \frac{(-1)^n a_0}{(r+2-\nu)(r+4-\nu)\cdots(r+2n-\nu)} \times \frac{1}{(r+2+\nu)(r+4+\nu)\cdots(r+2n+\nu)}.$$

3.1 The Case of Non-integer ν

If ν is not an integer and $\nu \neq 1/2$, we have the case (I). Two linearly independent solutions of Bessel's equation

$$J_\nu(x), \quad J_{-\nu}(x)$$

can be obtained by taking $r = \pm\nu$, $a_0 = 1/2^\nu\Gamma(\nu+1)$. Since, in this case,

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(r+1)(r+2)\cdots(r+n)},$$

we have for $r = \pm\nu$

$$J_r(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(r+n+1)} \left(\frac{x}{2}\right)^{2n+r}.$$

Recall that the Gamma function $\Gamma(x)$ is defined for $x \geq -1$ by

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x \, dt.$$

For $x \geq 0$ we have

$$\Gamma(x+1) = x\Gamma(x),$$

so that

$$\Gamma(n+1) = n!$$

for n an integer ≥ 0. We have

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t} t^{-1/2} dt = 2\int_0^\infty e^{-x^2} dt = \sqrt{\pi}.$$

The Gamma function can be extended uniquely to a function which satisfies the identity

$$\Gamma(x) = \Gamma(x)/x,$$

for all x, but

$$x \neq \{0, -1, -2, \dots, -n, \dots\}.$$

This is true even if x is taken to be complex. The resulting function is analytic except at zero and the negative integers where it has a simple pole.

These functions are called **Bessel functions of first kind of order ν.**

As an exercise the reader can show that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x), \quad J_{-\frac{1}{2}} = \sqrt{\frac{2}{\pi x}} \cos(x).$$

3.2 The Case of $\nu = -m$ with m an integer ≥ 0

For this case, the first solution $J_m(x)$ can be obtained as in the last section. As examples, we give some such solutions as follows:

- The Case of $m = 0$:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

- The case $m = 1$:

$$J_1(x) = \frac{1}{2} y_1(x) = \frac{x}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} n!(n+1)!} x^{2n}.$$

To derive the second solution, one has to proceed differently. For $\nu = 0$ the indicial equation has a repeated root, we have the case (II). One has a second solution of the form

$$y_2 = J_0(x) \ln(x) + \sum_{n=0}^{\infty} a'_{2n}(0) x^{2n}$$

where

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)^2(r+4)^2 \cdots (r+2n)^2}.$$

It follows that

$$\frac{a'_{2n}(r)}{a_{2n}} = -2 \left(\frac{1}{r+2} + \frac{1}{r+4} + \cdots \frac{1}{r+2n} \right)$$

so that

$$a'_{2n}(0) = -\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) a_{2n}(0) = -h_n a_{2n}(0),$$

where we have defined

$$h_n = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

Hence

$$y_2 = J_0(x)\ln(x) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} h_n}{2^{2n}(n!)^2} x^{2n}.$$

Instead of y_2, the second solution is usually taken to be a certain linear combination of y_2 and J_0. For example, the function

$$Y_0(x) = \frac{2}{\pi}\Big[y_2(x) + (\gamma - \ln 2)J_0(x)\Big],$$

where

$$\gamma = \lim_{n\to\infty}(h_n - \ln n) \approx 0.5772$$

, is known as the **Weber function of order** 0. The constant γ is known as Euler's constant; it is not known whether γ is rational or not.

If $\nu = -m$, with $m > 0$, the the roots of the indicial equation differ by an integer, we have the case (III). Then one has a solution of the form

$$y_2 = aJ_m(x)\ln(x) + \sum_{n=0}^{\infty} b'_{2n}(-m)x^{2n+m}$$

where $b_{2n}(r) = (r+m)a_{2n}(r)$ and $a = b_{2m}(-m)$. In the case $m = 1$ we have $a_0 = 1$,

$$a = b_2(-1) = -\frac{a_0}{2},$$

$$b_0(r) = (r - r_2)a_0$$

and for $n \geq 1$,

$$b_{2n}(r) = \frac{(-1)^n a_0}{(r+3)(r+5)\cdots(r+2n-1)(r+3)(r+5)\cdots(r+2n+1)}.$$

Subsequently, we have

$$b'_0(r) = a_0$$

and for $n \geq 1$,

$$b'_{2n}(r) = -\Big(\tfrac{1}{r+3} + \tfrac{1}{r+5} + \cdots + \tfrac{1}{r+2n-1} + \tfrac{1}{r+3} + \tfrac{1}{r+5} + \cdots + \tfrac{1}{r+2n+1}\Big)b_{2n}(r).$$

From here, we obtain

$$\begin{aligned} b_0'(-1) &= a_0 \\ b_{2n}'(-1) &= \tfrac{-1}{2}(h_n + h_{n-1})b_{2n}(-1), \quad (n \geq 1), \end{aligned} \tag{5.17}$$

where

$$b_{2n}(-1) = \frac{(-1)^n}{2^{2n}(n-1)!n!}a_0.$$

So that

$$\begin{aligned} y_2 &= \tfrac{-1}{2}y_1(x)\ln(x) \\ &\quad + \tfrac{1}{x}\Big[1 + \textstyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n+h_{n-1})}{2^{2n+1}(n-1)!n!}x^{2n}\Big] \\ &= -J_1(x)\ln(x) \\ &\quad + \tfrac{1}{x}\Big[1 + \textstyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(h_n+h_{n-1})}{2^{2n+1}(n-1)!n!}x^{2n}\Big] \end{aligned}$$

where, by convention, $h_0 = 0$, $(-1)! = 1$.

