Lecture Notes for Math250: Ordinary Differential Equations

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NB! These notes are used by myself. They are provided to students as a supplement to the textbook. They can not substitute the textbook.

Chapter 1. Introduction

Definition: A *differential equation* is an equation which contains derivatives of the unknown. (Usually it is a mathematical model of some physical phenomenon.)

Two classes of differential equations:

- O.D.E. (ordinary differential equations): linear and non-linear;
- P.D.E. (partial differential equations). (not covered in math250, but in math251)

Some concepts related to differential equations:

- system: a collection of several equations with several unknowns.
- order of the equation: the highest order of derivatives.

- linear or non-linear equations: Let $y(t)$ be the unknown. Then,

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t), \qquad (*)$$

is a linear equations. If the equation can not be written as $(*)$, the it's non-linear.

Two things you must know: identify the linearity and order of an equation.

Example 1. Let $y(t)$ be the unknown. Identify the order and linearity of the following equations.

(a). $(y+t)y' + y = 1$,

(b). $3y' + (t+4)y = t^2 + y''$,

(c). $y''' = \cos(2ty)$,

(d). $y^{(4)} + \sqrt{t}y''' + \cos t = e^y$.

Answer.

Problem	order	linear?
(a). $(y+t)y' + y = 1$	1	No
(b). $3y' + (t+4)y = t^2 + y''$	2	Yes
(c). $y''' = \cos(2ty)$	3	No
(d). $y^{(4)} + \sqrt{t}y''' + \cos t = e^y$	4	No

What is a solution? Solution is a function that satisfied the equation and the derivatives exist.

Example 2. Verify that $y(t) = e^{at}$ is a solution of the IVP (initial value problem)

$$y' = ay, \qquad y(0) = 1.$$

Here $y(0) = 1$ is called the initial condition.

Answer. Let's check if $y(t)$ satisfies the equation and the initial condition:

$$y' = ae^{at} = ay, \qquad y(0) = e^0 = 1.$$

They are both OK. So it is a solution.

Example 3. Verify that $y(t) = 10 - ce^{-t}$ with c a constant, is a solution to $y' + y = 10$.

Answer.

$$y' = -(-ce^{-t}) = ce^{-t}, \qquad y' + y = ce^{-t} + 10 - ce^{-t} = 10. \qquad \text{OK.}$$

Let's try to solve one equation.

Example 4. Consider the equation

$$(t+1)y' = t^2$$

We can rewrite it as (for $t \neq -1$)

$$y' = \frac{t^2}{t+1} = \frac{t^2 - 1 + 1}{t+1} = \frac{(t+1)(t-1)+1}{t+1} = (t-1) + \frac{1}{t+1}$$

To find y, we need to integrate y':

$$y = \int y'(t)dt = \int \left[(t-1) + \frac{1}{t+1}\right] dt = \frac{t^2}{2} - t + \ln|t+1| + c$$

where c is an integration constant which is arbitrary. This means there are infinitely many solutions.

Additional condition: initial condition $y(0) = 1$. (meaning: $y = 1$ when $t = 0$) Then

$$y(0) = 0 + \ln|1| + c = c = 1, \qquad \text{so} \qquad y(t) = \frac{t^2}{2} - t + \ln|t+1| + 1.$$

So for equation like $y' = f(t)$, we can solve it by integration: $y = \int f(t)dt$.

Review on integration:

$$\begin{aligned}
\int x^n \, dx &= \frac{1}{n+1} x^{n+1} + c, \quad (n \neq 1) \\
\int \frac{1}{x} \, dx &= \ln|x| + c \\
\int \sin x \, dx &= -\cos x + c \\
\int \cos x \, dx &= \sin x + c \\
\int e^x \, dx &= e^x + c \\
\int a^x dx &= \frac{a^x}{\ln a} + c
\end{aligned}$$

Integration by parts:

$$\int u\,dv = uv - \int v\,du$$

Chain rule:

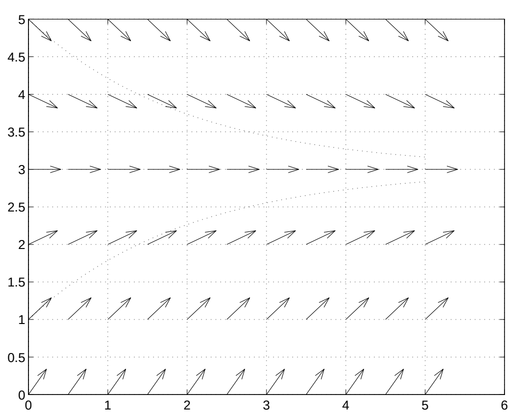
$$\frac{d}{dt}(f(g(t)) = f'(g(t)) \cdot g'(t)$$

Directional field: for first order equations $y' = f(t, y)$.
Interpret y' as the slope of the tangent to the solution $y(t)$ at point (t, y) in the $y - t$ plane.

Example 5. Consider the equation $y' = \dfrac{3-y}{2}$. We know the following:

- If $y = 3$, then $y' = 0$, flat slope,
- If $y > 3$, then $y' < 0$, down slope,
- If $y < 3$, then $y' > 0$, up slope.

See the directional field below (with some solutions sketched):

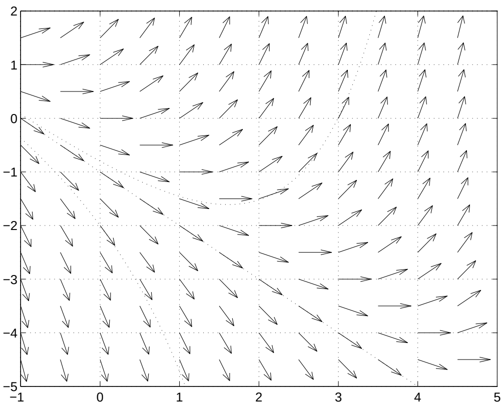


As $t \to \infty$, we have $y \to 3$.

Example 6. $y' = t + y$

- We have $y' = 0$ when $y = -t$,

- We have $y' > 0$ when $y > -t$,
- We have $y' < 0$ when $y < -t$.



What can we say about the solutions?

This depends on the initial condition $y(0) = y_0$.

- If $y(0) > -1$, then $y \to \infty$ as $t \to \pm\infty$.
- If $y(0) < -1$, then $y \to \mp\infty$ as $t \to \pm\infty$.
- If $y(0) = -1$, the $y(t) = -t - 1$.

Chapter 2: First order Differential Equations

We consider the equation

$$\frac{dy}{dt} = f(t, y)$$

Overview:

- Two special types of equations: linear, and separable;
- Linear vs. nonlinear;
- modeling;
- autonomous equations.

2.1: Linear equations; Method of integrating factors

The function $f(t, y)$ is a linear function in y, i.e, we can write

$$f(t, y) = -p(t)y + g(t).$$

So we will study the equation

$$y' + p(t)y = g(t). \tag{A}$$

We introduce the method of integrating factors (due to Leibniz): We multiply equation (A) by a function $\mu(t)$ on both sides

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

The function μ is chosen such that the equation is integrable, meaning the LHS (Left Hand Side) is the derivative of something. In particular, we require:

$$\mu(t)y' + \mu(t)p(t)y = (\mu(t)y)', \quad \Rightarrow \quad \mu(t)y' + \mu(t)p(t)y = \mu(t)y' + \mu'(t)y$$

which requires

$$\mu'(t) = \frac{d\mu}{dt} = \mu(t)p(t), \quad \Rightarrow \quad \frac{d\mu}{\mu} = p(t)\,dt$$

Integrating both sides

$$\ln \mu(t) = \int p(t)\,dt$$

which gives a formula to compute μ

$$\mu(t) = \exp\left(\int p(t)\,dy\right).$$

Therefore, this μ is called the *integrating factor*. Putting back into equation (A), we get

$$\frac{d}{dt}(\mu(t)y) = \mu(t)g(t), \quad \mu(t)y = \int \mu(t)g(t)\,dt + c$$

which give the formula for the solution

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)g(t)\,dt + c\right], \quad \text{where} \quad \mu(t) = \exp\left(\int p(t)\,dt\right).$$

Example 1. Solve $y' + ay = b$ $(a \neq 0)$.

Answer. We have $p(t) = a$ and $g(t) = b$. So

$$\mu = \exp(\int a\,dt) = e^{at}$$

so

$$y = e^{-at}\int e^{at}b\,dt = e^{-at}\left(\frac{b}{a}e^{at} + c\right) = \frac{b}{a} + ce^{-at},$$

where c is an arbitrary constant.

Example 2. Solve $y' + y = e^{2t}$.

Answer. We have $p(t) = 1$ and $g(t) = e^{2t}$. So

$$\mu(t) = \exp(\int 1\,dt) = e^t$$

and

$$y(t) = e^{-t} \int e^t e^{2t}\, dt = e^t \int e^{3t} dt = e^{-t} \left(\frac{1}{3} e^{3t} + c \right) = \frac{1}{3} e^{2t} + c e^{-t}.$$

Example 3. Solve

$$(1+t^2)y' + 4ty = (1+t^2)^{-2}, \quad y(0) = 1.$$

Answer. First, let's rewrite the equation into the normal form

$$y' + \frac{4t}{1+t^2} y = (1+t^2)^{-3},$$

so

$$p(t) = \frac{4t}{1+t^2}, \quad g(t) = (1+t^2)^{-3}.$$

Then

$$\begin{aligned} \mu(t) &= \exp\left(\int p(t)\, dt \right) = \exp\left(\int \frac{4t}{1+t^2} dt \right) \\ &= \exp(2\ln(1+t^2)) = \exp(\ln(1+t^2)^2) = (1+t^2)^2. \end{aligned}$$

Then

$$y = (1+t^2)^{-2} \int (1+t^2)^2 (1+t^2)^{-3} dt = (1+t^2)^{-2} \int (1+t^2)^{-1} dt = \frac{\arctan t + c}{(1+t^2)^2}.$$

By the IC $y(0) = 1$:

$$y(0) = \frac{0+c}{1} = c = 1, \quad \Rightarrow \quad y(t) = \frac{\arctan t + 1}{(1+t^2)^2}.$$

Example 4. Solve $ty' - y = t^2 e^{-t}$, $(t > 0)$.

Answer. Rewrite it into normal form

$$y' - \frac{1}{t} y = t e^{-t}$$

so

$$p(t) = -1/t, \qquad g(t) = t e^{-t}.$$

We have

$$\mu(t) = \exp(\int (-1/t)dt) = \exp(-\ln t) = \frac{1}{t}$$

and

$$y(t) = t\int \frac{1}{t} te^{-t}dt = t\int e^{-t}dt = t(-e^{-t}+c) = -te^{-t}+ct.$$

Example 5. Solve $y - \frac{1}{3}y = e^{-t}$, with $y(0) = a$, and discussion how the behavior of y as $t \to \infty$ depends on the initial value a.

Answer. Let's solve it first. We have

$$\mu = e^{-\frac{1}{3}t}$$

so

$$y = e^{\frac{1}{3}t}\int e^{-\frac{1}{3}t}e^{-t}dt = e^{\frac{1}{3}t}\int e^{-\frac{4}{3}t}dt = e^{\frac{1}{3}t}(-\frac{3}{4}e^{-\frac{4}{3}t}+c).$$

Plug in the IC to find c

$$y(0) = e^0(-\frac{3}{4}+c) = a, \quad c = a + \frac{3}{4}$$

so

$$y(t) = e^{\frac{1}{3}t}\left(-\frac{3}{4}e^{-\frac{4}{3}t} + a + \frac{3}{4}\right) = -\frac{3}{4}e^{-t} + (a+\frac{3}{4})e^{t/3}.$$

To see the behavior of the solution, we see that it contains two terms. The first term e^{-t} goes to 0 as t grows. The second term $e^{t/3}$ goes to ∞ as t grows, but the constant $a + \frac{3}{4}$ is multiplied on it. So we have

- If $a + \frac{3}{4} = 0$, i.e., if $a = -\frac{3}{4}$, we have $y \to 0$ as $t \to \infty$;
- If $a + \frac{3}{4} > 0$, i.e., if $a > -\frac{3}{4}$, we have $y \to \infty$ as $t \to \infty$;
- If $a + \frac{3}{4} < 0$, i.e., if $a < -\frac{3}{4}$, we have $y \to -\infty$ as $t \to \infty$;

Example 6. Solve $ty' + 2y = 4t^2$, $y(1) = 2$.

Answer. Rewrite the equation first

$$y' + \frac{2}{t}y = 4t, \quad (t \neq 0)$$

So $p(t) = 2/t$ and $g(t) = 4t$. We have

$$\mu(t) = \exp(\int 2/t\, dt) = \exp(2 \ln t) = t^2$$

and

$$y(t) = t^{-2} \int 4t \cdot t^2 dy = t^{-2}(t^4 + c)$$

By IC $y(1) = 2$,

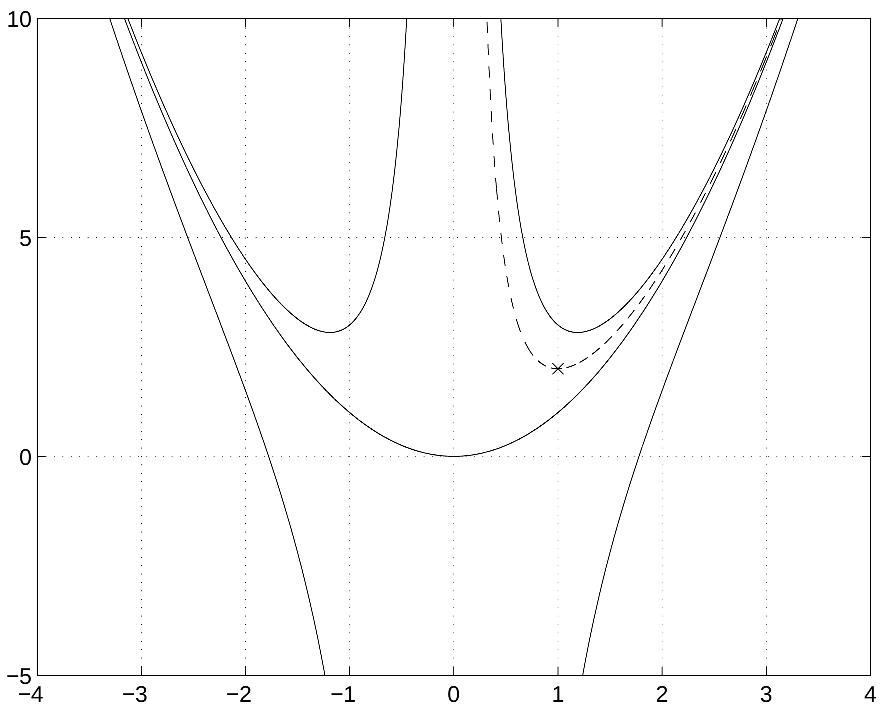
$$y(1) = 1 + c = 2, \quad c = 1$$

we get the solution:

$$y(t) = t^2 + \frac{1}{t^2}, \quad t > 0.$$

Note the condition $t > 0$ comes from the fact that the initial condition is given at $t = 1$, and we require $t \neq 0$.

In the graph below we plot several solutions in the $t - y$ plan, depending on initial data. The one for our solution is plotted with dashed line where the initial point is marked with a 'x'.



2.2: Separable Equations

We study first order equations that can be written as

$$\frac{dy}{dx} = f(x, y) = \frac{M(x)}{N(y)}$$

where $M(x)$ and $N(y)$ are suitable functions of x and y only. Then we have

$$N(y)\, dy = M(x)\, dx, \qquad \Rightarrow \quad \int N(y)\, dy = \int M(x)\, dx$$

and we get implicitly defined solutions of $y(x)$.

Example 1. Consider

$$\frac{dy}{dx} = \frac{\sin x}{1 - y^2}.$$

We can separate the variables:

$$\int (1 - y^2)\, dy = \int \sin x\, dx, \quad \Rightarrow \quad y - \frac{1}{3}y^3 = -\cos x + c.$$

If one has IC as $y(\pi) = 2$, then

$$2 - \frac{1}{3} \cdot 2^3 = -\cos \pi + c, \qquad \Rightarrow \quad c = -\frac{5}{3},$$

so the solution $y(x)$ is implicitly given as

$$y - \frac{1}{3}y^3 + \cos x + \frac{5}{3} = 0.$$

Example 2. Find the solution in explicit form for the equation

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y + 1)}, \qquad y(0) = -1.$$

Answer. Separate the variables

$$\int 2(y - 1)\, dy = \int (3x^2 + 4x + 2)\, dx\,, \quad \Rightarrow \quad (y - 1)^2 = x^3 + 2x^2 + 2x + c$$

Set in the IC $y(0) = -1$, i.e., $y = -1$ when $x = 0$, we get

$$(-1-1)^2 = 0 + c, \qquad c = 4, \qquad (y-1)^2 = x^3 + 2x^2 + 2x + 4.$$

In explicitly form, one has two choices:

$$y(t) = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

To determine which sign is the correct one, we check again by the initial condition:

$$y(0) = 1 \pm \sqrt{4} = 1 \pm 2 = -1$$

We see we must choose the '-' sign. The solution in explicitly form is:

$$y(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.$$

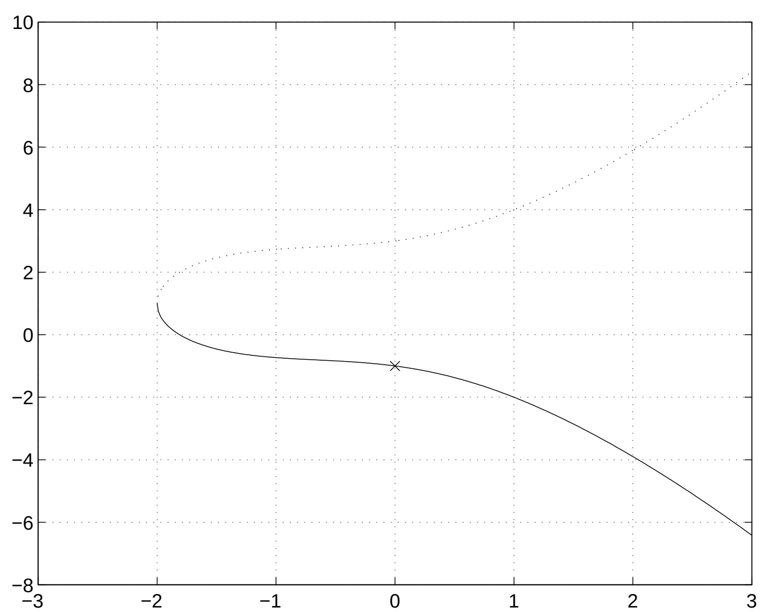
On which interval will this solution be defined?

$$x^3 + 2x^2 + 2x + 4 \geq 0, \quad \Rightarrow \quad x^2(x+2) + 2(x+2) \geq 0$$

$$\Rightarrow \quad (x^2+2)(x+2) \geq 0, \quad \Rightarrow \quad x \geq -2.$$

We can also argue that when $x = -2$, we have $y = 1$. At this point $|dy/dx| \to \infty$, therefore solution can not be defined at this point.

The plot of the solution is given below, where the initial data is marked with 'x'. We also include the solution with the '+' sign, using dotted line.



Example 3. Solve $y' = 3x^2 + 3x^2y^2$, $y(0) = 0$, and find the interval where the solution is defined.

Answer. Let's first separate the variables.

$$\frac{dy}{dx} = 3x^2(1+y^2), \quad \Rightarrow \quad \int \frac{1}{1+y^2} dy = \int 3x^2\, dx, \quad \Rightarrow \quad \arctan y = x^3 + c.$$

Set in the IC:

$$\arctan 0 = 0 + c, \quad \Rightarrow \quad c = 0$$

we get the solution

$$\arctan y = x^3, \quad \Rightarrow \quad y = \tan(x^3).$$

Since the initial data is given at $x = 0$, i.e., $x^3 = 0$, and tan is defined on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, we have

$$-\frac{\pi}{2} < x^3 < \frac{\pi}{2}, \quad \Rightarrow \quad -\left[\frac{\pi}{2}\right]^{1/3} < x < \left[\frac{\pi}{2}\right]^{1/3}.$$

Example 4. Solve

$$y' = \frac{1+3x^2}{3y^2-6y}, \qquad y(0) = 1$$

and identify the interval where solution is valid.

Answer. Separate the variables

$$\int (3y^2 - 6y)dy = \int (1+3x^2)dx \quad y^3 - 3y^2 = x + x^3 + c.$$

Set in the IC: $x = 0, y = 1$, we get

$$1 - 3 = c, \quad \Rightarrow \quad c = -2,$$

Then,

$$y^3 - 3y^2 = x^3 - x - 2.$$

Note that solution is given in implicitly form.

To find the valid interval of this solution, we note that y' is not defined is $3y^2 - 6y = 0$, i.e., when $y = 0$ or $y = 2$. These are the two so-called "bad points" where you can not define the solution. To find the corresponding values of x, we use the solution expression:

$$y = 0: \quad x^3 + x - 2 = 0,$$

$$\Rightarrow \quad (x^2 + x + 2)(x - 1) = 0, \quad \Rightarrow \quad x = 1$$

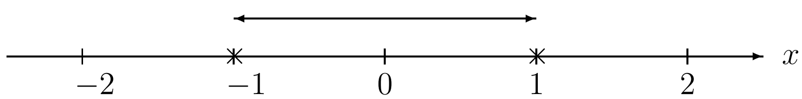
and

$$y = 2: \quad x^3 + x - 2 = -4, \quad \Rightarrow \quad x^3 + x + 2 = 0,$$

$$\Rightarrow \quad (x^2 - x + 2)(x + 1) = 0, \quad \Rightarrow \quad x = -1$$

(Note that we used the facts $x^2 + x + 2 \neq 0$ and $x^2 - x + 2 \neq 0$ for all x.)

Draw the real line and work on it as following:



Therefore the interval is $-1 < x < 1$.

2.4: Differences between linear and nonlinear equations

We will take this chapter before the modeling (ch. 2.3).

For a linear equation

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0,$$

we have the following existence and uniqueness theorem.

Theorem . If $p(t)$ and $g(t)$ are continuous and bounded on an open interval containing t_0, then it has an unique solution on that interval.

Example 1. Find the largest interval where the solution can be defined for the following problems.

(A). $ty' + y = t^3$, $y(-1) = 3$.

Answer. Rewrite: $y' + \frac{1}{t}y = t^2$, so $t \neq 0$. Since $t_0 = -1$, the interval is $t < 0$.

(B). $ty' + y = t^3$, $y(1) = -3$.

Answer. The equation is same as (A), so $t \neq 0$. $t_0 = 1$, the interval is $t > 0$.

(C). $(t-3)y' + (\ln t)y = 2t$, $y(1) = 2$

Answer. Rewrite: $y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$, so $t \neq 3$ and $t > 0$ for the ln function. Since $t_0 = 1$, the interval is then $0 < t < 3$.

(D). $y' + (\tan t)y = \sin t$, $y(\pi) = 100$.

Answer. Since $t_0 = \pi$, and for $\tan t$ to be defined we must have $t \neq \frac{2k+1}{2}\pi$, $k = \pm 1, \pm 2, \cdots$. So the interval is $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

For non-linear equation

$$y' = f(t, y), \quad y(t_0) = y_0,$$

we have the following theorem:

Theorem . If $f(t,y)$, $\frac{\partial f}{\partial y}(t,y)$ are continuous and bounded on an rectangle $(\alpha < t < \beta, a < y < b)$ containing (t_0, y_0), then there exists an open interval around t_0, contained in (α, β), where the solution exists and is unique.

We note that the statement of this theorem is not as strong as the one for linear equation.

Below we give two counter examples.

Example 1. Loss of uniqueness. Consider

$$\frac{dy}{dy} = f(t,y) = -\frac{t}{y}, \qquad y(-2) = 0.$$

We first note that at $y = 0$, which is the initial value of y, we have $y' = f(t,y) \to \infty$. So the conditions of the Theorem are not satisfied, and we expect something to go wrong.

Solve the equation as an separable equation, we get

$$\int y\,dy = -\int t\,dt, \quad y^2 + t^2 = c,$$

and by IC we get $c = (-2)^2 + 0 = 4$, so $y^2 + t^2 = 4$, and $y = \pm\sqrt{4 - t^2}$. Both are solutions. We lose uniqueness of solutions.

Example 2. Blow-up of solution. Consider a simple non-linear equation:

$$y' = y^2, \qquad y(0) = 1.$$

Note that $f(t,y) = y^2$, which is defined for all t and y. But, due to the non-linearity of f, solution can not be defined for all t.

This equation can be easily solved as a separable equation.

$$\int \frac{1}{y^2} dy = \int dt, \qquad -\frac{1}{y} = t + c, \quad y(t) = \frac{-1}{t+c}.$$

By IC $y(0) = 1$, we get $1 = -1/(0+c)$, and so $c = -1$, and

$$y(t) = \frac{-1}{t-1}.$$

We see that the solution *blows up* as $t \to 1$, and can not be defined beyond that point.

This kind of blow-up phenomenon is well-known for nonlinear equations.

2.3: Modeling with first order equations

General modeling concept: derivatives describe "rates of change".

<u>Model I</u>: Exponential growth/decay.

$Q(t)$ = amount of quantity at time t

Assume the rate of change of $Q(t)$ is proportional to the quantity at time t. We can write

$$\frac{dQ}{dt}(t) = r \cdot Q(t), \qquad r: \text{ rate of growth/decay}$$

If $r > 0$: exponential growth
If $r < 0$: exponential decay

Differential equation:

$$Q' = rQ, \quad Q(0) = Q_0\,.$$

Solve it: separable equation.

$$\int \frac{1}{Q} dQ = \int r\,dt, \quad \Rightarrow \quad \ln Q = rt + c, \quad \Rightarrow \quad Q(t) = e^{rt+c} = ce^{rt}$$

Here r is called the *growth rate*. By IC, we get $Q(0) = C = Q_0$. The solution is

$$Q(t) = Q_0 e^{rt}.$$

Two concepts:

- Doubling time T_D (only if $r > 0$): is the time that $Q(T_D) = 2Q_0$.

  $$Q(T_D) = Q_0 e^{rT_D} = 2Q_0, \qquad e^{rT_D} = 2, \quad rT_D = \ln 2, \quad T_D = \frac{\ln 2}{r}.$$

- Half life (or half time) T_H (only for $r < 0$): is the time that $Q(T_H) = \frac{1}{2}Q_0$.

  $$Q(T_H) = Q_0 e^{rT_H} = \frac{1}{2}Q_0, \quad e^{rT_D} = \frac{1}{2}, \quad rT_D = \ln\frac{1}{2} = -\ln 2, \quad T_D = \frac{\ln 2}{-r}.$$

  Note here that $T_H > 0$ since $r < 0$.

NB! T_D, T_H do not depend on Q_0. They only depend on r.

Example 1. If interest rate is 8%, compounded continuously, find doubling time.

Answer. Since $r = 0.08$, we have $T_D = \frac{\ln 2}{0.08}$.

Example 2. A radio active material is reduced to 1/3 after 10 years. Find its half life.

Answer. Model: $\frac{dQ}{dt} = rQ$, r is rate which is unknown. We have the solution $Q(t) = Q_0 e^{rt}$. So

$$Q(10) = \frac{1}{3} Q_0, \qquad Q_0 e^{10r} = \frac{1}{3} Q_0, \qquad r = \frac{-\ln 3}{10}.$$

To find the half life, we only need the rate r

$$T_H = -\frac{\ln 2}{r} = -\ln 2 \frac{10}{-\ln 3} = 10 \frac{\ln 2}{\ln 3}.$$

<u>Model II:</u> Interest rate/mortgage problems.

Example 3. Start an IRA account at age 25. Suppose deposit \$2000 at the beginning and \$2000 each year after. Interest rate 8% annually, but assume compounded continuously. Find total amount after 40 years.

Answer. Set up the model: Let $S(t)$ be the amount of money after t years

$$\frac{ds}{dt} = 0.08S + 2000, \qquad S(0) = 2000.$$

This is a first order linear equation. Solve it by integrating factor

$$S' - 0.08S = 2000, \qquad \mu = e^{-0.08t}$$

$$S(t) = e^{0.08t} \int 2000 \cdot e^{-0.08t} dt = e^{0.08t} \left[2000 \frac{e^{-0.08t}}{-0.08} + c \right] = \frac{2000}{-0.08} + ce^{0.08t}$$

By IC,

$$S(0) = \frac{2000}{-0.08} + c = 2000, \qquad C = 2000(1 + \frac{1}{0.08}) = 27000,$$

we get

$$S(t) = 27000e^{0.08t} - 25000.$$

When $t = 40$, we have

$$S(40) = 27000 \cdot e^{3.2} - 25000 \approx 637,378.$$

Compare this to the total amount invested: $2000 + 2000 * 40 = 82,000$.

Example 4: A home-buyer can pay $800 per month on mortgage payment. Interest rate is 9% annually, (but compounded continuously), mortgage term is 20 years. Determine maximum amount this buyer can afford to borrow.

Answer. Set up the model: Let $Q(t)$ be the amount borrowed (principle) after t years

$$\frac{dQ}{dt} = 0.09Q(t) - 800 * 12$$

The terminal condition is given $Q(20) = 0$. We must find $Q(0)$.

Solve the differential equation:

$$Q' - 0.09Q = -9600, \qquad \mu = e^{-0.09t}$$

$$Q(t) = e^{0.09t}\int (-9600)e^{-0.09t}dt = e^{0.09t}\left[-9600\frac{e^{-0.09t}}{-0.09} + c\right] = \frac{9600}{0.09} + ce^{0.09t}$$

By terminal condition

$$Q(20) = \frac{9600}{0.09} + ce^{0.09*20} = 0, \qquad c = -\frac{9600}{0.09 \cdot e^{1.8}}$$

so we get

$$Q(t) = \frac{9600}{0.09} - \frac{9600}{0.09 \cdot e^{1.8}}e^{0.09t}.$$

Now we can get the initial amount

$$Q(0) = \frac{9600}{0.09} - \frac{9600}{0.09 \cdot e^{1.8}} = \frac{9600}{0.09}(1 - e^{-1.8}) \approx 89,034.79.$$

<u>Model III</u>: Mixing Problem.

Example 5. At $t = 0$, a tank contains Q_0 lb of salt dissolved in 100 gal of water. Assume that water containing 1/4 lb of salt per gal is entering the tank at a rate of r gal/min. At the same time, the well-mixed mixture is draining from the tank at the same rate.

(1). Find the amount of salt in the tank at any time $t \geq 0$.

(2). When $t \to \infty$, meaning after a long time, what is the limit amount Q_L?

Answer. Set up the model:
$Q(t)$ = amount (lb) of salt in the tank at time t (min)
In-rate: r gal/min $\times$ 1/4 lb/gal $= \frac{r}{4}$ lb/min
Out-rate: r gal/min $\times$ $Q(t)/100$ lb/gal $= \frac{Q}{100} r$ lb/min

$$\frac{dQ}{dt} = [\text{In-rate}] - [\text{Out-rate}] = \frac{r}{4} - \frac{r}{100} Q, \qquad \text{IC.} \quad Q(0) = Q_0.$$

(1). Solve the equation

$$Q' + \frac{r}{100} Q = \frac{r}{4}, \qquad \mu = e^{(r/100)t}.$$

$$Q(t) = e^{-(r/100)t} \int \frac{r}{4} e^{(r/100)t} dt = e^{-(r/100)t} \left[\frac{r}{4} e^{(r/100)t} \frac{100}{r} + c \right] = 25 + c e^{-(r/100)t}.$$

By IC

$$Q(0) = 25 + c = Q_0, \qquad c = Q_0 - 25,$$

we get

$$Q(t) = 25 + (Q_0 - 25) e^{-(r/100)t}.$$

(2). As $t \to \infty$, the exponential term goes to 0, and we have

$$Q_L = \lim_{t \to \infty} Q(t) = 25 \text{lb}.$$

Example 6. Tank contains 50 lb of salt dissolved in 100 gal of water. Tank capacity is 400 gal. From $t = 0$, 1/4 lb of salt/gal is entering at a rate of 4 gal/min, and the well-mixed mixture is drained at 2 gal/min. Find:

(1) time t when it overflows;

(2) amount of salt before overflow;

(3) the concentration of salt at overflow.

Answer. (1). Since the inflow rate 4 gal/min is larger than the outflow rate 2 gal/min, the tank will be filled up at t_f:

$$t_f = \frac{400 - 100}{4 - 2} = 150\text{min}.$$

(2). Let $Q(t)$ be the amount of salt at t min.

In-rate: 1/4 lb/gal $\times$ 4 gal/min = 1 lb/min

Out-rate: 2 gal/min $\times$ $\frac{Q(t)}{100+2t}$ lb/gal = $\frac{Q}{50+t}$ lb/min

$$\frac{dQ}{dt} = 1 - \frac{Q}{50+t}, \qquad Q' + \frac{1}{50+t}Q = 1, \qquad Q(0) = 50$$

$$\mu = \exp(\int \frac{1}{50+t}dt) = \exp(\ln(50+t)) = 50+t$$

$$Q(t) = \frac{1}{50+t}\int (50+t)dt = \frac{1}{50+t}[50t + \frac{1}{2}t^2 + c]$$

By IC:

$$Q(0) = c/50 = 50, \qquad c = 2500,$$

We get

$$Q(t) = \frac{50t + t^2/2 + 2500}{50+t}.$$

(3). The concentration of salt at overflow time $t = 150$ is

$$\frac{Q(150)}{400} = \frac{50 \cdot 150 + 150^2/2 + 2500}{400(50+150)} = \frac{17}{64}\text{lb/gal}.$$

<u>Model IV</u>: Air resistance

Example 7. A ball with mass 0.5 kg is thrown upward with initial velocity 10 m/sec from the roof of a building 30 meter high. Assume air resistance is $|v|/20$. Find the max height above ground the ball reaches.

Answer. Let $S(t)$ be the position (m) of the ball at time t sec. Then, the velocity is $v(t) = dS/dt$, and the acceleration is $a = dv/dt$. Let upward be the positive direction. We have by Newton's Law:

$$F = ma = -mg - \frac{v}{20}, \qquad a = -g - \frac{v}{20m} = \frac{dv}{dt}$$

Here $g = 9.8$ is the gravity, and $m = 0.5$ is the mass. We have an equation for v:

$$\frac{dv}{dt} = -\frac{1}{10}v - 9.8 = -0.1(v + 98),$$

so

$$\int \frac{1}{v+98} dv = \int (-0.1) dt, \quad \Rightarrow \quad \ln|v + 98| = -0.1t + c$$

which gives

$$v + 98 = \bar{c}e^{-0.1t}, \quad \Rightarrow \quad v = -98 + \bar{c}e^{-0.1t}.$$

By IC:

$$v(0) = -98 + \bar{c} = 10, \quad \bar{c} = 108, \quad \Rightarrow \quad v = -98 + 108e^{-0.1t}.$$

To find the position S, we use $S' = v$ and integrate

$$S(t) = \int v(t)\, dt = \int (-98 + 108e^{-0.1t}) dt = -98t + 108e^{-0.1t}/(-0.1) + c$$

By IC for S,

$$S(0) = -1080 + c = 30, \qquad c = 1110, \qquad S(t) = -98t - 1080e^{-0.1t} + 1110.$$

At the maximum height, we have $v = 0$. Let's find out the time T when max height is reached.

$$v(T) = 0, \quad -98 + 108e^{-0.1T} = 0, \quad 98 = 108e^{-0.1T}, \quad e^{-0.1T} = 98/108,$$

$$-0.1T = \ln(98/108), \quad T = -10\ln(98/108) = \ln(108/98).$$

So the max height S_M is

$$\begin{aligned} S_M &= S(T) = -980\ln\frac{108}{98} - 1080e^{-0.1\ln(108/98)} + 1110 \\ &= -980\ln\frac{108}{98} - 1080(98/108) + 1110 \approx 34.78 \text{ m.} \end{aligned}$$

Other possible questions:

- Find the time when the ball hit the ground.
  Solution: Find the time $t = t_H$ for $S(t_H) = 0$.

- Find the speed when the ball hit the ground.
  Solution: Compute $|v(t_H)|$.

- Find the total distance traveled by the ball when it hits the ground.
  Solution: Add up twice the max height S_M with the height of the building.

2.5: Autonomous equations and population dynamics

Definition: An autonomous equation is of the form $y' = f(y)$, where the function f for the derivative depends only on y, not on t.

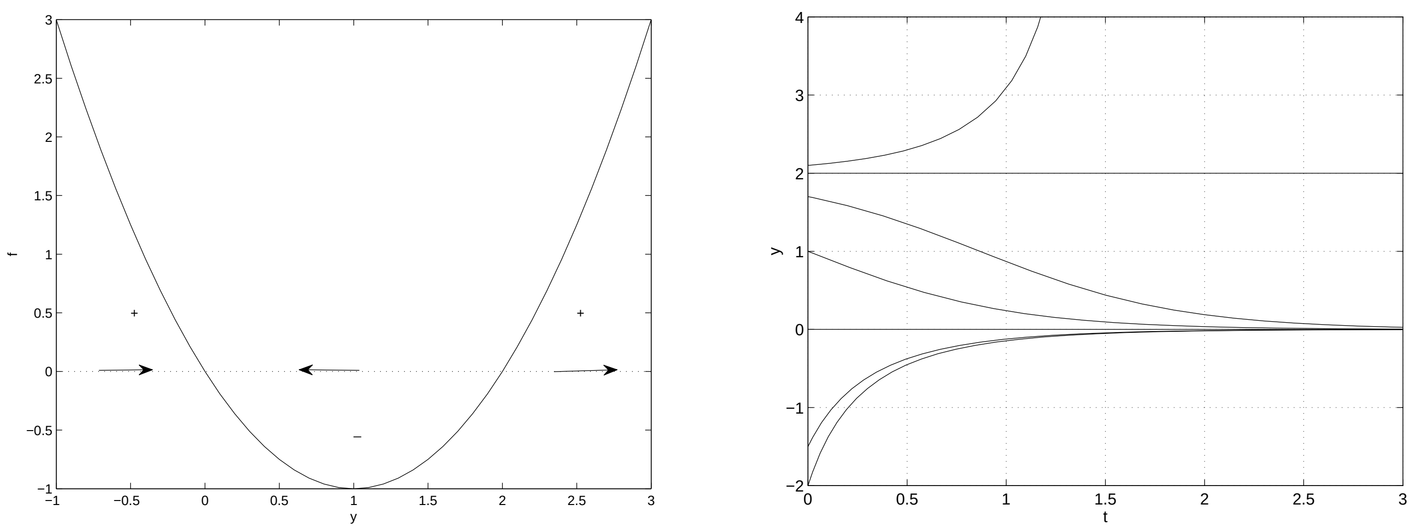
Simplest example: $y' = ry$, exponential growth/decay, where solution is $y = y_0 e^{rt}$.

<u>Definition</u>: Zeros of f where $f(y) = 0$ are called critical points or equilibrium points, or equilibrium solutions.

Why? Because if $f(y_0) = 0$, then $y(t) = y_0$ is a constant solution. It is called an equilibrium.

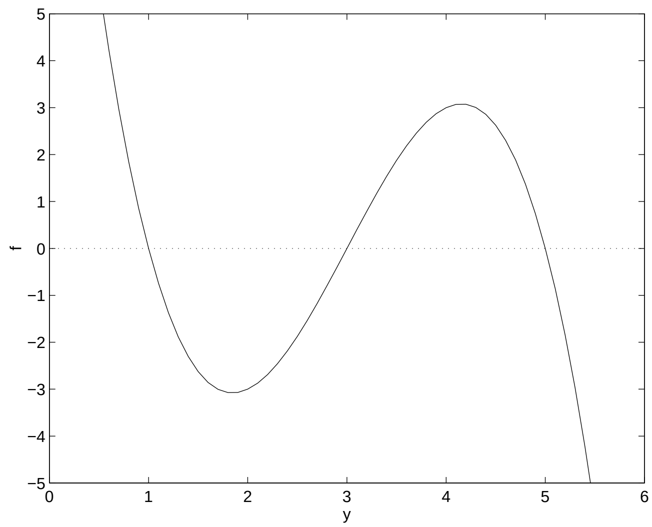
Question: Is an equilibrium stable or unstable?

Example 1. $y' = y(y-2)$. We have two critical points: $y_1 = 0, \quad y_2 = 2$.



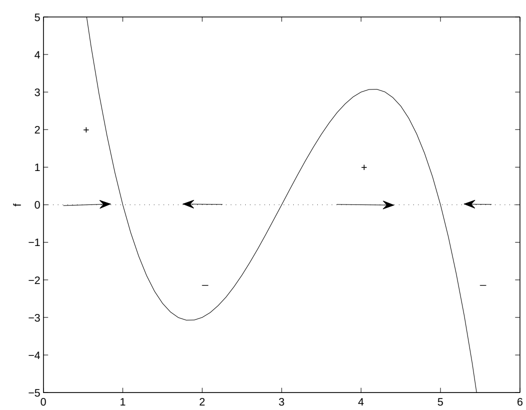
We see that $y_1 = 0$ is stable, and $y_2 = 2$ is unstable.

Example 2. For the equation $y' = f(y)$ where $f(y)$ is given in the following plot:



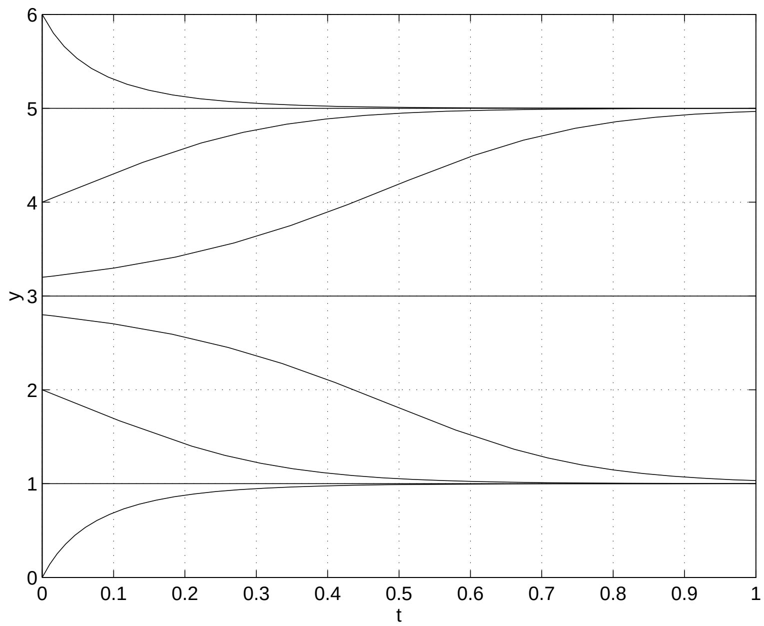
- (A). What are the critical points?
- (B). Are they stable or unstable?
- (C) Sketch the solutions in the $t-y$ plan, and describe the behavior of y as $t \to \infty$ (as it depends on the initial value $y(0)$.)