The **Weber function of order 1** is defined to be

$$Y_1(x) = \frac{4}{\pi}\Big[-y_2(x) + (\gamma - \ln 2)J_1(x)\Big].$$

The case $m > 1$ is slightly more complicated and will not be treated here.

The second solutions $y_2(x)$ of Bessel's equation of order $m \geq 0$ are unbounded as $x \to 0$.

It follows that

- Any solution of Bessel's equation of order $m \geq 0$ which is bounded as $x \to 0$ is a scalar multiple of J_m.
- The solutions which are unbounded as $x \to 0$ are called **Bessel functions of the second kind**.

Refer to the Reference Book by Boyce & DiPrima.

4. Behaviors of Solutions near the Regular Singular Point $x = 0$

In terms of the Frobenius Theorem, one may predict the behaviors of solutions near the singular point, say, $x = 0$, without solving the equation.

Assume that from the indicial equation, one finds the roots r_1, r_2. Then there are several possibilities.

4.1 Case (I): $r_1 - r_2 \neq N$

(i). ($r_1 \geq 0, r_2 < 0$): We have

$$y_1(x) \sim |x|^{r_1} \sim O(1), \quad (x \to 0)$$

bounded and

$$y_2(x) \sim |x|^{r_2} \sim \infty, \quad (x \to 0)$$

monotonically, unbounded.

(ii). ($r_1 > 0, r_2 \geq 0$): We have

$$y_1(x) \sim |x|^{r_1} \sim 0, \quad (x \to 0),$$

and

$$y_2(x) \sim |x|^{r_2} \sim O(1), \quad (x \to 0).$$

monotonically. All solutions are bounded.

(iii). The roots are complex conjugate, ($r_{1,2} = \lambda \pm \mathrm{i}\mu$):

If $\lambda < 0$, all solutions are unbounded and oscillatory,

$$\begin{aligned} y_{1,2}(x) &\sim |x|^{\lambda \pm \mathrm{i}\mu} \\ &= |x|^{\lambda}\{\cos(\mu \ln |x|) \pm \mathrm{i}\sin(\mu \ln |x|)\} \\ &\to \infty, \quad (x \to 0). \end{aligned}$$

If $\lambda \geq 0$, all solution are bounded and oscillatory,

$$\begin{aligned} y_{1,2}(x) &\sim |x|^{\lambda \pm \mathrm{i}\mu} \\ &= |x|^{\lambda}\{\cos(\mu \ln |x|) \pm \mathrm{i}\sin(\mu \ln |x|)\} \\ &= O(1), \quad (x \to 0). \end{aligned}$$

4.2 Case (II): $r_1 = r_2$

(i). ($r_1 > 0$): We have

$$y_1(x) \sim |x|^{r_1} \sim 0, \quad (x \to 0)$$

is bounded and

$$y_2(x) \sim |x|^{r_1} \ln |x| \sim 0, \quad (x \to 0)$$

is also bounded.

(ii). ($r_1 = r_2 = 0$): We have

$$y_1(x) = O(1), \quad (x \to 0),$$

bounded, but

$$y_2(x) \sim \ln |x| \to \infty, \quad (x \to 0).$$

is unbounded.

4.3 Case (III): $r_1 - r_2 = N \neq 0$

(i). $(r_1 > r_2 > 0)$: All solutions are bounded:

$$y_1(x) \sim |x|^{r_1} \sim 0, \quad (x \to 0)$$

is bounded and

$$y_2(x) \sim a|x|^{r_1} \ln|x| + |x|^{r_2} = 0, \quad (x \to 0).$$

(ii). $(r_1 > r_2 = 0)$: All solutions are bounded:

$$y_1(x) \to 0, \quad (x \to 0),$$

and

$$y_2(x) \to O(1), \quad (x \to 0).$$

(iii). $(r_1 > 0 > r_2)$: Some solutions are bounded:

$$y_1(x) \to 0, \quad (x \to 0),$$

and some solutions are unbounded:

$$y_2(x) \to \infty, \quad (x \to 0).$$

Chapter 6

(∗) SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS: EIGENVECTOR METHOD

1. Introduction

Let us consider the system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}. \tag{6.1}$$

with an arbitrary $n \times n$ matrix A.

Suppose that the solution can be written in the form:

$$\mathbf{x}(t) = \mathbf{v}\, e^{\lambda t}, \tag{6.2}$$

where λ is a parameter, while

$$\mathbf{v} = [v_1, v_2, \cdots, v_n]^T$$

is a constant column vector, both are to be determined. Substituting (6.2) into (6.1), we derive

$$A\mathbf{v} = \lambda \mathbf{v},$$

or

$$(A - \lambda I)\mathbf{v} = 0. \tag{6.3}$$

It is known from linear algebra that the linear homogeneous equation (6.3) has non-trivial solution, if and only if the determinant

$$\det(A - \lambda I) = P(\lambda) = 0. \tag{6.4}$$

The polynomial of λ of degree n-th, $P(\lambda)$, is called the **characteristic polynomial**. The roots of $P(\lambda)$ are called the **eigenvalues**, while the non-trivial solution $\mathbf{v}$ corresponding λ is called the **eigenvectors** associated with the eigenvalue λ.

In the following, we consider the special case: $n = 2$.