Answer. (A). There are three critical points: $y_1 = 1,\ y_2 = 3,\ y_3 = 5$.
(B). To see the stability, we add arrows on the y-axis:



We see that $y_1 = 1$ is stable, $y_2 = 3$ is unstable, and $y_3 = 5$ is stable.

(C). The sketch is given below:



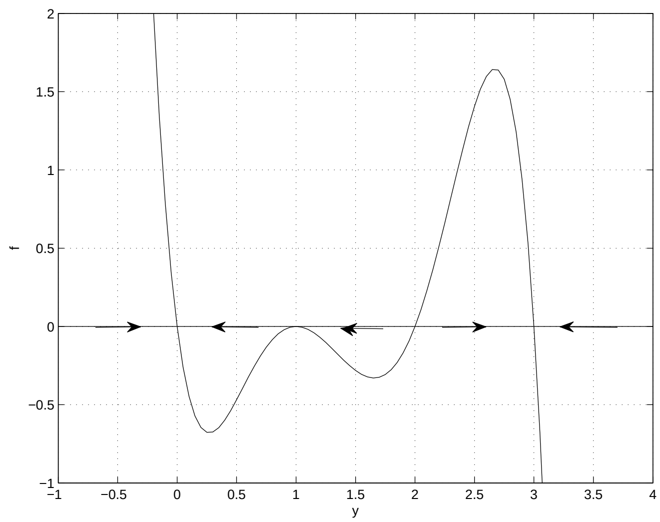
Asymptotic behavior for y as $t \to \infty$ depends on the initial value of y:

- If $y(0) < 1$, then $y(t) \to 1$,
- If $y(0) = 1$, then $y(t) = 1$;
- If $1 < y(0) < 3$, then $y(t) \to 1$;
- If $y(0) = 3$, then $y(t) = 3$;
- If $3 < y(0) < 5$, then $y(t) \to 5$;
- If $y(0) = 5$, then $y(t) = 5$;
- if $y(t) > 5$, then $y(t) \to 5$.

Stability: is not only stable or unstable.

Example 3. For $y' = y^2$, we have only one critical point $y_1 = 0$. For $y < 0$, we have $y' > 0$, and for $y > 0$ we also have $y' > 0$. So solution is increasing on both intervals. So on the interval $y < 0$, solution approaches $y = 0$ as t grows, so it is stable. But on the interval $y > 0$, solution grows and leaves $y = 0$, and it is unstable. This type of critical point is called *semi-stable*. This happens when one has a double root for $f(y) = 0$.

Example 4. For equation $y' = f(y)$ where $f(y)$ is given in the plot

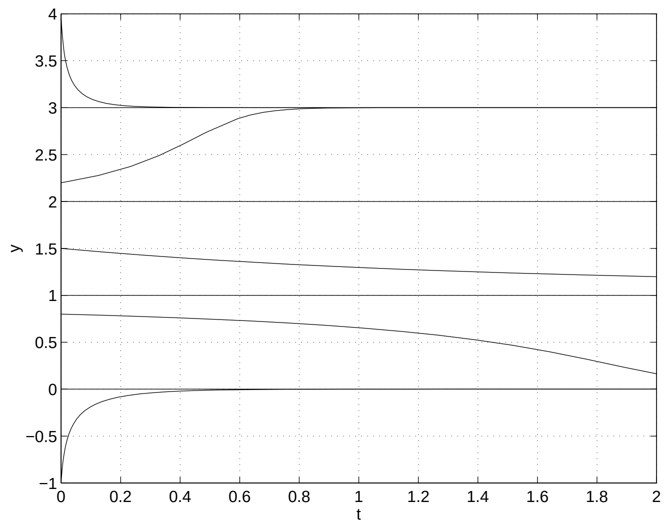


- (A). Identify equilibrium points;
- (B). Discuss their stabilities;
- (C). Sketch solution in $y - t$ plan;
- (D). Discuss asymptotic behavior as $t \to \infty$.

Answer. (A). $y = 0, y = 1, y = 2, y = 3$ are the critical points.

(B). $y = 0$ is stable, $y = 1$ is semi-stable, $y = 2$ is unstable, and $y = 3$ is stable.

(C). The Sketch is given in the plot:



(D). The asymptotic behavior as $t \to \infty$ depends on the initial data.

- If $y(0) < 1$, then $y \to 0$;
- If $1 \le y(0) < 2$, then $y \to 1$;
- If $y(0) = 2$, then $y(t) = 2$;
- If $y(0) > 2$, then $y \to 3$.

Application in population dynamics: let $y(t)$ be the population of a species.

$$\frac{dy}{dt} = (r - ay)y. \qquad \text{the logistic equation}$$

$$\frac{dy}{dt} = r(1 - \frac{y}{k}))y, \qquad k = \frac{r}{a},$$

r=intrinsic growth rate,
k=environmental carrying capacity.

critical points: $y = 0,\ y = k$. Here $y = 0$ is unstable, and $y = k$ is stable.

If $0 < y(0) < k$, then $y \to k$ as t grows.

Chapter 3: Second Order Linear Equations

General form of the equation:

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = b(t),$$

where

$$a_2(t) \neq 0, \qquad y(t_0) = y_0, \quad y'(t_0) = \bar{y}_0.$$

If $b(t) \equiv 0$, we call it *homogeneous*. Otherwise, it is called *non-homogeneous*.

3.1: Homogeneous equations with constant coefficients

This is the simplest case: a_2, a_1, a_0 are all constants, and $g = 0$. Let's write:

$$a_2 y'' + a_1 y' + a_0 = 0.$$

Example 1. Solve $y'' = y = 0$, (we have here $a_2 = 1, a_1 = 0, a_0 = 1$).

Answer. Guess $y_1(t) = e^t$.
Check: $y'' = e^t$, so $y'' - y = e^t - e^t = 0$, ok.

Guess another: $y_2(t) = e^{-t}$.
Check: $y' = -e^{-t}$, so $y'' = e^{-t}$, so $y'' - y = e^t - e^t = 0$, ok.

Observation: Another function $y = c_1 y_1 + c_2 y_2$ for any arbitrary constant c_1, c_2 (this is called a "linear combination of y_1, y_2.) is also a solution.

Check:

$$y = c_1 e^t + c_2 e^{-t},$$

then

$$y' = c_1 e^t - c_2 e^{-t}, \quad y'' = c_1 e^t + c_2 e^{-t}, \quad \Rightarrow \quad y'' - y = 0.$$

Actually this is a general property. It is called *the principle of superposition.*

Theorem *Let $y_1(t)$ and $y_2(t)$ be solutions of*

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0$$

Then, $y = c_1 y_1 + c_2 y_2$ for any constants c_1, c_2 is also a solution.

Proof: If y_1 solves the equation, then

$$a_2(t)y_1'' + a_1(t)y_1' + a_0(t)y_1 = 0. \qquad (I)$$

If y_2 solves the equation, then

$$a_2(t)y_2'' + a_1(t)y_2' + a_0(t)y_2 = 0. \qquad (II)$$

Multiple (I) by c_1 and (II) by c_2, and add them up:

$$a_2(t)(c_1 y_1 + c_2 y_2)'' + a_1(t)(c_1 y_1 + c_2 y_2)' + a_0(t)(c_1 y_1 + c_2 y_2) = 0.$$

Let $y = c_1 y_1 + c_2 y_2$, we have

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = 0$$

therefore y is also a solution to the equation.

How to find the solutions of $a_2 y'' + a_1 y' + a_0 y = 0$?
We seek solutions in the form $y(t) = e^{rt}$. Find r.

$$y' = re^{rt} = ry, \qquad y'' = r^2 e^{rt} = r^2 y$$

$$a_2 r^2 y + a_1 r y + a_0 y = 0$$

Since $y \neq 0$, we get

$$a_2 r^2 + a_1 r^1 + a_0 = 0$$

This is called the *characteristic equation.*

Conclusion: If r is a root of the characteristic equation, then $y = e^{rt}$ is a solution.

If there are two real and distinct roots $r_1 \neq r_2$, then the **general solution** is $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ where c_1, c_2 are two arbitrary constants to be determined by initial conditions (ICs).

Example 2. Consider $y'' - 5y' + 6y = 0$.

- (a). Find the general solution.

- (b). If ICs are given as: $y(0) = -1, y'(0) = 5$, find the solution.
- (c) What happens when $t \to \infty$?

Answer. (a). The characteristic equation is: $r^2 - 5r + 6 =$, so $(r-2)(r-3) = 0$, two roots: $r_1 = 2, r_2 = 3$. General solution is:

$$y(t) = c_1 e^{2t} + c_2 e^{3t}.$$

(b). $y(0) = -1$ gives: $c_1 + c_2 = -1$.
$y'(0) = 5$: we have $y' = 2c_1 e^{2t} + 3c_2 e^{3t}$, so $y'(0) = 2c_1 + 3c_2 = 5$.
Solve these two equations for c_1, c_2: Plug in $c_2 = -1 - c_1$ into the second equation, we get $2c_1 + 3(-1 - c_1) = 5$, so $c_1 = -8$. Then $c_2 = 7$. The solution is

$$y(t) = -8e^{2t} + 7e^{3t}.$$

(c). We see that $y(t) = e^{2t} \cdot (-8 + te^t)$, and both terms in the product go to infinity as t grows. So $y \to \infty$.

Example 3. Find the solution for $2y'' + y' - y = 0$, with initial conditions $y(1) = 0,\ y'(1) = 3$.

Answer. Characteristic equation:

$$2r^2 + r - 1 = 0, \quad \Rightarrow \quad (2r-1)(r+1) = 0, \quad \Rightarrow \quad r_1 = \frac{1}{2},\ r_2 = -1.$$

General solution is:

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 e^{-t}.$$

The ICs give

$$y(1) = 0: \quad c_1 e^{\frac{1}{2}} + c_2 e^{-1} = 0. \tag{A}$$

$$y'(1) = 3: \quad y'(t) = \frac{1}{2} c_1 e^{\frac{1}{2}t} - c_2 e^{-t}, \quad \frac{1}{2} c_1 e^{\frac{1}{2}} - c_2 e^{-1} = 3. \tag{B}$$

(A)+(B) gives

$$\frac{3}{2} c_1 e^{\frac{1}{2}} = 3, \quad c_1 = 2e^{-\frac{1}{2}}.$$

Plug this in (A):

$$c_2 = -ec_1 e^{\frac{1}{2}} = -e2e^{\frac{1}{2}}e^{\frac{1}{2}} = -2e.$$

The solution is

$$y(t) = 2e^{-\frac{1}{2}} e \frac{1}{2} t - 2ee^{-t} = 2e^{\frac{1}{2}(t-1)} - 2e^{-t+1},$$

and as $t \to \infty$ we have $y \to \infty$.

Summary of receipt:

1. Write the characteristic equation;
2. Find the roots;
3. Write the general solution;
4. Set in ICs to get the arbitrary constants c_1, c_2.

Example 4. Consider the equation $y'' - 5y = 0$.

- (a). Find the general solution.
- (b). If $y(0) = 1$, what should $y'(0)$ be such that y remain bounded as $t \to +\infty$?

Answer. (a). Characteristic equation

$$r^2 - 5 = 0, \quad \Rightarrow \quad r_1 = -\sqrt{5}, \quad r_2 = \sqrt{5}.$$

General solution is

$$y(t) = c_1 e^{-\sqrt{5}t} + c_2 e^{\sqrt{5}t}.$$

(b). If $y(t)$ remains bounded as $t \to \infty$, then the term $e^{\sqrt{5}t}$ must vanish, which means we must have $c_2 = 0$. This means $y(t) = c_1 e^{-\sqrt{5}t}$. If $y(0) = 1$, then $y(0) = c_1 = 1$, so $y(t) = e^{-\sqrt{5}t}$. This gives $y'(t) = -\sqrt{5}e^{-\sqrt{5}t}$ which means $y'(0) = -\sqrt{5}$.

Example 5. Consider the equation $2y'' + 3y' = 0$. The characteristic equation is

$$2r^2 + 3r = 0, \quad \Rightarrow \quad r(2r+3) = 0, \quad \Rightarrow \quad r_1 = -\frac{3}{2}, \; r_2 = 0$$

The general solutions is

$$y(t) = c_1 e^{-\frac{3}{2}t} + c_2 e^{0t} = c_1 e^{-\frac{3}{2}t} + c_2.$$

As $t \to \infty$, the first term in y vanished, and we have $y \to c_2$.

Example 6. Find a 2nd order equation such that $c_1 e^{3t} + c_2 e^{-t}$ is its general solution.

Answer. From the form of the general solution, we see the two roots are $r_1 = 3, r_2 = -1$. The characteristic equation could be $(r-3)(r+1) = 0$, or this equation multiplied by any non-zero constant. So $r^2 - 2r - 3 = 0$, which gives us the equation

$$y'' - 2y' - 3y = 0.$$

NB! This answer is not unique. Multiple it by any non-zero constant gives another equation.

3.2: Solutions of Linear Homogeneous Equations; the Wronskian

We consider some theoretical aspects of the solutions to a general 2nd order linear equations.

Theorem . (Existence and Uniqueness Theorem) *Consider the initial value problem*

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = \bar{y}_0.$$

If $p(t), q(t)$ and $g(t)$ are continuous and bounded on an open interval I containing t_0, then there exists exactly one solution $y(t)$ of this equation, valid on I.

Example 1. Given the equation

$$(t^2 - 3t)y'' + ty' - (t+3)y = e^t, \qquad y(1) = 2, \quad y'(1) = 1.$$

Find the largest interval where solution is valid.

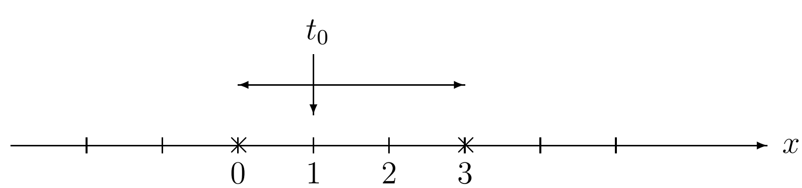
Answer. Rewrite the equation into the proper form:

$$y'' + \frac{t}{t(t-3)}y' - \frac{t+3}{t(t-3)}y = \frac{e^t}{t(t-3)},$$

so we have

$$p(t) = \frac{t}{t(t-3)}, \quad q(t) = -\frac{t+3}{t(t-3)}, \quad g(t) = \frac{e^t}{t(t-3)}.$$

We see that we must have $t \neq 0$ and $t \neq 3$. Since $t_0 = 1$, then the largest interval is $I = (0, 3)$, or $0 < t < 3$. See the figure below.



Definition. Given two functions $f(t), g(t)$, the **Wronskian** is defined as

$$W(f,g)(t) \doteq fg' - f'g.$$

Remark: One way to remember this definition could be using the determinant,

$$W(f,g)(t) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix}.$$

Main property of the Wronskian:

- If $W(f,g) \equiv 0$, then f anf g are *linearly dependent*.
- Otherwise, they are *linearly independent*.

Example 2. Check if the given pair of functions are linearly dependent or not.
(a). $f = e^t$, $g = e^{-t}$.

Answer. We have

$$W(f,g) = e^t(-e^{-t}) - e^t e^{-t} = -2 \neq 0$$

so they are linearly independent.
(b). $f(t) = \sin t$, $g(t) = \cos t$.

Answer. We have

$$W(f,g) = \sin t(\sin t) - \cos t \cos t = -1 \neq 0$$

and they are linearly independent.
(c). $f(t) = t + 1$, $g(t) = 4t + 4$.

Answer. We have

$$W(f,g) = (t+1)4 - (4t+4) = 0$$

so they are linearly dependent. (In fact, we have $g(t) = 4 \cdot f(t)$.)

(d). $f(t) = 2t$, $g(t) = |t|$.

Answer. Note that $g'(t) = \text{sign}(t)$ where sign is the sign function. So

$$W(f, g) = 2t \cdot \text{sign}(t) - 2|t| = 0$$

(we used $t \cdot \text{sign}(t) = |t|$). So they are linearly dependent.

Theorem . *Suppose $y_1(t), y_2(t)$ are two solutions of*

$$y'' + p(t)y' + q(t)y = 0.$$

Then

(I) We have either $W(y_1, y_2) \equiv 0$ or $W(y_1, y_2)$ never zero;

(II) If $W(y_1, y_2) \neq 0$, the $y = c_1y_1 + c_2y_2$ is the general solution. They are also called to form a fundamental set of solutions. *As a consequence, for any ICs $y(t_0) = y_0, y'(t_0) = \bar{y}_0$, there is a unique set of (c_1, c_2) that give a unique solution.*

The next Theorem is probably the most important one in this chapter.

Theorem (Abel's Theorem) *Let y_1, y_2 be two (linearly independent) solutions to $y'' + p(t)y' + q(t)y = 0$ on an open interval I. Then, the Wronskian $W(y_1, y_2)$ on I is given by*

$$W(y_1, y_2)(t) = C \cdot \exp(\int -p(t)\, dt),$$

for some constant C depending on y_1, y_2, but independent on t in I.

Proof. We skip this part. Read the book for a proof.

Example 3. Given

$$t^2y'' - t(t + 2)y' + (t + 2)y = 0.$$

Find $W(y_1, y_2)$ without solving the equation.

Answer. We first find the $p(t)$

$$p(t) = -\frac{t+2}{t}$$

which is valid for $t \neq 0$. By Abel's Theorem, we have

$$W(y_1, y_2) = C \cdot \exp(\int -p(t)\, dt) = C \cdot \exp(\int \frac{t+2}{t}\, dt) = Ce^{t+2\ln|t|} = Ct^2 e^t.$$

NB! The solutions are defined on either $(0, \infty)$ or $(-\infty, 0)$, depending on t_0.

From now on, when we say two solutions y_1, y_2 of the solution, we mean two linearly independent solutions that can form a fundamental set of solutions.

Example 4. If y_1, y_2 are two solutions of

$$ty'' + 2y' + te^t y = 0,$$

and $W(y_1, y_2)(1) = 2$, find $W(y_1, y_2)(5)$.

Answer. First we find that $p(t) = 2/t$. By Abel's Theorem we have

$$W(y_1, y_2)(t) = C \cdot \exp\left\{-\int \frac{2}{t}\, dt\right\} = C \cdot e^{-\ln t} = Ct^{-2}.$$

If $W(y_1, y_2)(1) = 2$, then $C1^{-2} = 2$, which gives $C = 2$. So we have

$$W(y_1, y_2)(5) = 25^{-2} = \frac{2}{25}.$$

Example 5. If $W(f, g) = 3e^{4t}$, and $f = e^{2t}$, find g.

Answer. By definition of the Wronskian, we have

$$W(f, g) = fg' - f'g = e^{2t}g' - 2e^{2t}g = 3e^{4t},$$

which gives a 1st order equation for g:

$$g' - 2g = 3e^{2t}.$$

Solve it for g:

$$\mu(t) = e^{-2t}, \quad g(t) = e^{2t} \int e^{-2t} 3e^{2t}\, dy = e^{2t}(3t + c).$$

We can choose $c = 0$, and get $g(t) = 3te^{2t}$.

Next example shows how Abel's Theorem can be used to solve 2nd order differential equations.

Example 6. Consider the equation $y'' + 2y' + y = 0$. Find the general solution.

Answer. The characteristic equation is $r^2 + 2r + 1 = 0$, which given double roots $r_1 = r_2 = -1$. So we know that $y_1 = e^{-t}$ is a solutions. How can we find another solution y_2 that's linearly independent?

By Abel's Theorem, we have

$$W(y_1, y_2) = C \exp\left\{\int -2\, dt\right\} = Ce^{-2t},$$

and we can choose $C = 1$ and get $W(y_1, y_2) = e^{-2t}$. By the definition of the Wronskian, we have

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{-t} y_2' - (-e^{-t} y_2) = e^{-t}(y_2' + y_2).$$

These two computation must have the same answer, so

$$e^{-t}(y_2' + y_2) = e^{-2t}, \qquad y_2' + y_2 = e^{-t}.$$

This is a 1st order equation for y_2. Solve it:

$$\mu(t) = e^{t}, \qquad y_2(t) = e^{-t} \int e^{t} e^{-t}\, dt = e^{-t}(t + c).$$

Choosing $c = 0$, we get $y_2 = te^{t}$. The general solution is

$$y(t) = c_1 y_1 + c_2 y_2 = c_1 e^{-t} + c_2 te^{-t}.$$

This is called *the method of reduction of order*. We will study it more later in chapter 3.4.