1.1 (2 × 2) System of Linear Equations

In case $n = 2$, we have

$$P(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0. \tag{6.5}$$

The characteristic equation (6.5) has three roots and there are three cases depending on whether the discriminant

$$\Delta = \text{tr}(A)^2 - 4\det(A)$$

of the characteristic polynomial $P(\lambda)$ is > 0, < 0, $= 0$.

1.2 Case 1: $\Delta > 0$

In this case the roots λ_1, λ_2 of the characteristic polynomial are real and unequal, say $\lambda_1 < \lambda_2$. Let $\mathbf{v}_i$ be an eigenvector with eigenvalues.

$$\lambda_{1,2} = \frac{\text{tr}(A)}{2} \pm \frac{\sqrt{\Delta}}{2}$$

By solving linear equation:

$$(A - \lambda_i I)\mathbf{v}_i = 0, \quad (i = 1, 2)$$

one can derive that

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ v_{12} \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ v_{22}. \end{pmatrix}$$

$$v_{12} = \frac{1}{a_{12}}\left[\lambda_1 - a_{11}\right],$$
$$v_{22} = \frac{1}{a_{12}}\left[\lambda_2 - a_{11}\right].$$

The two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear independent and the matrix

$$P = [\mathbf{v}_1, \mathbf{v}_2]$$

is invertible. In the other words, $\det(P) \neq 0$ and the inverse P^{-1} exists.

Note that the equations

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1,$$

$$A\mathbf{v}_2 = \lambda_1\mathbf{v}_2,$$

can be written in the matrix form:

$$A[\mathbf{v}_1, \mathbf{v}_2] = [\mathbf{v}_1, \mathbf{v}_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

or

$$AP = P\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}.$$

It shows that

$$P^{-1}AP = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}.$$

If we make the change of variable

$$\mathbf{x} = P\mathbf{u}$$

with

$$\mathbf{u} = \begin{pmatrix}u_1\\ u_2\end{pmatrix},$$

our system becomes

$$P\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = AP\mathbf{u}$$

or

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = P^{-1}AP\mathbf{u} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\mathbf{u}.$$

Hence, our system reduces to the uncoupled system

$$\begin{aligned}\frac{\mathrm{d}u_1}{\mathrm{d}t} &= \lambda_1 u_1,\\ \frac{\mathrm{d}u_2}{\mathrm{d}t} &= \lambda_2 u_2\end{aligned}$$

which has the general solution

$$\begin{aligned}u_1 &= c_1 e^{\lambda_1 t},\\ u_2 &= c_2 e^{\lambda_2 t}.\end{aligned}$$

Thus the general solution of the given system is

$$\mathbf{x} = P\mathbf{u} = u_1\mathbf{v}_1 + u_2\mathbf{v}_2 = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

1.3 Case 2: $\Delta < 0$

In this case the roots of the characteristic polynomial are complex numbers

$$\lambda = \alpha \pm i\omega = \mathrm{tr}(A)/2 \pm i\sqrt{|\Delta|}/2.$$

The corresponding eigenvectors of A, like the case (1), can be still expressed as a (complex) scalar multiples of

$$\mathbf{v}_1 = \begin{pmatrix}1\\ v_{12}\end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix}1\\ v_{22}.\end{pmatrix}$$

$$v_{12} = \frac{1}{a_{12}} [\lambda_1 - a_{11}],$$
$$v_{22} = \frac{1}{a_{12}} [\lambda_2 - a_{11}].$$

or say,

$$\mathbf{v}_{1,2} = \begin{pmatrix} 1 \\ \sigma \pm i\tau \end{pmatrix}$$

where

$$\sigma = (\alpha - a_{11})/a_{12}, \quad \tau = \omega/a_{12}.$$

The general complex solutions is

$$\mathbf{z} = \frac{1}{2}(c_1 + ic_2)e^{\alpha t}(\cos(\omega t) + i\sin(\omega t)) \begin{pmatrix} 1 \\ \sigma + i\tau \end{pmatrix},$$

where c_1, c_2 are arbitrary constants.

For $\mathbf{x}$ is the general real solution, we must have

$$\mathbf{x} = \mathbf{z} + \overline{\mathbf{z}}.$$

It follows that

$$\mathbf{x} = e^{\alpha t}(c_1 \cos(\omega t) - c_2 \sin(\omega t)) \begin{pmatrix} 1 \\ \sigma \end{pmatrix}$$
$$+ e^{\alpha t}(c_1 \sin(\omega t) + c_2 \cos(\omega t)) \begin{pmatrix} 0 \\ \tau \end{pmatrix}.$$

1.4 Case 3: $\Delta = 0$

Here the characteristic polynomial has only one root (multiple root)

$$\lambda = \lambda_1 = \mathrm{tr}(A)/2.$$

There were **two distinct cases**.

Case (I): $A = \lambda I$. From equation,

$$(A - \lambda_1 I)\mathbf{v} = 0.$$

In this case, $\mathrm{rank}\{A - \lambda_1 I\} = 0$. So that, the number of eigenvector corresponding to the eigenvalue λ_1 is $n_1 = 2 - 0 = 2$; any vector

$$\mathbf{v} \in \mathcal{R}^2$$

will be a eigenvector corresponding to the eigenvalue λ_1. Thus, corresponding to the multiple eigenvalue λ_1, one has **two linear independent eigenvectors**:

$$\mathbf{v}_1 = [1, 0], \quad \mathbf{v}_2 = [0, 1].$$

We derive the general solution:

$$\mathbf{x} = c_1 \mathbf{v}_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathbf{v}_2 \mathrm{e}^{\lambda_1 t}.$$

The same result can be obtained through another way. In fact, the system for this case is reduced to

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \lambda x_1, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = \lambda x_2.$$

which has the general solution

$$\begin{aligned} x_1 &= c_1 e^{\lambda_1 t} \\ x_2 &= c_2 e^{\lambda_1 t}. \end{aligned}$$

Case (II):

$$A \neq \lambda_1 I.$$

The eigenvector $\mathbf{v}_1$ corresponding to eigenvalue λ_1 satisfies:

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1.$$

In this case, $\operatorname{rank}\{A - \lambda_1 I\} = r = 1$, the number of eigenvector corresponding to the eigenvalue λ_1 is $n_1 = n - r = 2 - 1 = 1$; So, **only one eigenvector** $\mathbf{v}_1$ corresponding the multiple eigenvalue λ_1 can be found from

$$(A - \lambda_1 I)\mathbf{v}_2 = 0,$$

which leads to only one linearly independent solution:

$$\mathbf{x}_1 = \mathrm{e}^{\lambda_1 t} \mathbf{v}_1.$$

We assume the second solution has the following form:

$$\mathbf{x}_2 = \hat{\mathbf{x}}_1 = \mathrm{e}^{\lambda_1 t}(\hat{\mathbf{v}}_{1,0} + t\hat{\mathbf{v}}_{1,1}).$$

It follows that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{x}}_1 &= \mathrm{e}^{\lambda_1 t}\Big[(\lambda_1 \hat{\mathbf{v}}_{1,0} + \hat{\mathbf{v}}_{1,1}) + \lambda_1 t \hat{\mathbf{v}}_{1,1}\Big] \\ &= A\hat{\mathbf{x}}_1 = A\mathrm{e}^{\lambda_1 t}(\hat{\mathbf{v}}_{1,0} + t\hat{\mathbf{v}}_{1,1}). \end{aligned}$$

It holds for all t. Hence, we derive that the two linear independent vectors $\hat{\mathbf{v}}_{1,0}$ and $\hat{\mathbf{v}}_{1,1}$ can be obtained from the equations:

$$A\hat{\mathbf{v}}_{1,1} = \lambda_1 \hat{\mathbf{v}}_{1,1},$$
$$A\hat{\mathbf{v}}_{1,0} = \lambda_1 \hat{\mathbf{v}}_{1,0} + \hat{\mathbf{v}}_{1,1}.$$

It can be proved (the proof is omitted) that that for the case of double root, this second equation must be consistent, though the determinant

$$\det(A - \lambda_1 I) = 0.$$

Example: Given

$$\mathbf{x}' = A\mathrm{x}, \qquad A = \begin{bmatrix} 6 & -8 \\ 2 & -2 \end{bmatrix}.$$

We have

$$\mathrm{der}(A - \lambda I) = \begin{vmatrix} 6-\lambda & -8 \\ 2 & -2-\lambda \end{vmatrix} = (\lambda - 2)^2.$$

The system has a **double** eigenvalues $\lambda = \lambda_1 = 2$, $m_1 = 2$. In this case, for the second solution we need to solve the systems,

$$A\hat{\mathbf{v}}_{1,1} = \lambda_1 \hat{\mathbf{v}}_{1,1},$$
$$A\hat{\mathbf{v}}_{1,0} = \lambda_1 \hat{\mathbf{v}}_{1,0} + \hat{\mathbf{v}}_{1,1},$$

becomes

$$A\hat{\mathbf{v}}_{1,1} = 2\hat{\mathbf{v}}_{1,1},$$
$$A\hat{\mathbf{v}}_{1,0} = 2\hat{\mathbf{v}}_{1,0} + \hat{\mathbf{v}}_{1,1},$$

We solve $\hat{\mathbf{v}}_{1,1} = [c, d]^T$ from the first equation,

$$(A - 2I)\hat{\mathbf{v}}_{1,1} = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} [c, d]^T = 0.$$

It is derived that $\mathrm{rank}\{(A - 2I)\} = 1$ and

$$\hat{\mathbf{v}}_{1,1} = r[2, 1]^T.$$

The second equation for $\hat{\mathbf{v}}_{1,0} = [a, b]^T$ becomes

$$(A - 2I)\hat{\mathbf{v}}_{1,0} = \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} [a, b]^T = r[2, 1]^T.$$

It is seen that this equation is consistent and has the solutions:

$$\hat{\mathbf{v}}_{1,0} = \left[\frac{r}{2} + 2s, s\right],$$

where r, s are arbitrary constants. Let $r = 2, s = 0$, we get

$$\hat{\mathbf{v}}_{1,1} = [4, 2]^T, \quad \hat{\mathbf{v}}_{1,0} = [1, 0]^T.$$

Note: For the case under discussion, one may introduce the matrix

$$P = [\hat{\mathbf{v}}_{1,1}, \hat{\mathbf{v}}_{1,0}],$$

we have

$$AP = P \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Since P is invertible, we may write

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}.$$

Setting as before

$$\mathbf{x} = P\mathbf{u}$$

with

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

our system becomes

$$P\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = AP\mathbf{u}$$

or

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = P^{-1}AP\mathbf{u} = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix} \mathbf{u}.$$

Hence, our system reduces to the uncoupled system

$$\begin{aligned} \tfrac{\mathrm{d}u_1}{\mathrm{d}t} &= \lambda_1 u_1 + u_2, \\ \tfrac{\mathrm{d}u_2}{\mathrm{d}t} &= \lambda_1 u_2 \end{aligned}$$

which has the general solution

$$\begin{aligned} u_2 &= c_2 \mathrm{e}^{\lambda_1 t}, \\ u_1 &= c_1 e^{\lambda_1 t} + c_2 t \mathrm{e}^{\lambda_1 t}. \end{aligned}$$