3.3: Complex Roots

The roots of the characteristic equation can be complex numbers. Consider the equation

$$ay'' + by' + cy = 0, \qquad \to \quad ar^2 + br + c = 0.$$

The two roots are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If $b^2 - 4ac < 0$, the root are complex, i.e., a pair of complex conjugate numbers. We will write $r_{1,2} = \lambda \pm i\mu$. There are two solutions:

$$y_1 = e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t}, \qquad y_2 = y_1 = e^{(\lambda - i\mu)t} = e^{\lambda t} e^{-i\mu t}.$$

To deal with exponential function with pure imaginary exponent, we need the Euler's Formula:

$$e^{i\beta} = \cos\beta + i \sin\beta.$$

A couple of Examples to practice this formula:

$$e^{i\frac{5}{6}\pi} = \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi = -\frac{\sqrt{3}}{2} + i\frac{1}{2}.$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

$$e^{a+ib} = e^a e^{ib} = e^a(\cos b + i \sin b).$$

Back to y_1, y_2, we have

$$y_1 = e^{\lambda t}(\cos\mu t + i\sin\mu t), \qquad y_2 = e^{\lambda t}(\cos\mu t + i\sin\mu t).$$

But these solutions are complex valued. We want real-valued solutions! To achieve this, we use the Principle of Superposition. If y_1, y_2 are two solutions, then $\frac{1}{2}(y_1 + y_2), \frac{1}{2i}(y_1 - y_2)$ are also solutions. Let

$$\tilde{y}_1 \doteq \frac{1}{2}(y_1 + y_2) = e^{\lambda t}\cos\mu t, \qquad \tilde{y}_2 \doteq \frac{1}{2i}(y_1 - y_2) = e^{\lambda t}\sin\mu t.$$

To make sure they are linearly independent, we can check the Wronskian,

$$W(\tilde{y}_1, \tilde{y}_2) = \mu e^{2\lambda t} \neq 0. \quad \text{(home work problem).}$$

So y_1, y_2 are linearly independent, and we have the general solution

$$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t).$$

Example 1. (Perfect Oscillation: Simple harmonic motion.) Solve the initial value problem

$$y'' + 4y = 0, \qquad y(\frac{\pi}{6}) = 0, \quad y'(\frac{\pi}{6}) = 1.$$

Answer. The characteristic equation is

$$r^2 + 4 = 0, \quad \Rightarrow \quad r = \pm 2i, \quad \Rightarrow \quad \lambda = 0, \ \mu = 2.$$

The general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t.$$

Find c_1, c_2 by initial conditions: since $y' = -2c_1 \sin 2t + 2c_2 \cos 2t$, we have

$$y(\frac{\pi}{6}) = 0 : \quad c_1 \cos \frac{\pi}{3} + c_2 \sin \frac{\pi}{3} = \frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 = 0,$$

$$y'(\frac{\pi}{6}) = 1 : \quad -2c_1 \sin \frac{\pi}{3} + 2c_2 \cos \frac{\pi}{3} = -2c_1\frac{\sqrt{3}}{2} + 2c_2\frac{1}{2} = 1.$$

Solve these two equations, we get $c_1 = -\frac{\sqrt{3}}{4}$ and $c_2 = \frac{1}{4}$. So the solution is

$$y(t) = -\frac{\sqrt{3}}{4} \cos 2t + \frac{1}{4} \sin 2t,$$

which is a periodic oscillation. This is also called perfect oscillation or simple harmonic motion.

Example 2. (Decaying oscillation.) Find the solution to the IVP (Initial Value Problem)

$$y'' + 2y' + 101y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Answer. The characteristic equation is

$$r^2 + 2r + 101 = 0, \quad \Rightarrow \quad r_{1,2} = -1 \pm 10i, \quad \Rightarrow \quad \lambda = -1,\ \mu = 10.$$

So the general solution is

$$y(t) = e^{-t}(c_1 \cos 10t + c_2 \sin 10t),$$

so

$$y'(t) = -e^{-t}(c_1 \cos t + c_2 \sin t) + e^{-t}(-10c_1 \sin t + 10c_2 \cos t)$$

Fit in the ICs:

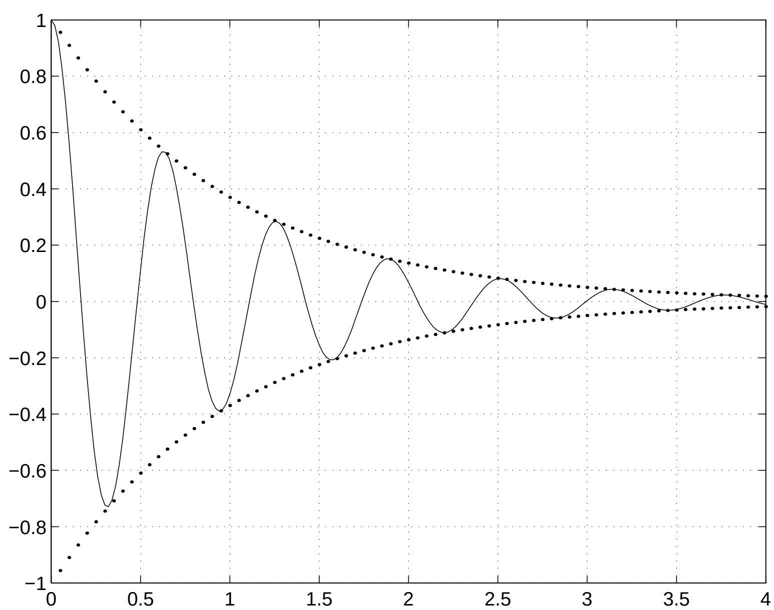
$$y(0) = 1: \qquad y(0) = e^0(c_1 + 0) = c_1 = 1,$$

$$y'(0) = 0: \qquad y'(0) = -1 + 10c_2 = 0, \quad c_2 = 0.1.$$

Solution is

$$y(t) = e^{-t}(\cos t + 0.1 \sin t).$$

The graph is given below:



We see it is a decaying oscillation. The sin and cos part gives the oscillation, and the e^{-t} part gives the decaying amplitude. As $t \to \infty$, we have $y \to 0$.

Example 3. (Growing oscillation) Find the general solution of $y'' - y' + 81.25y = 0$.

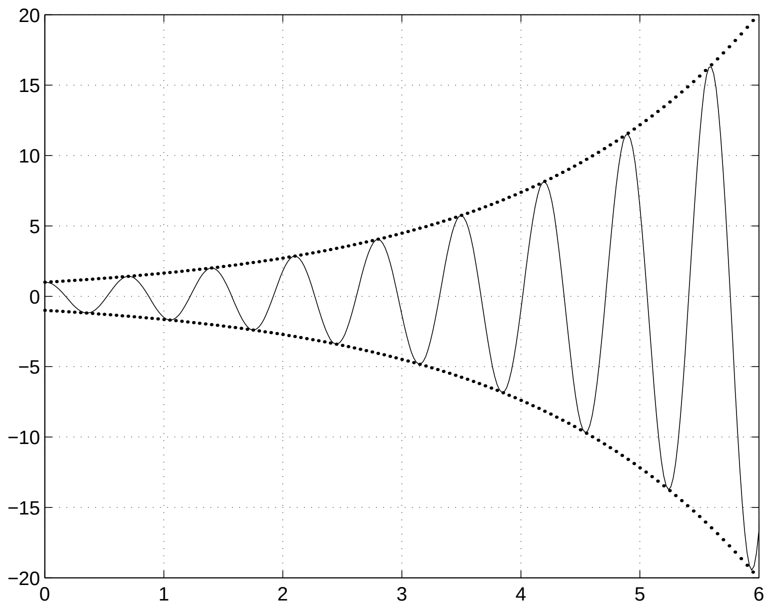
Answer.

$$r^2 - r + 81.25 = 0, \qquad \Rightarrow \quad r = 0.5 \pm 9i, \quad \Rightarrow \quad \lambda = 0.5, \quad \mu = 2.$$

The general solution is

$$y(t) = e^{0.5t}(c_1 \cos 9t + c_2 \sin 9t).$$

A typical graph of the solution looks like:



We see that y oscillate with growing amplitude as t grows. In the limit when $t \to \infty$, y oscillates between $-\infty$ and $+\infty$.

Conclusion: Sign of λ, the real part of the complex roots, decides the type of oscillation:

- $\lambda = 0$: perfect oscillation;
- $\lambda < 0$: decaying oscillation;

- $\lambda > 0$: growing oscillation.

We note that since $\lambda = \frac{-b}{2a}$, so the sign of λ follows the sign of $-b$.

3.4: Repeated roots; reduction of order

For the characteristic equation $ar^2 + br + c = 0$, if $b^2 = 4ac$, we will have two repeated roots

$$r_1 = r_2 = r = -\frac{b}{2a}.$$

We have one solution $y_1 = e^{rt}$. How can we find the second solution which is linearly independent of y_1?

Example 1. Consider the equation $y''+4y'+4y = 0$. We have $r^2+4r+4 = 0$, and $r_1 = r_2 = r = -2$. So one solution is $y_1 = e^{-2t}$. What is y_2?

Method 1. Use Wronskian and Abel's Theorem. By Abel's Theorem we have

$$W(y_1, y_2) = c\exp(-\int 4\,dt) = ce^{-4t} = e^{-4t}, \quad (\text{let } c = 1).$$

By the definition of Wronskian we have

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = e^{-2t}y_2' - (-2)e^{-2t}y_2 = e^{-2t}(y_2' + 2y_2).$$

They must equal to each other:

$$e^{-2t}(y_2' + 2y_2) = e^{-4t}, \quad y_2' + 2y_2 = e^{-2t}.$$

Solve this for y_2,

$$\mu = e^{2t}, \quad y_2 = e^{-2t}\int e^{2t}e^{-2t}\,dt = e^{-2t}(t + C)$$

Let $C = 0$, we get $y_2 = te^{-2t}$, and the general solution is

$$y(t) = c_1 y_1 + c_2 y_2 = c_1 e^{-2t} + c_2 te^{-2t}.$$

Method 2. This is the textbook's version. We guess a solution of the form $y_2 = v(t)y_1 = v(t)e^{-2t}$, and try to find the function $v(t)$. We have

$$y_2' = v'e^{-2t} + v(-2e^{-2t}) = e^{-2t}(v' - 2v), \qquad y_2'' = e^{-2t}(v'' - 4v' + 4v).$$

Put them in the equation

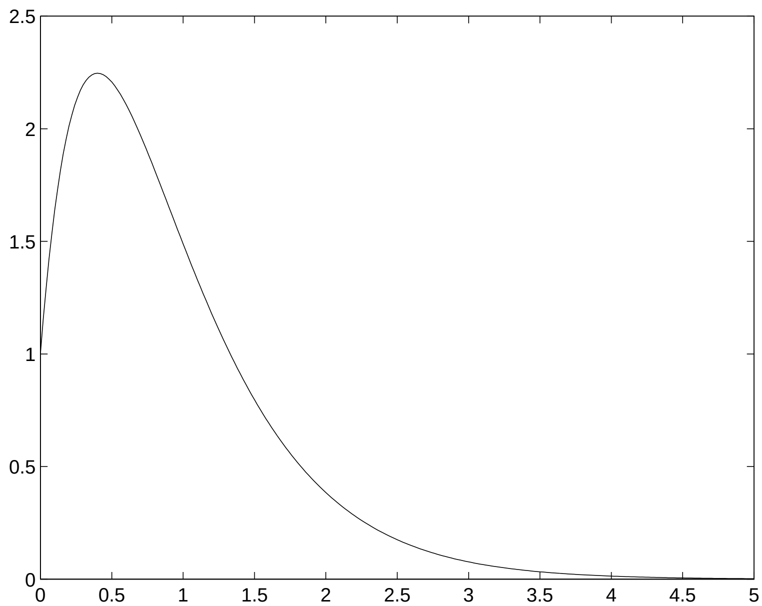
$$e^{-2t}(v'' - 4v' + 4v) + 4e^{-2t}(v' - 2v) + 4v(t)e^{-2t} = 0.$$

Cancel the term e^{-2t}, and we get $v'' = 0$, which gives $v(t) = c_1 t + c_2$. So

$$y_2(t) = vy_1 = (c_1 t + c_2)e^{-2t} = c_1 te^{-2t} + c_2 e^{-2t}.$$

Note that the term $c_2 e^{-2t}$ is already contained in cy_1. Therefore we can choose $c_1 = 1, c_2 = 0$, and get $y_2 = te^{-2t}$, which gives the same general solution as Method 1. We observe that this method involves more computation than Method 1.

A typical solution graph is included below:



We see if $c_2 > 0$, y increases for small t. But as t grows, the exponential (decay) function dominates, and solution will go to 0 as $t \to \infty$.

One can show that in general if one has repeated roots $r_1 = r_2 = r$, then $y_1 = e^{rt}$ and $y_2 = te^{rt}$, and the general solution is

$$y = c_1 e^{rt} + c_2 tr^{rt} = e^{rt}(c_1 + c_2 t).$$

Example 2. Solve the IVP

$$y'' - 2y' + y = 0, \qquad y(0) = 2, \quad y'(0) = 1.$$

Answer. This follows easily now

$$r^2 - 2r + 1 = 0, \quad \Rightarrow \quad r_1 = r_2 = 1, \quad \Rightarrow \quad y(t) = (c_1 + c_2 t)e^t.$$

The ICs give

$$y(0) = 2: \quad c_1 + 0 = 2, \quad \Rightarrow \quad c_1 = 2.$$

$$y'(t) = (c_1 + c_2 t)e^t + c_2 e^t, \quad y'(0) = c_1 + c_2 = 1, \quad \Rightarrow \quad c_2 = 1 - c_1 = -1.$$

So the solution is $y(t) = (2 - t)e^t$.

Summary: For $ay'' + by' + cy = 0$, and $ar^2 + br + c = 0$ has two roots r_1, r_2, we have

- If $r_1 \neq r_2$ (real): $\quad y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$;
- If $r_1 = r_2 = r$ (real): $\quad y(t) = (c_1 + c_2 t)e^{rt}$;
- If $r_{1,2} = \lambda \pm i\mu$ complex: $\quad y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$.

More on reduction of order: This method can be used to find a second solution y_2 if the first solution y_1 is given for a second order linear equation.

Example 3. For the equation

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0,$$

given one solution $y_1 = \frac{1}{t}$, find a second linearly independent solution.

Answer. **Method 1**: Use Abel's Theorem and Wronskian. By Abel's Theorem, and choose $C = 1$, we have

$$W(y_1, y_2) = \exp\left\{-\int p(t)\,dt\right\} = \exp\left\{-\int \frac{3t}{2t^2} dt\right\} = \exp\left\{-\frac{3}{2}\ln t\right\} = t^{-3/2}.$$

By definition of the Wronskian,

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \frac{1}{t} y_2' - (-\frac{1}{t^2}) y_2 = t^{-3/2}.$$

Solve this for y_2:

$$\mu = \exp(\int \frac{1}{t} dt) = \exp(\ln t) = t, \quad \Rightarrow \quad y_2 = \frac{1}{t}\int t \cdot t^{-\frac{3}{2}} dt = \frac{1}{t}(\frac{2}{3}t^{\frac{3}{2}} + C).$$

Let $C = 0$, we get $y_2 = \frac{2}{3}\sqrt{t}$. Since $\frac{2}{3}$ is a constant multiplication, we can drop it and choose $y_2 = \sqrt{t}$.

Method 2: This is the textbook's version. We saw in the previous example that this method is inferior to Method 1, therefore we will not focus on it at all. If you are interested in it, read the book.

Let's introduce another method that combines the ideas from Method 1 and Method 2.

Method 3. We will use Abel's Theorem, and at the same time we will seek a solution of the form $y_1 = vy_1$.

By Abel's Theorem, we have (worked out in M1) $W(y_1, y_2) = t^{-\frac{3}{2}}$. Now, seek $y_2 = vy_1$. By the definition of the Wronskian, we have

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = y_1(vy_1)' - y_1'(vy_1) = y_1(v'y_1 + vy_1'') - vy_1 y_1' = v'y_1^2.$$

Note that this is a general formula.

Now putting $y_1 = 1/t$, we get

$$v'\frac{1}{t^2} = t^{-\frac{3}{2}}, \qquad v' = t^{\frac{1}{2}}, \qquad v = \int t^{\frac{1}{2}} dt = \frac{2}{3}t^{\frac{3}{2}}.$$

Drop the constant $\frac{2}{3}$, we get

$$y_2 = vy_1 = t^{\frac{3}{2}}\frac{1}{t} = t^{\frac{1}{2}}.$$

We see that Method 3 is the most efficient one among all three methods. We will focus on this method from now on.

Example 4. Consider the equation

$$t^2 y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0.$$

Given $y_1 = t$, find the general solution.

Answer. We have

$$p(t) = -\frac{t(t+2)}{t^2} = -\frac{t+2}{t} = -1 - \frac{2}{t}.$$

Let y_2 be the second solution. By Abel's Theorem, choosing $c = 1$, we have

$$W(y_1, y_2) = \exp\left\{-\int (-1 - \frac{2}{t})dt\right\} = \exp\{t + 2\ln t\} = t^2 e^t.$$

Let $y_2 = vy_1$, the $W(y_1, y_2) = v' y_1^2 = t^2 v'$. Then we must have

$$t^2 v' = t^2 e^t, \qquad v' = e^t, \qquad v = e^t, \qquad y_2 = te^t.$$

(A cheap trick to double check your solution y_2 would be: plug it back into the equation and see if it satisfies it.) The general solution is

$$y(t) = c_1 y_2 + c_2 y_2 = c_1 t + c_2 te^t.$$

We observe here that Method 3 is very efficient.

Example 5. Given the equation $t^2 y'' - (t - \frac{3}{16})y = 0, \quad t > 0$, and $y_1 = t^{(1/4)} e^{2\sqrt{t}}$, find y_2.

Answer. We will always use method 3. We see that $p = 0$. By Abel's Theorem, setting $c = 1$, we have

$$W(y_1, y_2) = \exp(\int 0dt) = 1.$$

Seek $y_2 = vy_1$. Then, $W(y_1, y_2) = y_1^2 v' = t^{\frac{1}{2}} e^{4\sqrt{t}} v'$. So we must have

$$t^{\frac{1}{2}} e^{4\sqrt{t}} v' = 1, \quad \Rightarrow \quad v' = t^{-\frac{1}{2}} e^{-4\sqrt{t}}, \quad \Rightarrow \quad v = \int t^{-\frac{1}{2}} e^{-4\sqrt{t}} dt.$$

Let $u = -4\sqrt{t}$, so $du = -2t^{-\frac{1}{2}} dt$, we have

$$v = \int -\frac{1}{2} e^u \, du = -\frac{1}{2} e^u = -\frac{1}{2} e^{-4\sqrt{t}}.$$

So drop the constant $-\frac{1}{2}$, we get

$$y_2 = vy_1 = e^{-4\sqrt{t}} t^{\frac{1}{4}} e^{2\sqrt{t}} = t^{\frac{1}{4}} e^{-2\sqrt{t}}.$$

The general solution is

$$y(t) = c_1 y_1 + c_2 y_2 = t^{\frac{1}{4}} (c_1 e^{2\sqrt{t}} + c_2 e^{-2\sqrt{t}}).$$

3.6: Non-homogeneous equations; method of undetermined coefficients

Want to solve the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t), \tag{N}$$

Steps:

1. First solve the homogeneous equation

   $$y'' + p(t)y' + q(t)y = 0, \tag{H}$$

   i.e., find y_1, y_2, linearly independent of each other, and form the general solution

   $$y_H = c_1 y_1 + c_2 y_2.$$

2. Find a particular/specific solution Y for (N), by MUC (method of undetermined coefficients);

3. The general solution for (N) is then

   $$y = y_H + Y = c_1 y_1 + c_2 y_2 + Y.$$

   Find c_1, c_2 by initial conditions, if given.

Key step: step 2.

Why $y = y_H + Y$?
A quick proof: If y_H solves (H), then

$$y_H'' + p(t)y_H' + q(t)y_H = 0, \tag{A}$$

and since Y solves (N), we have

$$Y'' + p(t)Y' + q(t)Y = g(t), \tag{B}$$

Adding up (A) and (B), and write $y = y_H + Y$, we get $y'' + p(t)y' + q(t)y = g(t)$.
Main focus: constant coefficient case, i.e.,

$$ay'' + by' + cy = g(t).$$

Example 1. Find the general solution for $\quad y'' - 3y' + 4y = 3e^{2t}$.

Answer. Step 1: Find y_H.

$$r^2 - 3r - 4 = (r+1)(r-4) = 0, \quad \Rightarrow \quad r_1 = -1,\ r_2 = 4,$$

so

$$y_H = c_1 e^{-t} + c_2 e^{4t}.$$

Step 2: Find Y. We guess/seek solution of the same form as the source term $Y = Ae^{2t}$, and will determine the coefficient A.

$$Y' = 2Ae^{2t}, \quad Y'' = 4Ae^{2t}.$$

Plug these into the equation:

$$4Ae^{2t} - 3 \cdot 2Ae^{2t} - 4Ae^{2t} = 3e^{2t}, \quad \Rightarrow \quad -6A = 3, \quad \Rightarrow \quad A = -\frac{1}{2}.$$

So $Y = -\frac{1}{2}e^{2t}$.

Step 3. The general solution to the non-homogeneous solution is

$$y(t) = y_H + Y = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{2}e^{2t}.$$

Observation: The particular solution Y take the same *form* as the source term $g(t)$.

But this is not always true.

Example 2. Find general solution for $\quad y'' - 3y' + 4y = 2e^{-t}$.

Answer. The homogeneous solution is the same as Example 1: $y_H = c_1 e^{-t} + c_2 e^{4t}$. For the particular solution Y, let's first try the same form as g, i.e., $Y = Ae^{-t}$. So $Y' = -Ae^{-t}, Y'' = Ae^{-t}$. Plug them back in to the equation, we get

$$\text{LHS} = Ae^{-t} - 3(-Ae^{-t}) - 4Ae^{-t} = 0 \neq 2e^{-et} = \text{RHS}.$$

So it doesn't work. Why?

We see $r_1 = -1$ and $y_1 = e^{-t}$, which means our guess $Y = Ae^{-t}$ is a solution to the homogeneous equation. It will never work.

Second try: $Y = Ate^{-t}$. So

$$Y' = Ae^{-t} - Ate^{-t}, \quad Y'' = -Ae^{-t} - Ae^{-t} + Ate^{-t} = -2Ae^{-t} + Ate^{-t}.$$

Plug them in the equation

$$(-2Ae^{-t} + Ate^{-t}) - 3(Ae^{-t} - Ate^{-t}) - 4Ate^{-t} = -5Ae^{-t} = 2e^{-t},$$

we get

$$-5A = 2, \quad \Rightarrow \quad A = -\frac{2}{5},$$

so we have $Y = -\frac{2}{5}te^{-t}$.