Thus the general solution of the given system is

$$\begin{aligned} \mathbf{x} = P\mathbf{u} &= u_1 \hat{\mathbf{v}}_{1,1} + u_2 \hat{\mathbf{v}}_{1,0} \\ &= (c_1 + c_2 t)\mathrm{e}^{\lambda_1 t} \hat{\mathbf{v}}_{1,1} + c_2 \mathrm{e}^{\lambda_1 t} \hat{\mathbf{v}}_{1,0}. \end{aligned}$$

2. Solutions for $(n \times n)$ Homogeneous Linear System

One may easily extend the ideas and results given in the previous sections to the general $(n \times n)$ system of linear equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \quad t \in (I) \tag{6.6}$$

where A is $n \times n$ matrix with real elements.

Let us consider the solution in the form:

$$\mathbf{x}(\mathbf{t}) = \{x_1(t), \cdots, x_n(t)\} = \mathrm{e}^{\lambda t}\mathbf{v}.$$

It is derived that that

$$\lambda \mathrm{e}^{\lambda t}\mathbf{v} = A\mathrm{e}^{\lambda t}\mathbf{v}; \quad (A - \lambda I)\mathbf{v} = 0.$$

It implies that λ must be an eigenvalue of A, satisfying

$$P(\lambda) = \det(A - \lambda I) = 0,$$

while $\mathbf{v}$ must be its corresponding eigenvector. There are several cases should be discussed separately.

2.1 Case (I): (A) is non-defective matrix

THEOREM 2.1.1 *If A is non defective, having n real linear independent eigenvectors $\{\mathbf{v}_1, \cdots \mathbf{v}_n\}$ with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$ (not necessarily distinct)*, then the functions

$$\mathbf{x}_k(t) = \mathrm{e}^{\lambda_k t}\mathbf{v}_k, \quad (k = 1, 2, \cdots, n),$$

yield a set of linear independent solutions of (6.6). Hence, the general solution of EQ. is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t).$$

Proof: We have just shown that $\mathbf{x}_k(t)$ are solution. So, we only need to prove the n vector functions are linear independent. This can be done by shown their determinant

$$\det[\mathbf{x}_1, \cdots \mathbf{x}_n] = \mathrm{e}^{(\lambda_1 + \cdots + \lambda_n)t} \det[\mathbf{v}_1, \cdots, \mathbf{v}_n] \neq 0,$$

for all $t \in (I)$.

Example 1: Find the general solution for EQ. $\mathbf{x}' = A\mathbf{x}$ with

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}.$$

solution: It is calculated that

$$P(\lambda) = -(\lambda+1)(\lambda-2)^2,$$

so that A has simple EV $\lambda = -1$ and double EV $\lambda = 2$. We now proceed to find the corresponding eigenvectors.

- For $\lambda_1 = -1$, we solve the system:

$$\begin{aligned} v_1 + 2v_2 - 3v_3 &= 0, \\ -2v_1 + 5v_2 - 3v_3 &= 0 \\ -2v_1 + 2v_2 &= 0, \end{aligned}$$

  which can be reduce to the echelon form:

$$\begin{aligned} v_1 + 2v_2 - 3v_3 &= 0, \\ v_2 - v_3 &= 0 \\ 0 &= 0. \end{aligned}$$

  It is seen $\operatorname{rank}\{A - \lambda_1 I\} = r_1 = 2$, $n_1 = n - r_1 = 3 - 2 = 1$. One derives the solution $\mathbf{v} = (v_1, v_2, v_3) = r(1,1,1)$, where r is arbitrary constant. Thus, we have the corresponding eigenvector

$$\mathbf{v}_1 = (1,1,1).$$

- For $\lambda_2 = 2$, the multiplicity is $m_2 = 2$, the system may reduce to a single EQ:

$$2v_1 - 2v_2 + 3v_3 = 0,$$

  so that, $\operatorname{rank}\{A - \lambda_2 I\} = r_2 = 1$, $n_2 = n - r_2 = 3 - 1 = 2$. The dimension of eigen-space E_2 is $n_2 = \dim(E_2) = 2$.

  One may solve the general solution from the system $(A - \lambda_2 I)\mathbf{v} = 0$,

$$\mathbf{v} = r(1,1,0) + s(-3,0,2),$$

  where r, s are arbitrary constants. Thus, we may set the two eigenvectors

$$\mathbf{v}_2 = (1,1,0), \quad \mathbf{v}_3 = (-3,0,2).$$

Thus, the system is non-defective, we have three L.I. solutions:

$$\mathbf{x}_1 = e^{-t}(1,1,1)^T;$$
$$\mathbf{x}_2 = e^{2t}(1,1,0)^T;$$
$$\mathbf{x}_3 = e^{2t}(-3,0,2)^T.$$

The general solution is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t).$$

2.2 Case (II): (A) has a pair of complex conjugate eigen-values

THEOREM 2.2.1 *If A has a pair of complex conjugate EV's, $(\lambda, \bar{\lambda}) = a \pm ib$, with corresponding complex conjugate eigenvector are* $(\mathbf{v}, \bar{\mathbf{v}}) = \mathbf{v}_{\mathrm{R}} \pm i\mathbf{v}_{\mathrm{I}}$, then the functions

$$\mathbf{x}_1(t) = e^{at}\Big[\cos(bt)\mathbf{v}_{\mathrm{R}} - \sin(bt)\mathbf{v}_{\mathrm{I}}\Big],$$
$$\mathbf{x}_2(t) = e^{at}\Big[\sin(bt)\mathbf{v}_{\mathrm{R}} + \cos(bt)\mathbf{v}_{\mathrm{I}}\Big],$$

yield a pair of linear independent real solutions of (6.6).

Proof: The theorem for case (I) is applicable for the complex EV. Hence, the system has a pair of the complex solutions:

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}; \quad \bar{\mathbf{x}}(t) = e^{\bar{\lambda} t}\bar{\mathbf{v}}.$$

Taking real part and imaginary part, respectively, we obtain the real solutions:

$$\mathbf{x}_1(t) = \Re\{\mathbf{x}(t)\}; \quad \mathbf{x}_2(t) = \Im\{\mathbf{x}(t)\}.$$

2.3 Case (III): (A) is a defective matrix

THEOREM 2.3.1 *Assume that A is defective with multiple eigenvalue λ with the multiplicity $m \geq 1$; the dimension of the eigen-space* $\dim(E) = k < m$, *and the corresponding the eigenvectors in the eigen-space are* $(\mathbf{v}_1, \cdots \mathbf{v}_k)$. Then beside of solutions given by these eigenvectors:

$$\mathbf{x}_i(t) = e^{\lambda t}\mathbf{v}_i, \quad (i = 1, 2, \cdots k).$$

one may look for the following forms $(m-k)$ solutions for the system:

$$
\begin{aligned}
&\hat{\mathbf{x}}_1(t) = \mathrm{e}^{\lambda t}\Big[\hat{\mathbf{v}}_{1,0} + t\hat{\mathbf{v}}_{1,1}\Big],\\
&\hat{\mathbf{x}}_2(t) = \mathrm{e}^{\lambda t}\Big[\hat{\mathbf{v}}_{2,0} + t\hat{\mathbf{v}}_{2,1} + \tfrac{t^2}{2!}\hat{\mathbf{v}}_{2,2}\Big],\\
&\vdots\\
&\hat{\mathbf{x}}_j(t) = \mathrm{e}^{\lambda t}\Big[\hat{\mathbf{v}}_{j,0} + t\hat{\mathbf{v}}_{j,1} + \cdots + \tfrac{t^j}{j!}\hat{\mathbf{v}}_{j,j}\Big],\\
&\vdots\\
&(j = 1, 2, \cdots m-k).
\end{aligned}
$$

To determine the vectors $\{\hat{\mathbf{v}}_{j,i}, (i = 0, 1, \cdots j), (j = 1, 2, \cdots m-k)\}$, we may substitute $\hat{\mathbf{x}}_j(t)$ into EQ (6.6):

$$
\begin{aligned}
\hat{\mathbf{x}}_j'(t) &= \lambda \mathrm{e}^{\lambda t}\Big[\hat{\mathbf{v}}_{j,0} + t\hat{\mathbf{v}}_{j,1} + \cdots + \tfrac{t^j}{j!}\hat{\mathbf{v}}_{j,j}\Big]\\
&\quad + \mathrm{e}^{\lambda t}\Big[\hat{\mathbf{v}}_{j,1} + t\hat{\mathbf{v}}_{j,2} + \cdots + \tfrac{t^{j-1}}{(j-1)!}\hat{\mathbf{v}}_{j,j}\Big]\\
&= A\mathrm{e}^{\lambda t}\Big[\hat{\mathbf{v}}_{j,0} + t\hat{\mathbf{v}}_{j,1} + \cdots + \tfrac{t^j}{j!}\hat{\mathbf{v}}_{j,j}\Big].
\end{aligned}
$$

It holds for all $t \in (I)$. Equating the coefficients of each power term of t of both sides, if follows that

$$
\begin{aligned}
&(A - \lambda I)\hat{\mathbf{v}}_{j,j} = 0,\\
&(A - \lambda I)\hat{\mathbf{v}}_{j,j-1} = \hat{\mathbf{v}}_{j,j}\\
&(A - \lambda I)\hat{\mathbf{v}}_{j,j-2} = \hat{\mathbf{v}}_{j,j-1}\\
&\quad\vdots\\
&(A - \lambda I)\hat{\mathbf{v}}_{j,0} = \hat{\mathbf{v}}_{j,1},\\
&\quad\vdots\\
&\quad(j = 1, 2, \cdots m-k).
\end{aligned}
\tag{6.7}
$$

It is seen that $\hat{\mathbf{v}}_{j,j} \in (E)$ is **one** of the eigenvector corresponding to λ to be determined. It can be proved that the system (6.7) allows a set of non-zero solutions $\{\hat{\mathbf{v}}_{j,i}, (i = 0, 1, \cdots j), (j = 1, 2, \cdots m-k)\}$.

The whole set of solutions $\{\mathbf{x}_i(t); \hat{\mathbf{x}}_j(t)\}$ yield m linear independent solutions of (6.6).

We do not give the proof of the above statement, but show the ideas via the following example.