Summary 1. If $g(t) = ae^{\alpha t}$, then the form of the particular solution Y depends on r_1, r_2 (the roots of the characteristic equation).

case	form of the particular solution Y
$r_1 \neq \alpha$ and $r_2 \neq \alpha$	$Y = Ae^{\alpha t}$
$r_1 = \alpha$ or $r_2 = \alpha$, but $r_1 \neq r_2$	$Y = Ate^{\alpha t}$
$r_1 = r_2 = \alpha$	$Y = At^2e^{\alpha t}$

Example 3. Find the general solution for

$$y'' - 3y' - 4y = 3t^2 + 2.$$

Answer. The y_H is the same $y_H = c_1e^{-t} + c_2e^{4t}$.
Note that $g(t)$ is a polynomial of degree 2. We will try to guess/seek a particular solution of the same form:

$$Y = At^2 + Bt + C, \qquad Y' = 2At + B, \qquad Y'' = 2A$$

Plug back into the equation

$$2A - 3(2At + b) - 4(At^2 + Bt + C) = -4At^2 - (6A + 4B)t + (2A - 3B - 4C) = 3t^2 + 2.$$

Compare the coefficient, we get three equations for the three coefficients A, B, C:

$$\begin{aligned} -4A = 3 \quad & \rightarrow \quad A = -\frac{3}{4} \\ -(6A+4B) = 0, \quad & \rightarrow \quad B = \frac{9}{8} \\ 2A - 3B - 4C = 2, \quad & \rightarrow \quad C = \frac{1}{4}(2A - 3B - 2) = -\frac{55}{32} \end{aligned}$$

So we get

$$Y(t) = -\frac{3}{4}t^2 + \frac{9}{8}t - \frac{55}{32}.$$

But sometimes this guess won't work.

Example 4. Find the particular solution for $y'' - 3y' = 3t^2 + 2$.

Answer. We see that the form we used in the previous example $Y = At^2 + Bt + C$ won't work because $Y'' - 3Y'$ will not have the term t^2.

New try: multiply by a t. So we guess $Y = t(At^2 + Bt + C) = At^3 + Bt^2 + Ct$. Then

$$Y' = 3At^2 + 2Bt + C, \quad Y'' = 6At + 2B.$$

Plug them into the equation

$$(6At + 2B) - 3(3At^2 + 2Bt + C) = -9At^2 + (6A - 6B)t + (2B - 3C) = 3t^2 + 2.$$

Compare the coefficient, we get three equations for the three coefficients A, B, C:

$$\begin{aligned} -9A = 3 \quad & \rightarrow \quad A = -\frac{1}{3} \\ (6A - 6B) = 0, \quad & \rightarrow \quad B = A = -\frac{1}{3} \\ 2B - 3C = 2, \quad & \rightarrow \quad C = \frac{1}{3}(2B - 2) = -\frac{8}{9} \end{aligned}$$

So $Y = t(-\frac{1}{3}t^2 - \frac{1}{3}t - \frac{8}{9})$.

Summary 2. If $g(t)$ is a polynomial of degree n, i.e.,

$$g(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0$$

the particular solution for

$$ay'' + by' + cy = g(t)$$

(where $a \neq 0$) depends on b, c:

case	form of the particular solution Y
$c \neq 0$	$Y = P_n(t) = A_n t^n + \cdots + A_1 t + A_0$
$c = 0$ but $b \neq 0$	$Y = tP_n(t) = t(A_n t^n + \cdots + A_1 t + A_0)$
$c = 0$ and $b = 0$	$Y = t^2 P_n(t) = t^2(A_n t^n + \cdots + A_1 t + A_0)$

Example 5. Find a particular solution for

$$y'' - 3y' - 4y = \sin t.$$

Answer. Since $g(t) = \sin t$, we will try the same form. Note that $(\sin t)' = \cos t$, so we must have the $\cos t$ term as well. So the form of the particular solution is

$$Y = A \sin t + B \cos t.$$

Then

$$Y' = A\cos t - B \sin t, \qquad Y'' = -A \sin t - B \cos t.$$

Plug back into the equation, we get

$$\begin{aligned} &(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + b\cos t) \\ = \;& (-5A + 3B)\sin t + (-3A - 5B)\cos t = \sin t. \end{aligned}$$

So we must have

$$-5A + 3B = 1, \quad -3A - 5B = 0, \qquad \to \quad A = \frac{5}{34}, \quad B = \frac{3}{34}.$$

So we get

$$Y(t) = -\frac{5}{34}\sin t + \frac{3}{34}\cos t.$$

But this guess won't work if the form is a solution to the homogeneous equation.

Example 6. Find a general solution for $y'' + y = \sin t$.

Answer. Let's first find y_H. We have $r^2 + 1 = 0$, so $r_{1,2} = \pm i$, and $y_H = c_1 \cos t + c_2 \sin t$.

For the particular solution Y: We see that the form $Y = A \sin t + B \cos t$ won't work because it solves the homogeneous equation.

Our new guess: multiply it by t, so

$$Y(t) = t(A \sin t + B \cos t).$$

Then

$$Y' = (A \sin t + B \cos t) + t(A \cos t + B \sin t),$$
$$Y'' = (-2B - At) \sin t + (2A - Bt) \cos t.$$

Plug into the equation

$$Y'' + Y = -2B \sin t + 2A \cos t = \sin t, \quad \Rightarrow \quad A = 0, \quad B = -\frac{1}{2}$$

So

$$Y(y) = -\frac{1}{2} t \cos t.$$

The general solution is

$$y(t) = y_H + Y = c_1 \cos t + c_2 \sin t - \frac{1}{2} t \cos t.$$

Summary 3. If $g(t) = a \sin \alpha t + b \cos \alpha t$, the form of the particular solution depends on the roots r_1, r_2.

case	form of the particular solution Y
$r_{1,2} \neq \pm \alpha i$	$Y = A \sin \alpha t + B \cos \alpha t$
$r_{1,2} = \pm \alpha i$	$Y = t(A \sin \alpha t + B \cos \alpha t)$

Next we study a couple of more complicated forms of g.

Example 7. Find a particular solution for

$$y'' - 3y' - 4y = te^t.$$

Answer. We see that $g = P_1(t)e^{at}$, where P_1 is a polynomial of degree 1. Also we see $r_1 = -1, r_2 = 4$, so $r_1 \neq a$ and $r_2 \neq a$. For a particular solution we will try the same form as g, i.e., $Y = (At + B)e^t$. So

$$Y' = Ae^t + (At + b)e^t = (A + b)e^t + Ate^t,$$

$$Y'' = \cdots = (2A + B)e^t + Ate^t.$$

Plug them into the equation,

$$[(2A+B)e^t+Ate^t]-3[(A+b)e^t+Ate^t]-4(At+B)e^t = (-6At-A-6B)e^t = te^t.$$

We must have $-6At - A - 6B = t$, i.e.,

$$-6A = 1, \quad -A-6B = 0, \quad \Rightarrow \quad A = -\frac{1}{6}, B = \frac{1}{36}, \quad \Rightarrow \quad Y = (-\frac{1}{6}t+\frac{1}{36})e^t.$$

However, if the form of g is a solution to the homogeneous equation, it won't work for a particular solution. We must multiply it by t in that case.

Example 8. Find a particular solution of

$$y'' - 3y' - 4y = te^{-t}.$$

Answer. Since $a = -1 = r_1$, so the form we used in Example 7 won't work here. Try

$$Y = t(At + B)e^{-t} = (At^2 + Bt)e^{-t}.$$

Then

$$Y' = \cdots = [-At^2 + (2A - B)t + B]e^{-t},$$

$$Y'' = \cdots = [At^2 + (B - 4A)t + 2A - 2B]e^{-t}.$$

Plug into the equation

$$[At^2 + (B - 4A)t + 2A - 2B]e^{-t} - 3[-At^2 + (2A - B)t + B]e^{-t} - 4(At^2 + Bt)e^{-t}$$
$$= \quad [-10At + 2A - 5B]e^{-t} \quad = \quad te^t.$$

So we must have $-10At + 2A - 5B = t$, which means

$$-10A = 1, \quad 2A - 5B = 0, \qquad \Rightarrow \qquad A = -\frac{1}{10}, \quad B = -\frac{1}{25}.$$

Then

$$Y = \left(-\frac{1}{10}t^2 - \frac{1}{25}t\right) e^{-t}.$$

Summary 4. If $g(t) = P_n(t)e^{at}$ where $P_n(t) = \alpha_n t^n + \cdots + \alpha_1 t + \alpha_0$ is a polynomial of degree n, then the form of a particular solution depends on the roots r_1, r_2.

case	form of the particular solution Y
$r_1 \neq a$ and $r_2 \neq a$	$Y = \tilde{P}_n(t)e^{at} = (A_n t^n + \cdots + A_1 t + A_0)e^{at}$
$r_1 = a$ or $r_2 = a$ but $r_1 \neq r_2$	$Y = t\tilde{P}_n(t)e^{at} = t(A_n t^n + \cdots + A_1 t + A_0)e^{at}$
$r_1 = r_2 = a$	$Y = t^2\tilde{P}_n(t)e^{at} = t^2(A_n t^n + \cdots + A_1 t + A_0)e^{at}$

Other cases of g are treated in a similar way: Check if the form of g is a solution to the homogeneous equation. If not, then use it as the form of a particular solution. If yes, then multiply it by t or t^2.

We summarize a few cases below.

Summary 5. If $g(t) = e^{\alpha t}(a\cos\beta t + b\sin\beta t)$, and r_1, r_2 are the roots of the characteristic equation. Then

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm i\beta$	$Y = e^{\alpha t}(A\cos\beta t + B\sin\beta t)$
$r_{1,2} = \alpha \pm i\beta$	$Y = t \cdot e^{\alpha t}(A\cos\beta t + B\sin\beta t)$

Summary 6. If $g(t) = P_n(t)e^{\alpha t}(a\cos\beta t + b\sin\beta t)$ where $P_n(t)$ is a polynomial of degree n, and r_1, r_2 are the roots of the characteristic equation. Then

case	form of the particular solution Y
$r_{1,2} \neq \alpha \pm i\beta$	$Y = e^{\alpha t}[(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t]$
$r_{1,2} = \alpha \pm i\beta$	$Y = t \cdot e^{\alpha t}[(A_n t^n + \cdots + A_0)\cos\beta t + (B_n t^n + \cdots + B_0)\sin\beta t]$

If the source $g(t)$ has several terms, we treat each separately and add up later. Let $g(t) = g_1(t) + g_2(t) + \cdots g_n(t)$, then, find a particular solution Y_i for each $g_i(t)$ term as if it were the only term in g, then $Y = Y_1 + Y_2 + \cdots Y_n$. This claim follows from the principle of superposition.

In the examples below, we want to write the form of a particular solution.

Example 9. $\quad y'' - 3y' - 4y = \sin 4t + 2e^{4t} + e^{5t} - t.$

Answer. Since $r_1 = -1, r_2 = 2$, we treat each term in g separately and the add up:

$$Y(t) = A\sin 4t + B\cos 4tCte^{4t} + De^{5t} + (Et + F).$$

Example 10. $\quad y'' + 16y = \sin 4t + \cos t - 4\cos 4t + 4.$

Answer. The char equation is $r^2 + 16 = 0$, with roots $r_{1,2} = \pm 4i$, and

$$y_H = c_1 \sin 4t + c_2 \cos 4t.$$

We also note that the terms $\sin 4t$ and $-4\cos 4t$ are of the same type, and we must multiply it by t. So

$$Y = t(A\sin 4t + B\cos 4t) + (C\cos t + D\sin t) + E.$$

Example 11. $\quad y'' - 2y' + 2y = e^t \cos t + 8e^t \sin 2t + te^{-t} + 4e^{-t} + t^2 - 3.$

Answer. The char equation is $r^2 - 2r + 2 = 0$ with roots $r_{1,2} = 1 \pm i$. Then, for the term $e^t \cos t$ we must multiply by t.

$$Y = te^t(A_1\cos t + A_2 \sin t) + e^t(B_1 \cos 2t + B_2 \sin 2t) + (C_1 t + C_0)e^{-t} + De^{-t} + (F_2 t^2 + F_1 t + F_0).$$

3.7: Mechanical vibrations

In this chapter we study some applications of the IVP

$$ay'' + by' + cy = g(t), \qquad y(0) = y_0, \quad y'(0) = \bar{y}_0.$$

The spring-mass system: See figure below.

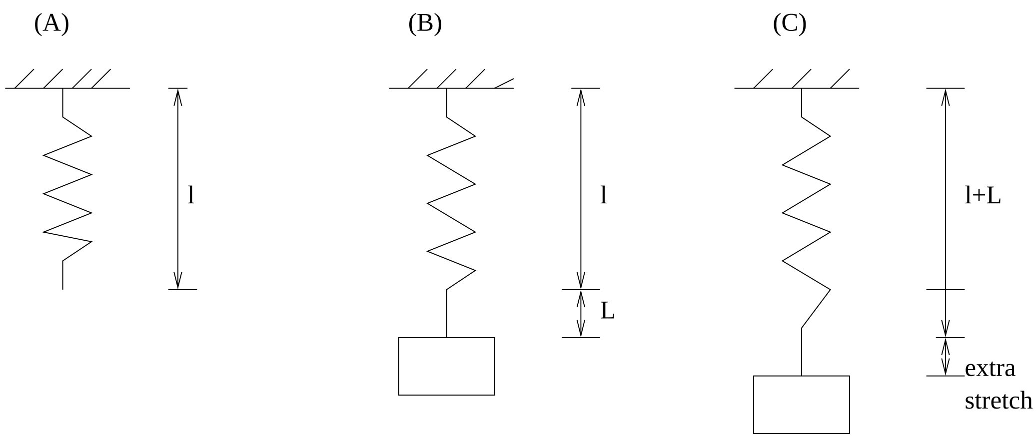
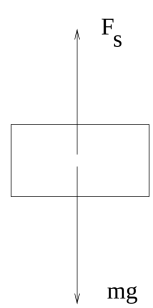


Figure (A): a spring in rest, with length l.

Figure (B): we put a mass m on the spring, and the spring is stretched. We call length L the elongation

Figure (C): The spring-mass system is set in motion by stretch/squueze it extra, with initial velocity, or with external force.

Force diagram at equilibrium position: $mg = Fs$.



Hooke's law: Spring force $F_s = -kL$, where L =elongation and k =spring constant.

So: we have $mg = kL$ which give

$$k = \frac{mg}{L}$$

which gives a way to obtain k by experiment: hang a mass m and measure the elongation L.

Model the motion: Let $u(t)$ be the displacement/position of the mass at time t, assuming the origin $u = 0$ at the equilibrium position, and downward the positive direction.
Total elongation: $L + u$
Total spring force: $F_s = -k(L + u)$

Other forces:
* damping/resistent force: $F_d(t) = -\gamma v = -\gamma u'(t)$, where γ is the damping constant, and v is the velocity
* External force applied on the mass: $F(t)$, given function of t

Total force on the mass: $\sum f = mg + F_s + F_d + F$.

Newton's law of motion $ma = \sum f$ gives

$$ma = mu'' = \sum f = mg + F_s + F_d + F, \qquad mu'' = mg - k(L+u) - \gamma u' + F.$$

Since $mg = kL$, by rearranging the terms, we get

$$mu'' + \gamma u' + ku = F$$

where m ia the mass, γ is the damping constant, k is the spring constant, and F is the external force.

Next we study several cases.

Case 1: <u>Undamped free vibration</u> (simple harmonic motion). We assume no damping $(\gamma = 0)$ and no external force $(F = 0)$. So the equation becomes

$$mu'' + ku = 0.$$

Solve it

$$mr^2 + k = 0, \quad r^2 = -\frac{k}{m}, \quad r_{1,2} = \pm\sqrt{\frac{k}{m}}i = \pm\omega_0 i, \quad \text{where } \omega_0 = \sqrt{\frac{k}{m}}.$$

General solution

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

Four terms of this motion, frequency, period, amplitude and phase, defined below:

Frequency: $\omega_0 = \sqrt{\frac{k}{m}}$

Period: $T = \frac{2\pi}{\omega_0}$

Amplitude and phase: We need to work on this a bit. We can write

$$u(t) = \sqrt{c_1^2 + c_2^2} \left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}} \cos \omega_0 t + \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \sin \omega_0 t \right).$$

Now, define δ, such that $\tan \delta = c_2/c_1$, then

$$\sin \delta = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}, \qquad \cos \delta = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}$$

so we have

$$u(t) = \sqrt{c_1^2 + c_2^2}(\cos \delta \cdot \cos \omega_0 t + \sin \delta \cdot \sin \omega_0 t) = \sqrt{c_1^2 + c_2^2} \cos(\omega_0 t - \delta).$$

So amplitude is $R = \sqrt{c_1^2 + c_2^2}$ and phase is $\delta = \arctan \frac{c_2}{c_1}$.

A few words on units:

force (f)	weight (mg)	length (u)	mass (m)	gravity (g)
lb	lb	ft	lb $\cdot$ sec^2/ft	32 ft/sec^2
newton	newton	m	kg	9.8 m/sec^2

Example 1. A mass weighing 10 lb stretches a spring 2 in. If the mass is displaced an additional 2 in, and is then set in motion with initial upward velocity of 1 ft/sec, determine the position, frequency, period, amplitude and phase of the motion.

Answer. We see this is free harmonic oscillation. We have

$$mg = 10, \quad g = 32, \quad m = \frac{10}{g} = \frac{10}{32} = \frac{5}{16}.$$

And the elongation is $L = 2\text{in} = \frac{1}{6}\text{ft}$. So $k = mg/L = 60$. Let $u(t)$ be the position from equilibrium, we get the equation

$$mu'' + ku = 0, \qquad \frac{5}{16}u'' + 60u = 0,$$

therefore

$$u'' + 192u = 0, \qquad u(0) = \frac{1}{6}, \quad u'(0) = -1.$$

So the frequency is $\omega_0 = \sqrt{192}$, and the general solution is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

By the ICs:

$$u(0) = c_1 = \frac{1}{6}, \quad u'(0) = \omega_0 c_2 = -1, \quad c_2 = -\frac{1}{\omega_0} = -\frac{1}{\sqrt{192}}.$$

(Note that $c_1 = u(0)$ and $c_2 = u'(0)/\omega_0$.) Now we have the position at any time t

$$u(t) = \frac{1}{6}\cos \omega_0 t - \frac{1}{\sqrt{192}} \sin \omega_0 t.$$

The four terms of the motion are

$$\omega_0 = \sqrt{192}, \quad T = \frac{2\pi}{\omega_0} = \frac{\pi}{\sqrt{48}}, \quad R = \sqrt{c_1^2 + c_2^2} = \sqrt{\frac{19}{576}} \approx 0.18,$$

and

$$\delta = \arctan \frac{c_2}{c_1} = \arctan -\frac{6}{\sqrt{192}} = -\arctan \frac{\sqrt{3}}{4}.$$

Case II: Damped free vibration. We assume that $\gamma \neq 0 (> 0)$ and $F = 0$.

$$mu'' + \gamma u' + ku = 0$$

then

$$mr^2 + \gamma r + k = 0, \qquad r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m}.$$

We see the type of root depends on the sign of $\gamma^2 - 4km$.

- If $\gamma^2 - 4km > 0$, (i.e., $\gamma > \sqrt{4km}$) we have two real roots, and the general solution is $u = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, with $r_1 < 0, r_2 < 0$.

  Due to the large damping force, there will be no vibration in the motion. The mass will simply return to the equilibrium position exponentially. This kind of motion is called **overdamped**.

- If $\gamma^2 - 4km = 0$, (i.e., $\gamma = \sqrt{4km}$) we have double roots $r_1 = r_2 = r < 0$. So $u = (c_1 + c_2 t)e^{rt}$.

  Depending on the sign of c_1, c_2 (which is determined by the ICs), the mass may cross the equilibrium point maximum once. This kind of motion is called **critically damped**, and this value of γ is called **critical damping**.

- If $\gamma^2 - 4km < 0$, (i.e., $\gamma < \sqrt{4km}$) we have complex roots

  $$r_{1,2} = -\lambda \pm \mu i, \quad \lambda = \frac{\gamma}{2m}, \quad \mu = \frac{\sqrt{4km - \gamma^2}}{2m}.$$

  So the position is

  $$u = e^{-\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t).$$

  This motion is **damped oscillation**. We can write

  $$u(t) = e^{-\lambda t} R \cdot \cos(\mu t - \delta), \quad R = \sqrt{c_1^2 + c_2^2}, \quad \delta = \arctan \frac{c_2}{c_1}.$$

  Here the term $e^{-\lambda t} R$ is the amplitude, and μ is called the quasi frequency, and the quasi period is $\frac{2\pi}{\mu}$. The graph of the solution looks like the one for complex roots with negative real part.

Summary: For all cases, since the real part of the roots are always negative, u will go to zero as t grow. This means, if there is damping, no matter how big or small, the motion will eventually come to a rest.

Example 2. A mass of 9.8 kg is hanging on a spring with $k = 1$. The mass is in a medium that exerts a viscous resistance of 6 lb when the mass has a velocity of 48 ft/s. The mass is then further stretched for another 2ft, then released from rest. Find the position $u(t)$ of the mass.