Example 2: Find the general solution for EQ: $\mathbf{x}' = A\mathbf{x}$ with

$$
A = \begin{bmatrix} 6 & 3 & 6 \\ 1 & 4 & 2 \\ -2 & -2 & -1 \end{bmatrix}.
$$

Solution: It is calculate that $P(\lambda) = (\lambda - 3)^3$. One has EV, $\lambda = 3$, multiplicity $m = 3$. The corresponding eigenvectors $\mathbf{v} = r(-1,1,0) + s(-2,0,1)$, where (r,s) are two arbitrary constants. Hence, we have $\mathbf{v}_1 = (-1,1,0)^T$; $\mathbf{v}_2 = (-2,0,1)^T$, and $\dim(E) = k = 2 < m$. (A) is defective matrix. Thus, we have two solutions:

$$\mathbf{x}_1(t) = \mathrm{e}^{3t}\mathbf{v}_1; \quad \mathbf{x}_2(t) = \mathrm{e}^{3t}\mathbf{v}_2.$$

The third solution can be expressed in the form:

$$\hat{\mathbf{x}}_1(t) = \mathrm{e}^{3t}(\hat{\mathbf{v}}_{1,0} + t\hat{\mathbf{v}}_{1,1}),$$

where $(\hat{\mathbf{v}}_1; \hat{\mathbf{v}}_2)$ are subject to the system:

$$\begin{aligned} &(A - 3I)\hat{\mathbf{v}}_{1,1} = 0, \\ &(A - 3I)\hat{\mathbf{v}}_{1,0} = \hat{\mathbf{v}}_{1,1}. \end{aligned} \tag{6.8}$$

It is seen that $\hat{\mathbf{v}}_{1,1} = r(-1,1,0) + s(-2,0,1)$, r, s are to be determined. Furthermore, let us denote $\hat{\mathbf{v}}_{1,0} = (a,b,c)^T$. (6.8) can be written as

$$\begin{aligned} 3a + 3b + 6c &= -r - 2s \\ a + b + c &= r \\ -2a - 2b - 4c &= s \end{aligned}$$

This is a non-homogeneous linear system with zero determinant. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 3 & 6 & -r-2s \\ 1 & 1 & 2 & r \\ -1 & -2 & -4 & s \end{array}\right]$$

To have a solution, the system must satisfy the consistency condition $\operatorname{rank}\{(A - \lambda I)\} = \operatorname{rank}\{(A - \lambda I)^{\#}\}$, which determines the constants (r,s). By using Gauss elimination method, it is derived that the system is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & r \\ 0 & 0 & 0 & -2(2r+s) \\ 0 & 0 & 0 & 2r+s \end{array}\right] \tag{6.9}$$

The consistency condition requires that

$$2r + s = 0.$$

One may let $r = 1, s = -2$. So that, $\hat{\mathbf{v}}_{1,1} = (3, 1, -2)$. The solution $\hat{\mathbf{v}}_{1,0} = (a, b, c)^T$ can be obtained by solving (6.9):

$$a + b + 2c = 1.$$

It is obtained that $a = 1, b = c = 0$, so that, $\hat{\mathbf{v}}_{1,0} = (1, 0, 0)$.

$$\hat{\mathbf{x}}_1(t) = \mathrm{e}^{3t}[(1, 0, 0)^T + t(3, 1, -2)^T].$$

The three linear independent solutions are $\{\mathbf{x}_1, \mathbf{x}_2, \hat{\mathbf{x}}_1\}$.

3. Solutions for Non-homogeneous Linear System

Given non-homogeneous linear system:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x} + \mathbf{b}(t), \quad t \in (I) \tag{6.10}$$

where A is $n \times n$ matrix with real elements. Assume that the associated homogeneous system:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \quad t \in (I) \tag{6.11}$$

has linear independent solutions: $\{\mathbf{x}_1(t), \cdots \mathbf{x}_n(t)\}$. The general solution of (6.10) is

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) + \mathbf{x}_p(t),$$

$\mathbf{x}_p(t)$ is a particular solution of EQ. (6.10).

3.1 Variation of Parameters Method

To determine a particular solution $\mathbf{x}_p(t)$, we assume

$$\mathbf{x}_p(t) = u_1(t)\mathbf{x}_1(t) + \cdots + u_n(t)\mathbf{x}_n(t) = \mathbf{X}(t)\mathbf{u}(t),$$

where

$$\mathbf{X}(t) = [\mathbf{x}_1, \cdots, \mathbf{x}_n], \quad \mathbf{u} = [u_1(t), \cdots, u_n(t)]^T.$$

By substituting the above into EQ, we have

$$\begin{aligned}&[\mathbf{X}(t)\mathbf{u}(t)]' = \mathbf{X}'(t)\mathbf{u}(t) + \mathbf{X}(t)\mathbf{u}'(t)\\ &= A[\mathbf{X}(t)\mathbf{u}(t)] + \mathbf{b}(t).\end{aligned}$$

Noting that

$$\begin{aligned}&\mathbf{X}'(t) = [\mathbf{x}_1', \cdots, \mathbf{x}_n'] = [A\mathbf{x}_1, \cdots, A\mathbf{x}_n]\\ &= A\mathbf{X}(t),\end{aligned}$$

we derive

$$A\mathbf{X}(t)\mathbf{u}(t) + \mathbf{X}(t)\mathbf{u}'(t) = A[\mathbf{X}(t)\mathbf{u}(t)] + \mathbf{b}(t)$$

or

$$\mathbf{X}(t)\mathbf{u}'(t) = \mathbf{b}(t).$$

Consequently, we obtain:

$$\mathbf{u}'(t) = \mathbf{X}^{-1}\mathbf{b}(t),$$

so that,

$$\mathbf{u}(t) = \int^t \mathbf{X}^{-1}(s)\mathbf{b}(s)\mathrm{d}s,$$

and

$$\mathbf{x}_p(t) = \mathbf{X}(t)\int^t \mathbf{X}^{-1}(s)\mathbf{b}(s)\mathrm{d}s.$$