Answer. We have $\gamma = \frac{6}{48} = \frac{1}{8}$. So the equation for u is

$$mu'' + \gamma u' + ku = 0, \quad u'' + \frac{1}{8}u' + u = 0, \quad u(0) = 2, \quad u'(0) = 0.$$

Solve it

$$r^2 + \frac{1}{8}r + 1 = 0, \quad r_{1,2} = -\frac{1}{16} \pm \frac{\sqrt{255}}{16}i, \quad \omega_0 = \frac{\sqrt{255}}{16}$$

$$u(t) = e^{-\frac{1}{16}t}(c_1 \cos\omega_0 t + c_2 \sin\omega_0 t).$$

By ICs, we have $u(0) = c_1 = 2$, and

$$u'(t) = -\frac{1}{16}u(t) + e^{-\frac{1}{16}t}(-\omega_0 c_1 \sin\omega_0 t + \omega_0 c_2 \cos\omega_0 t),$$

$$u'(0) = -\frac{1}{16}u(0) + \omega_0 c_2 = 0, \quad c_2 = \frac{2}{\sqrt{255}}.$$

So the position at any time t is

$$u(t) = e^{-t/16}(2\cos\omega_0 t - \frac{2}{\sqrt{255}}\sin\omega_0 t).$$

3.9: Forced vibrations

In this chapter we assume the external force is $F(t) = F_0 \cos \omega t$. (The case where $F(t) = F_0 \sin \omega t$ is totally similar.)

Case 1: With damping.

$$mu'' + \gamma u' + ku = F_0 \cos \omega t.$$

Solution consists of two parts:

$$u = u_H + U,$$

u_H: the solution of the homogeneous equation,
U: a particular solution.

From discussion is the previous chapter, we know that $u_H \to 0$ as $t \to \infty$ for systems with damping. Therefore, this part of the solution is called the *transient solution*.

The appearance of U is due to the force term F. Therefore it is called *the forced response*. The form is $U = R\cos(\omega t - \delta)$. We see it is a periodic oscillation for all time t.

As time $t \to \infty$, we have $u \to U$. So U is called the *steady state*.

Case 2: Without damping.

$$mu'' + ku = F_0 \cos \omega t$$

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad u_H = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t$$

The form of the particular solution depends on the value of w. We have two cases.

Case 2A: if $w \neq w_0$. The particular solution should be

$$U = A \cos wt + B \cos wt$$

But there is no u' term, so we only need $U = A \cos wt$. And $U'' = -w^2 A \cos wt$. Plug in the equation

$$m(-w^2 A \cos wt) + kA \cos wt = F_0 \cos wt,$$

$$(k - mw^2)A = F_0, \quad A = \frac{F_0}{k - mw^2} = \frac{F_0}{m(w_0^2 - w^2)}.$$

General solution

$$u(t) = c_1 \cos w_0 t + c_2 \sin w_0 t + A \cos wt$$

Assume ICs: $u(0) = 0$, $u'(0) = 0$. Find c_1, c_2.

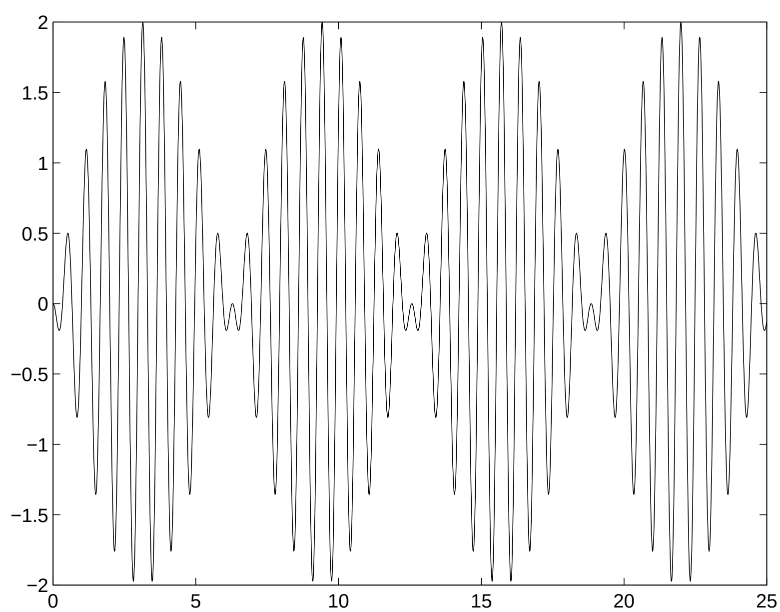
$$u(0) = 0 : \quad c_1 + A = 0, \quad c_1 = -A$$

$$u'(0) = 0 : \quad 0 + w_0 c_2 + 0 = 0, \quad c_2 = 0$$

Solution

$$u(t) = -A \cos w_0 t + A \cos wt = A(\cos wt - \cos w_0 t) = 2A \sin \frac{w_0 - w}{2} t \cdot \sin \frac{w_0 + w}{2} t.$$

(We used the trig identity: $\cos a - \cos b = 2 \sin \frac{b-a}{2} \sin \frac{a+b}{2}$.)
We see the first term $2A \sin \frac{w_0 - w}{2} t$ can be viewed as the varying amplitude, and the second term $\sin \frac{w_0+w}{2} t$ is the vibration.
One particular situation: if $w_0 \neq w$ but $w_o \approx w$, then $|w_0 - w| << |w_0 + w|$.
The plot looks like (we choose $w_0 = 9, w = 10$)

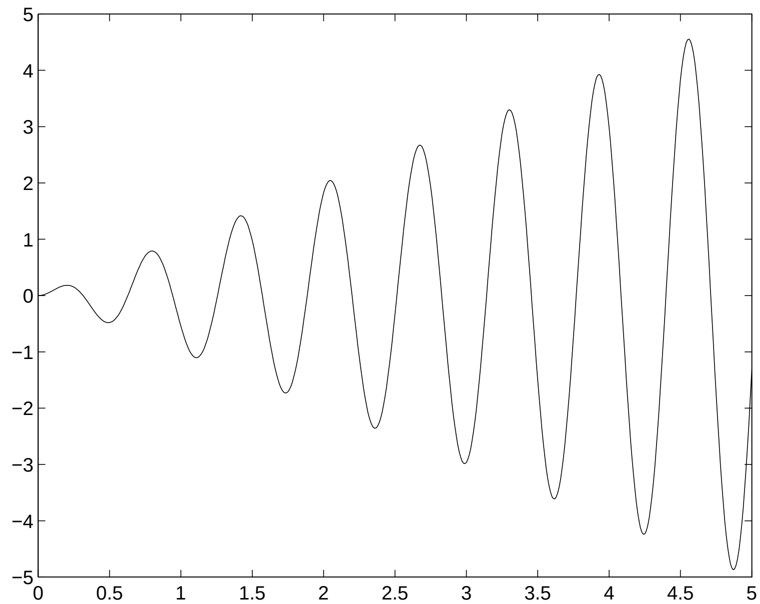


This is called a *beat*. (One observes it by hitting two nearby keys on a piano, for example.)

Case 2B: If $w = w_0$. The particular solution is

$$U = At\cos w_0 t + Bt\sin w_0 t$$

A typical plot looks like:



This is called *resonance*. If the frequency of the source term ω equals to the frequency of the system ω_0, then, small source term could make the solution grow very large!

Summary:

- With damping: Transient solution plus the forced response term,
- Without damping:
  if $w = w_0$: resonance.
  if $w \neq w_0$ but $w \approx w_0$: beat.

Chapter 6. The Laplace Transform

—used to handle piecewise continuous or impulsive force.

6.1: Definition of the Laplace transform

Topics:

- Definition of Laplace transform,
- Compute Laplace transform by definition, including piecewise continuous functions.

Definition: Given a function $f(t)$, $t \geq 0$, its Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is defined as

$$F(s) = \mathcal{L}\{f(t)\} \doteq \int_0^\infty e^{-st} f(t)\, dt \doteq \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt$$

We say the transform converges if the limit exists, and diverges if not.

Next we will give examples on computing the Laplace transform of given functions by definition.

Example 1. $f(t) = 1$ for $t \geq 0$.

Answer.

$$F(s) = \mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st}\, dt = \lim_{A\to\infty} -\frac{1}{s} e^{-st}\Big|_0^A = \lim_{A\to\infty} -\frac{1}{s}\left[e^{-sA} - 1\right] = \frac{1}{s}, \quad (s > 0)$$

Example 2. $f(t) = e^t$.

Answer.

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} = \lim_{A\to\infty} \int_0^A e^{-st} e^{at} dt = \lim_{A\to\infty} \int_0^A e^{-(s-a)t} dt = \lim_{A\to\infty} -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^A \\ &= \lim_{A\to\infty} -\frac{1}{s-a}\left(e^{-(s-a)A} - 1\right) = \frac{1}{s-a}, \quad (s > a) \end{aligned}$$

Example 3. $f(t) = t^n$, for $n \geq 1$ integer.

Answer.

$$\begin{aligned} F(s) &= \lim_{A\to\infty} \int_0^A e^{-st} t^n dt \;=\; \lim_{A\to\infty} \left\{ t^n \left. \frac{e^{-st}}{-s} \right|_0^A - \int_0^A \frac{n t^{n-1} e^{-st}}{-s} dt \right\} \\ &= 0 + \frac{n}{s} \lim_{A\to\infty} \int_0^A e^{-st} t^{n-1} dt \;=\; \frac{n}{s} \mathcal{L}\{t^{n-1}\}. \end{aligned}$$

So we get a recursive relation

$$\mathcal{L}\{t^n\} = \frac{n}{s} \mathcal{L}\{t^{n-1}\}, \quad \forall n,$$

which means

$$\mathcal{L}\{t^{n-1}\} = \frac{n-1}{s} \mathcal{L}\{t^{n-2}\}, \quad \mathcal{L}\{t^{n-2}\} = \frac{n-2}{s} \mathcal{L}\{t^{n-3}\}, \cdots$$

By induction, we get

$$\begin{aligned} \mathcal{L}\{t^n\} &= \frac{n}{s} \mathcal{L}\{t^{n-1}\} \;=\; \frac{n}{s} \frac{(n-1)}{s} \mathcal{L}\{t^{n-2}\} \;=\; \frac{n}{s} \frac{(n-1)}{s} \frac{(n-2)}{s} \mathcal{L}\{t^{n-3}\} \\ &= \cdots \;=\; \frac{n}{s} \frac{(n-1)}{s} \frac{(n-2)}{s} \cdots \frac{1}{s} \mathcal{L}\{1\} \;=\; \frac{n!}{s^n} \frac{1}{s} \;=\; \frac{n!}{s^{n+1}}, \quad (s > 0) \end{aligned}$$

Example 4. Find the Laplace transform of $\sin at$ and $\cos at$.

Answer. Method 1. Compute by definition, with integration-by-parts, twice. (lots of work...)

Method 2. Use the Euler's formula

$$e^{iat} = \cos at + i \sin at, \qquad \Rightarrow \qquad \mathcal{L}\{e^{iat}\} = \mathcal{L}\{\cos at\} + i\mathcal{L}\{\sin at\}.$$

By Example 2 we have

$$\mathcal{L}\{e^{iat}\} = \frac{1}{s - ia} = \frac{1(s + ia)}{(s - ia)(s + ia)} = \frac{s + ia}{s^2 + a^2} = \frac{s}{s^2 + a^2} + i \frac{a}{s^2 + a^2}.$$

Comparing the real and imaginary parts, we get

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}, \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}, \quad (s > 0).$$

Remark: Now we will use $\int_0^\infty$ instead of $\lim_{A\to\infty}\int_0^A$, without causing confusion.

For piecewise continuous functions, Laplace transform can be computed by integrating each integral and add up at the end.

Example 5. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ t-2, & 2 \le t. \end{cases}$$

We do this by definition:

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \;=\; \int_0^2 e^{-st}dt + \int_2^\infty (t-2)e^{-st}dt \\
&= \frac{1}{-s}e^{-st}\Big|_{t=0}^{2} + (t-2)\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} - \int_2^A \frac{1}{-s}e^{-st}dt \\
&= \frac{1}{-s}(e^{-2s}-1) + (0-0) + \frac{1}{s}\,\frac{1}{-s}e^{-st}\Big|_{t=2}^{\infty} \;=\; \frac{1}{-s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s}
\end{aligned}$$

6.2: Solution of initial value problems

Topics:

- Properties of Laplace transform, with proofs and examples
- Inverse Laplace transform, with examples, review of partial fraction,
- Solution of initial value problems, with examples covering various cases.

Properties of Laplace transform:

1. Linearity: $\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathcal{L}\{f(t)\} + c_2\mathcal{L}\{g(t)\}$.
2. First derivative: $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.
3. Second derivative: $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.
4. Higher order derivative:

   $$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$
5. $\mathcal{L}\{-tf(t)\} = F'(s)$ where $F(s) = \mathcal{L}\{f(t)\}$. This also implies $\mathcal{L}\{tf(t)\} = -F'(s)$.
6. $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ where $F(s) = \mathcal{L}\{f(t)\}$. This implies $e^{at}f(t) = \mathcal{L}^{-1}\{F(s-a)\}$.

Remarks:

- Note property 2 and 3 are useful in differential equations. It shows that each derivative in t caused a multiplication of s in the Laplace transform.
- Property 5 is the counter part for Property 2. It shows that each derivative in s causes a multiplication of $-t$ in the inverse Laplace transform.
- Property 6 is also known as the Shift Theorem. A counter part of it will come later in chapter 6.3.

Proof:

1. This follows by definition.

2. By definition

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st}f'(t)dt = e^{-st}f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st}f(t)dt = -f(0)+s\mathcal{L}\{f(t)\}.$$

3. This one follows from Property 2. Set f to be f' we get

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\}-f'(0) = s(s\mathcal{L}\{f(t)\}-f(0))-f'(0) = s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0).$$

4. This follows by induction, using property 2.

5. The proof follows from the definition:

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty \frac{\partial}{\partial s}(e^{-st})f(t)dt = \int_0^\infty (-t)e^{-st}f(t)dt = \mathcal{L}\{-tf(t)\}.$$

6. This proof also follows from definition:

$$\mathcal{L}\{e^{at}f(t)\} \int_0^\infty e^{-st}e^{at}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt = F(s-a).$$

By using these properties, we could find more easily Laplace transforms of many other functions.

Example 1.

From $\quad \mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad$ we get $\quad \mathcal{L}\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}.$

Example 2.

From $\quad \mathcal{L}\{\sin bt\} = \dfrac{b}{s^2+b^2}, \quad$ we get $\quad \mathcal{L}\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}.$

Example 3.

From $\quad \mathcal{L}\{\cos bt\} = \dfrac{s}{s^2+b^2}, \quad$ we get $\quad \mathcal{L}\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}.$

Example 4.

$$\mathcal{L}\{t^3+5t-2\} = \mathcal{L}\{t^3\} + 5\mathcal{L}\{t\} - 2\mathcal{L}\{1\} = \frac{3!}{s^4} + 5\frac{1}{s^2} - 2\frac{1}{s}.$$

Example 5.

$$\mathcal{L}\{e^{2t}(t^3+5t-2)\} = \frac{3!}{(s-2)^4} + 5\frac{1}{(s-2)^2} - 2\frac{1}{s-2}.$$

Example 6.

$$\mathcal{L}\{(t^2+4)e^{2t} - e^{-t}\cos t\} = \frac{2}{(s-2)^3} + \frac{4}{s-2} - \frac{s+1}{(s+1)^2+1},$$

because

$$\mathcal{L}\{t^2+4\} = \frac{2}{s^3} + \frac{4}{s}, \qquad \Rightarrow \qquad \mathcal{L}\{(t^2+4)e^{2t}\} = \frac{2}{(s-2)^3} + \frac{4}{s-2}.$$

Next are a few examples for Property 5.

Example 7.

Given $\quad \mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}, \quad$ we get $\quad \mathcal{L}\{te^{at}\} = -\left(\dfrac{1}{s-a}\right)' = \dfrac{1}{(s-a)^2}$

Example 8.

$$\mathcal{L}\{t\sin bt\} = -\left(\frac{b}{s^2+b^2}\right)' = \frac{-2bs}{(s^2+b^2)^2}.$$

Example 9.

$$\mathcal{L}\{t\cos bt\} = -\left(\frac{s}{s^2+b^2}\right)' = \cdots = \frac{s^2-b^2}{(s^2+b^2)^2}.$$

Inverse Laplace transform. Definition:

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \qquad \text{if} \qquad F(s) = \mathcal{L}\{f(t)\}.$$

Technique: find the way back.

Some simple examples:

Example 10.

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2}\cdot\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \frac{3}{2}\sin 2t.$$

Example 11.

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+5)^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{6}{(s+5)^4}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3!}{(s+5)^4}\right\} = \frac{1}{3}e^{-5t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3}e^{-5t}t^3.$$

Example 12.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \cos 2t\frac{1}{2}\sin 2t.$$

Example 13.

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/4}{s-2} + \frac{1/4}{s+2}\right\} = \frac{3}{4}e^{2t}+\frac{1}{4}e^{-2t}.$$

Here we used partial fraction to find out:

$$\frac{s+1}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}, \qquad A = 3/4, \quad B = 1/4.$$

Solutions of initial value problems.

We will go through one example first.

Example 14. (Two distinct real roots.) Solve the initial value problem by Laplace transform,

$$y'' - 3y' - 10y = 2, \qquad y(0) = 1, y'(0) = 2.$$

Answer. Step 1. Take Laplace transform on both sides: Let $\mathcal{L}\{y(t)\} = Y(s)$, and then

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0) = sY - 1, \qquad \mathcal{L}\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = s^2Y - s - 2.$$

Note the initial conditions are the first thing to go in!

$$\mathcal{L}\{y''(t)\} - 3\mathcal{L}\{y'(t)\} - 10\mathcal{L}\{y(t)\} = \mathcal{L}\{2\}, \qquad \Rightarrow \qquad s^2Y - s - 2 - 3(sY - 1) - 10Y = \frac{2}{s}.$$

Now we get an algebraic equation for $Y(s)$.

Step 2: Solve it for $Y(s)$:

$$(s^2 - 3s - 10)Y(s) = \frac{2}{s} + s + 2 - 3 = \frac{s^2 - s + 2}{s}, \qquad \Rightarrow \qquad Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)}.$$

Step 3: Take inverse Laplace transform to get $y(t) = \mathcal{L}^{-1}\{Y(s)\}$. The main technique here is **partial fraction**.

$$Y(s) = \frac{s^2 - s + 2}{s(s-5)(s+2)} = \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s+2} = \frac{A(s-5)(s+2) + Bs(s+2) + Cs(s-5)}{s(s-5)(s+2)}.$$

Compare the numerators:

$$s^2 - s + 2 = A(s-5)(s+2) + Bs(s+2) + Cs(s-5).$$

The previous equation holds for all values of s.
Set $s = 0$: we get $-10A = 2$, so $A = -\frac{1}{5}$.
Set $s = 5$: we get $35B = 22$, so $B = \frac{22}{35}$.
Set $s = -2$: we get $14C = 8$, so $C = \frac{4}{7}$.

Now, $Y(s)$ is written into sum of terms which we can find the inverse transform:

$$y(t) = A\mathcal{L}^{-1}\{\frac{1}{s}\} + B\mathcal{L}^{-1}\{\frac{1}{s-5}\} + C\mathcal{L}^{-1}\{\frac{1}{s+2}\} = -\frac{1}{5} + \frac{22}{35}e^{5t} + \frac{4}{7}e^{-2t}.$$

Structure of solutions:

- Take Laplace transform on both sides. You will get an algebraic equation for Y.
- Solve this equation to get $Y(s)$.
- Take inverse transform to get $y(t) = \mathcal{L}^{-1}\{Y\}$.

Example 15. (Distinct real roots, but one matches the source term.) Solve the initial value problem by Laplace transform,

$$y'' - y' - 2y = e^{2t}, \qquad y(0) = 0, \quad y'(0) = 1.$$

Answer. Take Laplace transform on both sides of the equation, we get

$$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - \mathcal{L}\{2y\} = \mathcal{L}\{e^{2t}\}, \qquad \Rightarrow \qquad s^2Y(s) - 1 - sY(s) - 2Y(s) = \frac{1}{s-2}.$$

Solve it for Y:

$$(s^2 - s - 2)Y(s) = \frac{1}{s-2} + 1 = \frac{s-1}{s-2}, \qquad \Rightarrow \qquad Y(s) = \frac{s-1}{(s-2)(s^2-s-2)} = \frac{s-1}{(s-2)^2(s+1)}.$$

Use partial fraction:

$$\frac{s-1}{(s-2)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}.$$

Compare the numerators:

$$s - 1 = A(s-2)^2 + B(s+1)(s-2) + C(s+1)$$

Set $s = -1$, we get $A = -\frac{2}{9}$.
Set $s = 2$, we get $C = \frac{1}{3}$.
Set $s = 0$ (any convenient values of s can be used in this step), we get $B = \frac{2}{9}$.
So

$$Y(s) = -\frac{2}{9}\frac{1}{s+1} + \frac{2}{9}\frac{1}{s-2} + \frac{1}{3}\frac{1}{(s-2)^2}$$

and

$$y(t) = \mathcal{L}^{-1}\{Y\} = -\frac{2}{9}e^{-t} + \frac{2}{9}e^{2t} + \frac{1}{3}te^{2t}.$$

Compare this to the method of undetermined coefficient: general solution of the equation should be $y = y_H + Y$, where y_H is the general solution to the homogeneous equation and Y is a particular solution. The characteristic equation is $r^2 - r - 2 = (r+1)(r-2) = 0$, so $r_1 = -1, r_2 = 2$, and $y_H = c_1 e^{-t} + c_2 e^{2t}$. Since 2 is a root, so the form of the particular solution is $Y = Ate^{2t}$. This discussion concludes that the solution should be of the form

$$y = c_1 e^{-t} + c_2 e^{2t} + Ate^{2t}$$

for some constants c_1, c_2, A. This fits well with our result.

Example 16. (Complex roots.) Solve

$$y'' - 2y' + 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 1.$$

Answer. Before we solve it, let's use the method of undetermined coefficients to find out which terms will be in the solution.

$$r^2 - 2r + 2 = 0, \qquad (r-1)^1 + 1 = 0, \qquad r_{1,2} = 1 \pm i,$$

$$y_H = c_1 e^t \cos t + c_2 e^t \sin t, \qquad Y = Ae^{-t},$$

so the solution should have the form:

$$y = y_H + Y = c_1 e^t \cos t + c_2 e^t \sin t + Ae^{-t}.$$

The Laplace transform would be

$$Y(s) = c_1 \frac{s-1}{(s-1)^2+1} + c_2 \frac{1}{(s-1)^2+1} + A\frac{1}{s+1} = \frac{c_1(s-1)+c_2}{(s-1)^2+1} + \frac{A}{s+1}.$$

This gives us some idea on which terms to look for in partial fraction.

Now let's use the Laplace transform:

$$Y(s) = \mathcal{L}\{y\}, \quad \mathcal{L}\{y'\} = sY - y(0) = sY,$$

$$\mathcal{L}\{y''\} = s^2 Y - sy(0) - y(0) = s^2 Y - 1.$$

$$s^2Y - 1 - 2sY + 2Y = \frac{1}{s+1}, \qquad \Rightarrow \qquad (s^2 - 2s + 2)Y(s) = \frac{1}{s+1} + 1 = \frac{s+2}{s+1}$$

$$Y(s) = \frac{s+2}{(s+1)(s^2-2s+2)} = \frac{s+2}{(s+1)((s-1)^2+1)} = \frac{A}{s+1} + \frac{B(s-1)+C}{(s-1)^2+1}$$

Compare the numerators:

$$s + 2 = A((s-1)^2 + 1) + (B(s-1) + C)(s+1).$$

Set $s = -1$: $5A = 1$, $A = \frac{1}{5}$.
Compare coefficients of s^2-term: $A + B = 0$, $B = -A = -\frac{1}{5}$.
Set any value of s, say $s = 0$: $2 = 2A - B + C$, $C = 2 - 2A + B = \frac{9}{5}$.

$$Y(s) = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{9}{5}\frac{1}{(s-1)^2+1}$$

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^t \cos t + \frac{9}{5}e^t \sin t.$$

We see this fits our prediction.

Example 17. (Pure imaginary roots.) Solve

$$y'' + y = \cos 2t, \qquad y(0) = 2, \quad y'(0) = 1.$$

Answer. Again, let's first predict the terms in the solution:

$$r^2 + 1 = 0, \qquad r_{1,2} = \pm i, \qquad y_H = c_1 \cos t + c_2 \sin t, \qquad Y = A \cos 2t$$

so

$$y = y_H + Y = c_1 \cos t + c_2 \sin t + A \cos 2t,$$

and the Laplace transform would be

$$Y(s) = c_1 \frac{s}{s^1+1} + c_2 \frac{1}{s^2+1} + A\frac{s}{s^2+4}.$$

Now, let's take Laplace transform on both sides:

$$s^2Y - 2s - 1 + Y = \mathcal{L}\{\cos 2t\} = \frac{s}{s^2+4}$$

$$(s^2+1)Y(s) = \frac{s}{s^2+4} + 2s + 1 = \frac{2s^3+s^2+9s+4}{s^2+4}$$
$$Y(s) = \frac{2s^3+s^2+9s+4}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}.$$

Comparing numerators, we get

$$2s^3+s^2+9s+4 = (As+B)(s^2+4) + (Cs+D)(s^2+1).$$

One may expand the right-hand side and compare terms to find A, B, C, D, but that takes more work.

Let's try by setting s into complex numbers.

Set $s = i$, and remember the facts $i^2 = -1$ and $i^3 = -i$, we have

$$-2i - 1 + 9i + 4 = (Ai + B)(-1+4),$$

which gives

$$3 + 7i = 3B + 3Ai, \quad \Rightarrow \quad B = 1, \quad A = \frac{7}{3}.$$

Set now $s = 2i$:

$$-16i - 4 + 18i + 4 = (2Ci + D)(-3),$$

then

$$0 + 2i = -3D - 6Ci, \quad \Rightarrow \quad D = 0, C = -\frac{1}{3}.$$

So

$$Y(s) = \frac{7}{3}\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

and

$$y(t) = \frac{7}{3}\cos t + \sin t - \frac{1}{3}\cos 2t.$$

A very brief review on partial fraction, targeted towards inverse Laplace transform.

Goal: rewrite a fractional form $\frac{P_n(s)}{P_m(s)}$ (where P_n is a polynomial of degree n) into sum of "simpler" terms. We assume $n < m$.

The type of terms appeared in the partial fraction is solely determined by the denominator $P_m(s)$. First, fact out $P_m(s)$, write it into product of terms of (i) $s-a$, (ii) s^2+a^2, (iii) $(s_a)^2+b^2$. The following table gives the terms in the partial fraction and their corresponding inverse Laplace transform.

term in $P_M(s)$	from where?	term in partial fraction	inverse L.T.
$s-a$	real root, or $g(t)=e^{at}$	$\dfrac{A}{s-a}$	Ae^{at}
$(s-a)^2$	double roots, or $r=a$ and $g(t)=e^{at}$	$\dfrac{A}{s-a}+\dfrac{B}{(s-a)^2}$	$Ae^{at}+Bte^{at}$
$(s-a)^3$	double roots, and $g(t)=e^{at}$	$\dfrac{A}{s-a}+\dfrac{B}{(s-a)^2}+\dfrac{C}{(s-a)^3}$	$Ae^{at}+Bte^{at}+\dfrac{C}{2}t^2e^{at}$
$s^2+\mu^2$	imaginary roots or $g(t)=\cos\mu t$ or $\sin\mu t$	$\dfrac{As+B}{s^2+\mu^2}$	$A\cos\mu t+B\sin\mu t$
$(s-\lambda)^2+\mu^2$	complex roots, or $g(t)=e^{\lambda t}\cos\mu t(\text{or }\sin\mu t)$	$\dfrac{A(s-\lambda)+B}{(s-\lambda)^2+\mu^2}$	$e^{\lambda t}(A\cos\mu t+B\sin\mu t)$

In summary, this table can be written

$$\begin{aligned}&\frac{P_n(s)}{(s-a)(s-b)^2(s-c)^3((s-\lambda)^2+\mu^2)}\\ =\quad&\frac{A}{s-a}+\frac{B_1}{s-b}+\frac{B_2}{(s-b)^2}+\frac{C_1}{s-c}+\frac{C_2}{(s-c)^2}+\frac{C_3}{(s-c)^3}+\frac{D_1(s-\lambda)+D_2}{(s-\lambda)^2+\mu^2}.\end{aligned}$$

6.3: Step functions

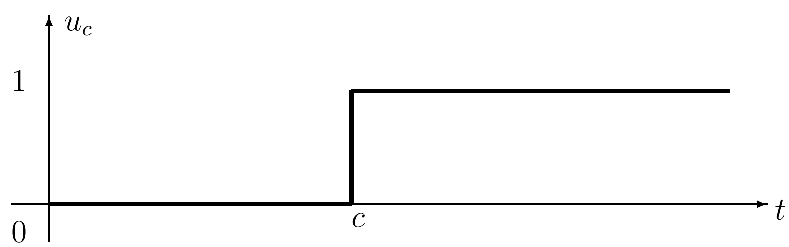
Topics:

- Definition and basic application of unit step (Heaviside) function,
- Laplace transform of step functions and functions involving step functions (piecewise continuous functions),
- Inverse transform involving step functions.

We use steps functions to form piecewise continuous functions.
Unit step function(Heaviside function):

$$u_c t = \begin{cases} 0, & 0 \le t < c, \\ 1, & c \le t. \end{cases}$$

for $c \ge 0$. A plot of $u_c(t)$ is below:



For a given function $f(t)$, if it is multiplied with $u_c(t)$, then

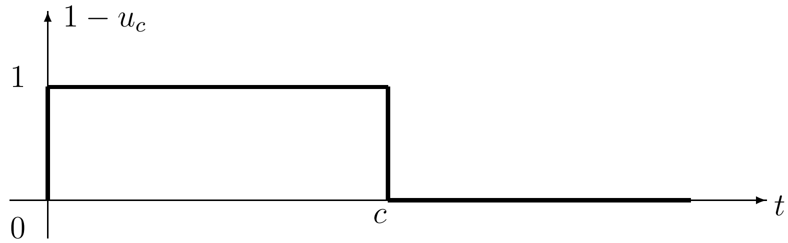
$$u_c t f(t) = \begin{cases} 0, & 0 < t < c, \\ f(t), & c \le t. \end{cases}$$

We say u_c *picks up* the interval $[c, \infty)$.

Example 1. Consider

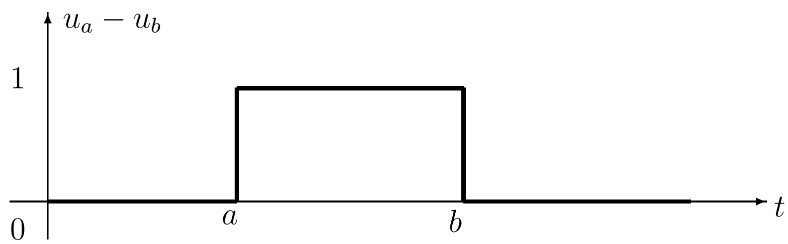
$$1 - u_c(t) = \begin{cases} 1, & 0 \le t < c, \\ 0, & c \le t. \end{cases}$$

A plot of this is given below



We see that this function picks up the interval $[0, c)$.

Example 2. Rectangular pulse. The plot of the function looks like



for $0 \leq a < b < \infty$. We see it can be expressed as

$$u_a(t) - u_b(t)$$

and it picks up the interval $[a, b)$.

Example 3. For the function

$$g(t) = \begin{cases} f(t), & a \leq t < b \\ 0, & \text{otherwise} \end{cases}$$

We can rewrite it in terms of the unit step function as

$$g(t) = f(t) \cdot \Big(u_a(t) - u_b(t)\Big).$$

Example 4. For the function

$$ft = \begin{cases} \sin t, & 0 \leq t < 1, \\ e^t, & 1 \leq t < 5, \\ t^2 & 5 \leq t, \end{cases}$$

we can rewrite it in terms of the unit step function as we did in Example 3, treat each interval separately

$$f(t) = \sin t \cdot \Big(u_0(t) - u_1(t)\Big) + e^t \cdot \Big(u_1(t) - u_5(t)\Big) + t^2 \cdot u_5(t).$$

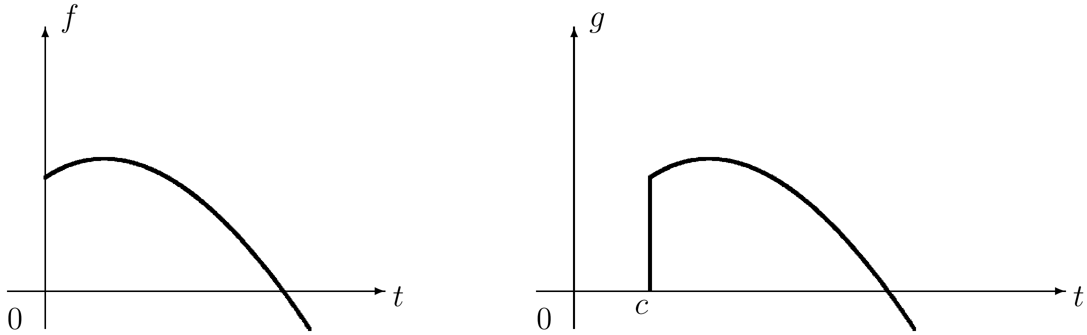
Laplace transform of $u_c(t)$: by definition

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st} \cdot 1\, dt = \left. \frac{e^{-st}}{-s} \right|_{t=c}^{\infty} = 0 - \frac{e^{-sc}}{-s} = \frac{e^{-st}}{s}, \qquad (s > 0).$$

Shift of a function: Given $f(t)$, $t > 0$, then

$$g(t) = \begin{cases} f(t-c), & c \le t, \\ 0, & 0 \le t < c, \end{cases}$$

is the shift of f by c units. See figure below.



Let $F(s) = \mathcal{L}\{f(t)\}$ be the Laplace transform of $f(t)$. Then, the Laplace transform of $g(t)$ is

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_c(t) \cdot f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt = \int_c^\infty e^{-st} f(t-c)\, dt.$$

Let $y = t - c$, so $t = y + c$, and $dt = dy$, and we continue

$$\mathcal{L}\{g(t)\} = \int_0^\infty e^{-s(y+c)} f(y)\, dy = e^{-sc} \int_0^\infty e^{-sy} f(y)\, dy = e^{-cs} F(s).$$

So we conclude

$$\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s)\,,$$

which is equivalent to

$$\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c)\,.$$

Note now we are only considering the domain $t \geq 0$. So $u_0(t) = 1$ for all $t \geq 0$.

In following examples we will compute Laplace transform of piecewise continuous functions with the help of the unit step function.

Example 5. Given

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \dfrac{\pi}{4}, \\ \sin t + \cos(t - \dfrac{\pi}{4}), & \dfrac{\pi}{4} \leq t. \end{cases}$$

It can be rewritten in terms of the unit step function as

$$f(t) = \sin t + u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4}).$$

(Or, if we write out each intervals

$$f(t) = \sin t(1 - u_{\frac{\pi}{4}}(t)) + (\sin t + \cos(t - \frac{\pi}{4}))u_{\frac{\pi}{4}}(t) = \sin t + u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4}).$$

which gives the same answer.)
And the Laplace transform of f is

$$F(s) = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\frac{\pi}{4}}(t) \cdot \cos(t - \frac{\pi}{4})\} = \frac{1}{s^2+1} + e^{-\frac{\pi}{4}s}\frac{s}{s^2+1}\,.$$

Example 6. Given

$$f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 1, & 1 \leq t. \end{cases}$$

It can be rewritten in terms of the unit step function as

$$f(t) = t(1 - u_1(t)) + 1 \cdot u_1(t) = t - (t-1)u_1(t)\,.$$

The Laplace transform is

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{(t-1)u_1(t)\} = \frac{1}{s^2} - e^{-s}\frac{1}{s^2}\,.$$

Example 7. Given

$$f(t) = \begin{cases} 0, & 0 \le t < 2, \\ t+3, & 2 \le t. \end{cases}$$

We can rewrite it in terms of the unit step function as

$$f(t) = (t+3)u_2(t) = (t-2+5)u_2(t) = (t-2)u_2(t) + 5u_2(t)\,.$$

The Laplace transform is

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-2)u_2(t)\} + 5\mathcal{L}\{u_2(t)\} = e^{-2s}\frac{1}{s^2} + 5e^{-2s}\frac{1}{s}\,.$$

Example 8. Given

$$g(t) = \begin{cases} 1, & 0 \le t < 2, \\ t^2, & 2 \le t. \end{cases}$$

We can rewrite it in terms of the unit step function as

$$g(t) = 1 \cdot (1 - u_2(t)) + t^2 u_2(t) = 1 + (t^2 - 1)u_2(t)\,.$$

Observe that

$$t^2 - 1 = (t-2+2)^2 - 1 = (t-2)^2 + 4(t-2) + 4 - 1 = (t-2)^2 + 4(t-2) + 3\,,$$

we have

$$g(t) = 1 + \big((t-2)^2 + 4(t-2) + 3\big)u_2(t)\,.$$

The Laplace transform is

$$\mathcal{L}\{g(t)\} = \frac{1}{s} + e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right)\,.$$

Example 9. Given

$$f(t) = \begin{cases} 0, & 0 \le t < 3, \\ e^t, & 3 \le t < 4, \\ 0, & 4 \le t. \end{cases}$$

We can rewrite it in terms of the unit step function as

$$f(t) = e^t \left(u_3(t) - u_4(t)\right) = u_3(t)e^{t-3}e^3 - u_4(t)e^{t-4}e^4 \,.$$

The Laplace transform is

$$\mathcal{L}\{g(t)\} = e^3 e^{-3s}\frac{1}{s-1} - e^4 e^{-4s}\frac{1}{s-1} = \frac{1}{s-1}\left[e^{-3(s-1)} - e^{-4(s-1)}\right] \,.$$

Inverse transform: We use two properties:

$$\mathcal{L}\{u_c(t)\} = e^{-cs}\frac{1}{s}, \qquad \text{and} \qquad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\cdot\mathcal{L}\{f(t)\} \,.$$

In the following examples we want to find $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Example 10.

$$F(s) = \frac{1-e^{-2s}}{s^3} = \frac{1}{s^3} - e^{-2s}\frac{1}{s^3}.$$

We know that $\mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{1}{2}t^2$, so we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}t^2 - u_2(t)\frac{1}{2}(t-2)^2 = \begin{cases} \frac{1}{2}t^2, & 0 \le t < 2, \\ \frac{1}{2}t^2 - \frac{1}{2}(t-2)^2, & 2 \le t. \end{cases}$$

Example 11. Given

$$F(s) = \frac{e^{-3s}}{s^2+s-12} = e^{-3s}\frac{1}{(s+4)(s+3)} = e^{-3s}\left(\frac{A}{s+4} + \frac{B}{s-3}\right).$$

By partial fraction, we find $A = -\frac{1}{7}$ and $B = \frac{1}{7}$. So

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = u_3(t)\left[Ae^{-4(t-3)} + Be^{3(t-3)}\right] = \frac{1}{7}u_3(t)\left[-e^{-4(t-3)} + e^{3(t-3)}\right]$$

which can be written as a p/w continuous function

$$f(t) = \begin{cases} 0, & 0 \le t < 3, \\ -\frac{1}{7}e^{-4(t-3)} + \frac{1}{7}e^{3(t-3)}, & 3 \le t. \end{cases}$$

Example 12. Given

$$F(s) = \frac{se^{-s}}{s^2+4s+5} = e^{-s}\frac{s+2-2}{(s+2)^2+1} = s^{-s}\left[\frac{s+2-2}{(s+2)^2+1} + \frac{s+2-2}{(s+2)^2+1}\right].$$

So

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = u_1(t)\left[e^{-2(t-1)}\cos(t-1) - 2e^{-2(t-1)}\sin(t-1)\right]$$

which can be written as a p/w continuous function

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ e^{-2(t-1)}\left[\cos(t-1) - 2\sin(t-1)\right], & 1 \le t. \end{cases}$$

6.4: Differential equations with discontinuous forcing functions

Topics:

- Solve initial value problems with discontinuous force, examples of various cases,
- Describe behavior of solutions, and make physical sense of them.

Next we study initial value problems with discontinuous force. We will start with an example.

Example 1. (Damped system with force, complex roots) Solve the following initial value problem

$$y'' + y' + y = g(t), \qquad g(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1, & 1 \le t, \end{cases} , \qquad y(0) = 1, \quad y'(0) = 0\,.$$

Answer. Let $\mathcal{L}\{y(t)\} = Y(s)$, so $\mathcal{L}\{y'\} = sY - 1$ and $\mathcal{L}\{y''\} = s^2Y - s$. Also we have $\mathcal{L}\{g(t)\} = \mathcal{L}\{u_1(t)\} = e^{-s}\frac{1}{s}$. Then

$$s^2Y - s + sY - 1 + Y = e^{-s}\frac{1}{s},$$

which gives

$$Y(s) = \frac{e^{-s}}{s(s^2+s+1)} + \frac{s+1}{s^2+s+1}.$$

Now we need to find the inverse Laplace transform for $Y(s)$. We have to do partial fraction first. We have

$$\frac{1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}.$$

Compare the numerators on both sides:

$$1 = A(s^2+s+1) + (Bs+C)\cdot s$$

Set $s = 0$, we get $A = 1$.

Compare s^2-term: $0 = A + B$, so $B = -A = -1$.

Compare s-term: $0 = A + C$, so $C = -A = -1$.

So

$$Y(s) = e^{-s}\left(\frac{1}{s} - \frac{s+1}{s^2+s+1}\right) + \frac{s+1}{s^2+s+1}\,.$$

We work out some detail

$$\frac{s+1}{s^2+s+1} = \frac{s+1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \frac{(s+\frac{1}{2}) + \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2},$$

so

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right).$$

We conclude

$$\begin{aligned} y(t) &= u_1(t)\left[1 - e^{-\frac{1}{2}(t-1)}\left(\cos\frac{\sqrt{3}}{2}(t-1) - \sin\frac{\sqrt{3}}{2}(t-1)\right)\right] \\ &\quad + e^{-\frac{1}{2}t}\left[\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right]\,. \end{aligned}$$

Remark: There are other ways to work out the partial fractions.

Extra question: What happens when $t \to \infty$?

Answer: We see all the terms with the exponential function will go to zero, so $y \to 1$ in the limit. We can view this system as the spring-mass system with damping. Since $g(t)$ becomes constant 1 for large t, and the particular solution (which is also the steady state) with 1 on the right hand side is 1, which provides the limit for y.

Further observation:

- We see that the solution to the homogeneous equation is

$$e^{-\frac{1}{2}t}\left[c_1\cos\frac{\sqrt{3}}{2}t + c_2\sin\frac{\sqrt{3}}{2}t\right],$$

  and these terms do appear in the solution.

- Actually the solution consists of two part: the forced response and the homogeneous solution.
- Furthermore, the g has a discontinuity at $t = 1$, and we see a jump in the solution also for $t = 1$, as in the term $u_1(t)$.