Example 3: Solve the IVP: $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, $\mathbf{x}(0) = (3, 0)^T$, where

$$A = \begin{bmatrix} 1, 2 \\ 4, 3 \end{bmatrix}; \mathbf{b} = (12\mathrm{e}^{3t}, 18\mathrm{e}^{2t})^T.$$

Solution: $P(\lambda) = \det(A - \lambda I) = (\lambda - 5)(\lambda + 1) = 0$. Hence, EV's of A are $\lambda = -1, 5$. The eigenvectors: one may find that

- For $\lambda = -1$: $\mathbf{v}_1 = r(-1, 1)$;
- For $\lambda = 5$: $\mathbf{v}_2 = s(1, 2)$

This, we have the fundamental solutions:

$$\mathbf{x}_1 = \mathrm{e}^{-t}(-1, 1)^T, \quad \mathbf{x}_2 = \mathrm{e}^{5t}(1, 2)^T.$$

and the matrix of fundamental solutions:

$$X = \begin{bmatrix} -\mathrm{e}^{-t} & \mathrm{e}^{5t} \\ \mathrm{e}^{-t} & 2\mathrm{e}^{5t} \end{bmatrix}.$$

Therefore, the particular solution $\mathbf{x}_p = \mathbf{X}\mathbf{u}$ is subject to the system:

$$\mathbf{X}\mathbf{u}' = \mathbf{b},$$

or

$$\begin{bmatrix} -\mathrm{e}^{-t} & \mathrm{e}^{5t} \\ \mathrm{e}^{-t} & 2\mathrm{e}^{5t} \end{bmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} 12\mathrm{e}^{3t} \\ 18\mathrm{e}^{2t} \end{pmatrix}.$$

This gives the solutions:

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} -8e^{4t} + 6e^{3t} \\ 4e^{-2t} + 6e^{-3t} \end{pmatrix}.$$

So that, we may derive

$$\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -2e^{4t} + 2e^{3t} \\ -2e^{-2t} - 2e^{-3t} \end{pmatrix},$$

and

$$\begin{aligned} \mathbf{x}_p &= \mathbf{Xu} \\ &= \begin{bmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{bmatrix} \begin{pmatrix} -2e^{4t} + 2e^{3t} \\ -2e^{-2t} - 2e^{-3t} \end{pmatrix} \\ &= \begin{pmatrix} -4e^{2t} \\ -2e^{2t} - 6e^{3t} \end{pmatrix}. \end{aligned} \tag{6.12}$$

Example 4: Solve EQ $\mathbf{x}' = A\mathbf{x} + \mathbf{b}(t)$ with

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix},$$

and $\mathbf{b}(t) = (0, 0, e^t)^T$.

Solution: $P(\lambda) = \det(A - \lambda I) = -(\lambda - 2)^2(\lambda + 1) = 0$. It has been found that EV's of A are $\lambda = -1, 2$ and

- For $\lambda_1 = -1$: $\dim(E_1) = n_1 = 1$, $\mathbf{v}_1 = (1, 1, 1)$;
- For $\lambda = 2$: $\dim(E_1) = n_2 = 2$,
  $\mathbf{v}_2 = s(1, 1, 0)$, $\mathbf{v}_3 = (-3, 0, 2)$.

This, we have the fundamental solutions:

$$\begin{aligned} &\mathbf{x}_1 = e^{-t}(1, 1, 1)^T, \quad , \mathbf{x}_2 = e^{2t}(1, 1, 0)^T, \\ &\mathbf{x}_3 = e^{2t}(-3, 0, 2)^T, \end{aligned}$$

and the matrix of fundamental solutions:

$$X(t) = \begin{bmatrix} -e^{-t} & e^{2t} & -3e^{2t} \\ e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & 2e^{2t} \end{bmatrix}.$$

Thus, we have the particular solution $\mathbf{x}_p = \mathbf{X}\mathbf{u}$, which is subject to the system:

$$\begin{bmatrix} -e^{-t} & e^{2t} & -3e^{2t} \\ e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & 2e^{2t} \end{bmatrix} \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e^{t} \end{pmatrix}.$$

This gives the solutions:

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \\ u_3'(t) \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}e^{2t} \\ \frac{1}{2}e^{-t} \\ e^{-t} \end{pmatrix}.$$

So that, we may derive

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{pmatrix} -\frac{3}{4}e^{2t} \\ -\frac{1}{2}e^{-t} \\ -e^{-t} \end{pmatrix}.$$

and

$$\begin{aligned} \mathbf{x}_p &= \mathbf{X}\mathbf{u} \\ &= \begin{bmatrix} -e^{-t} & e^{2t} & -3e^{2t} \\ e^{-t} & e^{2t} & 0 \\ e^{-t} & 0 & 2e^{2t} \end{bmatrix} \begin{pmatrix} -\frac{3}{4}e^{2t} \\ -\frac{1}{2}e^{-t} \\ -e^{-t} \end{pmatrix} \qquad (6.13) \\ &= \begin{pmatrix} \frac{13}{4}e^{t} \\ -\frac{5}{4}e^{t} \\ -\frac{11}{4}e^{t} \end{pmatrix}. \qquad (6.14) \end{aligned}$$

The general solution is:

$$\mathbf{x}(t) = \mathbf{x}_p(t) + C_1\mathbf{x}_1 + +C_2\mathbf{x}_2 + +C_3\mathbf{x}_3.$$