Example 2. (Undamped system with force, pure imaginary roots) Solve the following initial value problem

$$y'' + 4y = g(t) = \begin{cases} 0, & 0 \le t < \pi, \\ 1, & \pi \le t < 2\pi, \\ 0, & 2\pi \le t, \end{cases} \qquad y(0) = 1, \quad y'(0) = 0\,.$$

Rewrite

$$g(t) = u_\pi(t) - u_{2\pi}(t), \qquad \mathcal{L}\{g\} = e^{-\pi s}\frac{1}{s} - e^{-2\pi}\frac{1}{s}\,.$$

So

$$s^2Y - s + 4Y = \frac{1}{s}\left(e^{-\pi} - e^{-2\pi}\right).$$

Solve it for Y:

$$Y(s) = \frac{e^{-\pi} - e^{-2\pi}}{s(s^2+4)} + \frac{s}{s^2+4} = \frac{e^{-\pi}}{s(s^2+4)} - \frac{e^{-2\pi}}{s(s^2+4)} + \frac{s}{s^2+4}\,.$$

Work out partial fraction

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}, \qquad A = \frac{1}{4}, \quad B = -\frac{1}{4}, \quad C = 0.$$

So

$$\mathcal{L}^{-1}\{\frac{1}{s(s^2+4)}\} = \frac{1}{4} - \frac{1}{4}\cos 2t\,.$$

Now we take inverse Laplace transform of Y

$$\begin{aligned} y(t) &= u_\pi(t)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-\pi)\right) - u_{2\pi}(t)\left(\frac{1}{4} - \frac{1}{4}\cos 2(t-2\pi)\right) + \cos 2t \\ &= (u_\pi(t) - u_{2\pi})\frac{1}{4}(1 - \cos 2t) + \cos 2t \\ &= \cos 2t + \begin{cases} \frac{1}{4}(1-\cos 2t), & \pi \le t < 2\pi, \\ 0, & \text{otherwise}, \end{cases} \\ &= \text{homogeneous solution} + \text{forced response} \end{aligned}$$

Example 3. In Example 2, let

$$g(t) = \begin{cases} 0, & 0 \le t < 4, \\ e^t, & 4 \le 5 < 2\pi, \\ 0, & 5 \le t. \end{cases}$$

Find $Y(s)$.

Answer. Rewrite

$$g(t) = e^t(u_4(t) - u_5(t)) = u_4(t)e^{t-4}e^4 - u_5(t)e^{t-5}e^5 \,,$$

so

$$G(s) = \mathcal{L}\{g(t)\} = e^4 e^{-4s}\frac{1}{s-1} - e^5 e^{-5s}\frac{1}{s-1} \,.$$

Take Laplace transform of the equation, we get

$$(s^2+4)Y(s) = G(s)+s, \qquad Y(s) = \left(e^4e^{-4s} - e^5e^{-5s}\right)\frac{1}{(s-1)(s^2+4)} + \frac{s}{s^2+4} \,.$$

Remark: We see that the first term will give the forced response, and the second term is from the homogeneous equation.

The students may work out the inverse transform as a practice.

Example 4. (Undamped system with force, example 2 from the book p. 334)

$$y'' + 4y = g(t), \quad y(0) = 0, y'(0) = 0, \quad g(t) = \begin{cases} 0, & 0 \le t < 5, \\ (t-5)/5, & 5 \le 5 < 10, \\ 1, & 10 \le t. \end{cases}$$

Let's first work on $g(t)$ and its Laplace transform

$$g(t) = \frac{t-5}{5}(u_5(t) - u_{10}(t)) + u_{10}(t) = \frac{1}{5}u_5(t)(t-5) - \frac{1}{5}u_{10}(t)(t-10),$$

$$G(s) = \mathcal{L}\{g\} = \frac{1}{5}e^{-5s}\frac{1}{s^2} - \frac{1}{5}e^{-10s}\frac{1}{s^2}$$

Let $Y(s) = \mathcal{L}\{y\}$, then

$$(s^2+4)Y(s) = G(s), \qquad Y(s) = \frac{G(s)}{s^2+4} = \frac{1}{5}e^{-5s}\frac{1}{s^2(s^2+4)} - \frac{1}{5}e^{-10s}\frac{1}{s^2(s^2+4)}$$

Work out the partial fraction:

$$H(s) \doteq \frac{1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+2D}{s^2+4}$$

one gets $A = 0$, $B = \frac{1}{4}$, $C = 0$, $D = -\frac{1}{8}$. So

$$h(t) \doteq \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s^2} - \frac{1}{8}\cdot\frac{2}{s^2+2^2}\right\} = \frac{1}{4}t - \frac{1}{8}\sin 2t.$$

Go back to $y(t)$

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y\} = \frac{1}{5}u_5(t)h(t-5) - \frac{1}{5}u_{10}(t)h(t-10) \\
&= \frac{1}{5}u_5(t)\left[\frac{1}{4}(t-5) - \frac{1}{8}\sin 2(t-5)\right] - \frac{1}{5}u_{10}(t)\left[\frac{1}{4}(t-10) - \frac{1}{8}\sin 2(t-10)\right] \\
&= \begin{cases} 0, & 0 \le t < 5, \\ \frac{1}{20}(t-5) - \frac{1}{40}\sin 2(t-5), & 5 \le 5 < 10, \\ \frac{1}{4} - \frac{1}{40}(\sin 2(t-5) - \sin 2(t-10)), & 10 \le t. \end{cases}
\end{aligned}$$

Note that for $t \ge 10$, we have $y(t) = \frac{1}{4} + R\cdot\cos(2t+\delta)$ for some amplitude R and phase δ.

The plots of g and y are given in the book. Physical meaning and qualitative nature of the solution:

The source $g(t)$ is known as *ramp loading*. During the interval $0 < t < 5$, $g = 0$ and initial conditions are all 0. So solution remains 0. For large time t, $g = 1$. A particular solution is $Y = \frac{1}{4}$. Adding the homogeneous solution, we should have $y = \frac{1}{4} + c_1 \sin 2t + c_2 \cos 2t$ for t large. We see this is actually the case, the solution is an oscillation around the constant $\frac{1}{4}$ for large t.

Chapter 7. Systems of two linear differential equations

7.1: Introduction to systems of differential equations

Given

$$ay'' + by' + cy = g(t), \qquad y(0) = \alpha, \quad y'(0) = \beta$$

we can do a variable change: let

$$x_1 = y, \qquad x_2 = x_1' = y'$$

then

$$\begin{cases} x_1' &=& x_2 \\ x_2' &=& y'' = \dfrac{1}{a}(g(t) - bx_2 - cx_1) \end{cases} \qquad \begin{cases} x_1(0) &=& \alpha \\ x_2(0) &=& \beta \end{cases}$$

Observation: For any 2nd order equation, we can rewrite it into a system of 2 first order equations.

Example 1. Given

$$y'' + 5y' - 10y = \sin t, \qquad y(0) = 2, \quad y'(0) = 4$$

Rewrite it into a system of first order equations: let $x_1 = y$ and $x_2 = y' = x_1'$, then

$$\begin{cases} x_1' &=& x_2 \\ x_2' &=& y'' = -5x_2 + 10x_1 + \sin t \end{cases} \qquad \text{I.C.'s:} \quad \begin{cases} x_1(0) &=& 2 \\ x_2(0) &=& 4 \end{cases}$$

We can do the same thing to any high order equations. For n-th order differential equation:

$$y^{(n)} = F(t, y, y', \cdots, y^{(n-1)})$$

define the variable change:

$$x_1 = y, \quad x_2 = y', \quad \cdots \quad x_n = y^{(n-1)}$$

we get

$$\begin{cases} x_1' &= y' = x_2 \\ x_2' &= y'' = x_3 \\ \vdots \\ x_{n-1}' &= y^{(n-1)} = x_n \\ x_n' &= y^{(n)} = F(t, x_1, x_2, \cdots, x_n) \end{cases}$$

with corresponding source terms.

Reversely, we can convert a 1st order system into a high order equation.

Example 2. Given

$$\begin{cases} x_1' &= 3x_1 - 2x_2 \\ x_2' &= 2x_1 - 2x_2 \end{cases} \qquad \begin{cases} x_1(0) &= 3 \\ x_2(0) &= \frac{1}{2} \end{cases}$$

Eliminate x_2: the first equation gives

$$2x_2 = 3x_1 - x_1', \qquad x_2 = \frac{3}{2}x_1 - \frac{1}{2}x_1'.$$

Plug this into second equation, we get

$$\left(\frac{3}{2}x_1 - \frac{1}{2}x_1'\right)' = 2x_1 - 2x_2 = -x_1 + x_1'$$

$$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = -x_1 + x_1'$$

$$x_1'' - x_1' - 2x_1 = 0$$

with the initial conditions:

$$x_1(0) = 3, \quad x_1'(0) = 3x_1(0) - 2x_2(0) = 8.$$

This we know how to solve!

Definition of a solution: a set of functions $x_1(t), x_2(t), \cdots, x_n(t)$ that satisfy the differential equations and the initial conditions.

7.2: Review of matrices

A matrix of size $m \times n$:

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix} = (a_{i,j}), \quad 1 \le i \le m, \ 1 \le j \le n.$$

We consider only square matrices, i.e., $m = n$, in particular for $n = 2$ and 3.

Basic operations: A, B are two square matrices of size n.

- Addition: $A + B = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$
- Scalar multiple: $\alpha A = (\alpha \cdot a_{ij})$
- Transpose: A^T switch the $a_{i,j}$ with a_{ji}. $(A^T)^T = A$.
- Product: For $A \cdot B = C$, it means $c_{i,j}$ is the inner product of (ith row of A) and (jth column of B). Example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x & y \\ u & v \end{pmatrix} = \begin{pmatrix} ax + bu & ay + bv \\ cx + du & cy + dv \end{pmatrix}$$

We can express system of linear equations using matrix product.

Example 1.

$$\begin{cases} x_1 - x_2 + 3x_3 &= 4 \\ 2x_1 \quad\quad + 5x_3 &= 0 \\ \quad\quad x_2 - x_3 &= 7 \end{cases} \quad \text{can be expressed as:} \quad \begin{pmatrix} 1 & -1 & 3 \\ 2 & 0 & 5 \\ 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 7 \end{pmatrix}$$

Example 2.

$$\begin{cases} x_1' &= a(t)x_1 + b(t)x_2 + g_1(t) \\ x_2' &= c(t)x_1 + d(t)x_2 + g_2(t) \end{cases} \quad \Rightarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Some properties:

- Identity I: $I = \text{diag}(1, 1, \cdots, 1)$, $AI = IA = A$.

- Determinant $\det(A)$:

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

$$\det\begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix} = avx + bwx + cuy - xvc - ywa - zub.$$

- Inverse $\mathrm{inv}(A) = A^{-1}$: $A^{-1}A = AA^{-1} = I$.

- The following statements are all equivalent:
  - (1) A is invertible;
  - (2) A is non-singular;
  - (3) $\det(A) \neq 0$;
  - (4) row vectors in A are linearly independent;
  - (5) column vectors in A are linearly independent.
  - (6) All eigenvalues of A are non-zero.

7.3: Eigenvalues and eigenvectors

Eigenvalues and eigenvectors of A (A is 2×2 or 3×3.)

λ: scalar value, $\vec{v}$: column vector, $\vec{v} \not\equiv 0$.
If $A\vec{v} = \lambda\vec{v}$, then $(\lambda, \vec{v})$ is the (eigenvalue, eigenvector) of A.

They are also called an eigen-pair of A.

Remark: If $\vec{v}$ is an eigenvector, then $\alpha\vec{v}$ for any $\alpha \neq 0$ is also an eigenvector, because

$$A(\alpha\vec{v}) = \alpha A\vec{v} = \alpha\lambda\vec{v} = \lambda(\alpha\vec{v}).$$

How to find (λ, v):

$$A\vec{v} - \lambda\vec{v} = 0, \qquad (A - \lambda I)\vec{v} = 0, \qquad \det(A - \lambda I) = 0.$$

We see that $\det(A - \lambda I)$ is a polynomial of degree 2 (or 3) in λ, and it is also called the characteristic polynomial of A. We need to find its roots.

Example 1: Find the eigenvalues and the eigenvectors of A where

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

Answer. Let's first find the eigenvalues.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = 0, \qquad \lambda_1 = -1, \ \lambda_2 = 3.$$

Now, let's find the eigenvector $\vec{v}_1$ for $\lambda_1 = -1$: let $\vec{v}_1 = (a, b)^T$

$$(A - \lambda_1 I)\vec{v}_1 = 0, \quad \Rightarrow \quad \begin{pmatrix} 1-(-1) & 1 \\ 4 & 1-(-1) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\Rightarrow \quad \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so

$$2a + b = 0, \qquad \text{choose } \ a = 1, \ \text{then we have } b = -2, \quad \Rightarrow \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Finally, we will compute the eigenvector $\vec{v}_2 = (c, d)^T$ for $\lambda_2 = 3$:

$$(A - \lambda_1 I)\vec{v}_2 = 0, \quad \Rightarrow \quad \begin{pmatrix} 1-3 & 1 \\ 4 & 1-3 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\Rightarrow \quad \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so

$$2c - d = 0, \qquad \text{choose} c = 1, \text{ then we have } d = 2, \qquad \Rightarrow \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Example 2. Eigenvalues can be complex numbers.

$$A = \begin{pmatrix} 2 & -9 \\ 4 & 2 \end{pmatrix}.$$

Let's first find the eigenvalues.

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -9 \\ 4 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 + 36 = 0, \quad \Rightarrow \quad \lambda_{1,2} = 2 \pm 6i$$

We see that $\lambda_2 = \bar{\lambda}_1$, complex conjugate. The same will happen to the eigenvectors, i.e., $\vec{v}_1 = \bar{\vec{v}}_2$. So we need to only find one. Take $\lambda_1 = 2 + 6i$, we compute $\vec{v} = (v^1, v^2)^T$:

$$(A - \lambda_1 I)\vec{v} = 0, \qquad \begin{pmatrix} -i6 & -9 \\ 4 & -i6 \end{pmatrix} \cdot \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = 0,$$

$$-6iv^1 - 9v^2 = 0, \qquad \text{choose } v^1 = 1, \text{ so } v^2 = -\frac{2}{3}i,$$

so

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -\frac{2}{3}i \end{pmatrix}, \qquad \vec{v}_2 = \bar{\vec{v}}_1 = \begin{pmatrix} 1 \\ \frac{2}{3}i \end{pmatrix}.$$

7.4: Basic theory of systems of first order linear equation

General form of a system of first order equations written in matrix-vector form:

$$\vec{x}' = P(t)\vec{x} + \vec{g}.$$

If $\vec{g} = 0$, it is homogeneous. We only consider this case, so

$$\vec{x}' = P(t)\vec{x}.$$

Superposition: If $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are two solutions of the homogeneous system, then any linear combination $c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution.

Wronskian of vector-valued functions are defined as

$$W[\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)] = \det X(t)$$

where X is a matrix whose columns are the vectors $\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)$.

If $\det X(t) \neq 0$, then $\big(\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)\big)$ is a set of linearly independent functions.

A set of linearly independent solutions $\big(\vec{x}_1(t), \vec{x}_2(t), \cdots, \vec{x}_n(t)\big)$ is said to be a *fundamental set of solutions*.

The general solution is the linear combination of these solutions, i.e.

$$\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t).$$

7.5: Homogeneous systems of two equations with constant coefficients.

We consider the following initial value problem:

$$\left\{ \begin{array}{lcl} x_1' & = & ax_1 + bx_2 \\ x_2' & = & cx_1 + dx_2 \end{array} \right. \qquad \text{I.C.'s:} \quad \left\{ \begin{array}{lcl} x_1(0) & = & \bar{x}_1 \\ x_2(0) & = & \bar{x}_2 \end{array} \right.$$

In matrix vector form:

$$\vec{x}' = A\vec{x}, \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \vec{x}(0) = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Claim: If $(\lambda, \vec{v})$ is an eigen-pair for A, then $\vec{z} = e^{\lambda t}\vec{v}$ is a solution to $\vec{x}' = A\vec{x}$.
Proof.

$$\vec{z}' = (e^{\lambda t}\vec{v})' = (e^{\lambda t})'\vec{v} = \lambda e^{\lambda t}\vec{v}$$
$$A\vec{z} = A(e^{\lambda t}\vec{v}) = e^{\lambda t}(A\vec{v}) = e^{\lambda t}\lambda\vec{v}$$

Therefore $\vec{z}' = A\vec{z}$ so $\vec{z}$ is a solution.
Steps to solve the initial value problem:

- Step I: Find eigenvalues of A: λ_1, λ_2.
- Step II: Find the corresponding eigenvectors $\vec{v}_1, \vec{v}_2$.
- Step III: Form two solutions: $\vec{z}_1 = e^{\lambda_1 t}\vec{v}_1$, $\vec{z}_2 = e^{\lambda_2 t}\vec{v}_2$.
- Step IV: Check that $\vec{z}_1, \vec{z}_2$ are linearly independent: the Wronskian

  $$W(\vec{z}_1, \vec{z}_2) = \det(\vec{z}_1, \vec{z}_2) \neq 0.$$

  (This step is usually OK in our problems.)
- Step V: Form the general solution: $\vec{x} = c_1\vec{z}_1 + c_2\vec{z}_2$.
- If initial condition $\vec{x}(0)$ is given, then use it to determine c_1, c_2.

We will start with an example.

Example 1. Solve

$$\vec{x}' = A\vec{x}, \qquad A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}.$$

First, find out the eigenvalues of A. By an example in 7.3, we have

$$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

So the general solution is

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Write it out in components:

$$\begin{cases} x_1(t) & = & c_1 e^{-t} + c_2 e^{3t} \\ x_2(t) & = & -2c_1 e^{-t} + 2c_2 e^{3t} \end{cases}.$$

Qualitative property of the solutions:

- What happens when $t \to \infty$?

  If $c_2 > 0$, then $x_1 \to \infty, x_2 \to \infty$.

  If $c_2 < 0$, then $x_1 \to -\infty, x_2 \to -\infty$.

  Asymptotic relation between x_1, x_2: look at $\frac{x_1}{x_2}$:

  $$\frac{x_1}{x_2} = \frac{c_1 e^{-t} + c_2 e^{3t}}{-2c_1 e^{-t} + 2c_2 e^{3t}}.$$

  As $t \to \infty$, we have

  $$\frac{x_1}{x_2} = \frac{c_2 e^{3t}}{2c_2 e^{3t}} = \frac{1}{2}.$$

  This means, $x_1 \to 2x_2$ asymptotically.

- What happens when $t \to -\infty$?

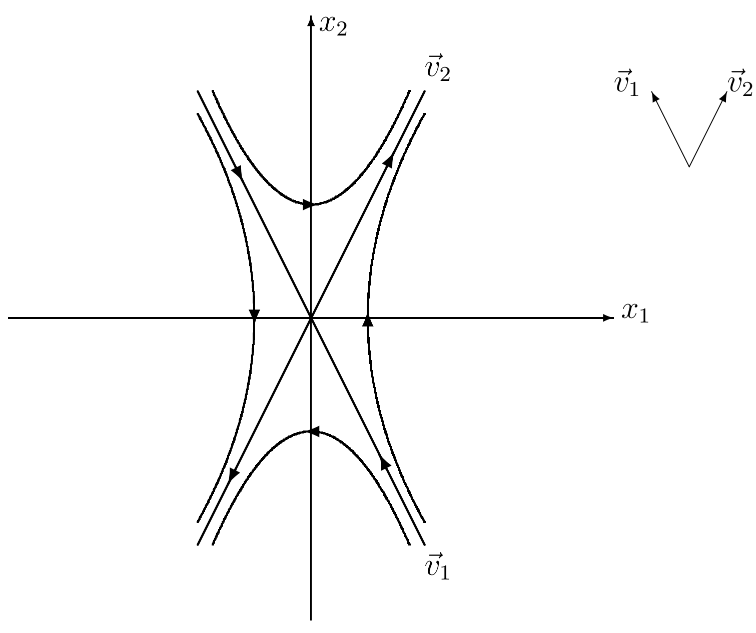
  Looking at $\frac{x_1}{x_2}$, we see as $t \to -\infty$ we have

  $$\frac{x_1}{x_2} = \frac{c_1 e^{-t}}{-2c_1 e^{-t}} = -\frac{1}{2},$$

  which means, $x_1 \to -2x_2$ asymptotically as $t \to -\infty$.

Phase portrait. is the trajectories of various solutions in the $x_2 - x_1$ plane.

- Since A is non-singular, then $\vec{x} = \vec{0}$ is the only critical point such that $\vec{x}' = A\vec{x} = 0$.
- If $c_1 = 0$, then $\dfrac{x_1}{x_2} = \dfrac{c_2 e^{3t}}{2c_2 e^{3t}} = \dfrac{1}{2}$, so the trajectory is a straight line $x_1 = 2x_2$.
  Note that this is exactly the direction of $\vec{v}_2$.
  Since $\lambda_2 = 3 > 0$, the trajectory is going away from 0.
- If $c_2 = 0$, then $\frac{x_1}{x_2} = \frac{c_1 e^{-t}}{-2c_1 e^{-t}} = -\frac{1}{2}$, so the trajectory is another straight line $x_1 = -2x_2$.
  Note that this is exactly the direction of $\vec{v}_1$.
  Since $\lambda_2 = -1 < 0$, the trajectory is going towards 0.
- For general cases where c_1, c_2 are not 0, the trajectories should start (asymptotically) from line $x_1 = -2x_2$, and goes to line $x_1 = 2x_2$ asymptotically as t grows.



Definition: If A has two real eigenvalues of opposite signs, the origin (critical point) is called a **saddle point**. A saddle point is unstable.

Tips for drawing phase portrait for saddle point: only need the eigenvalues and eigenvectors!

General case: If two eigenvalues of A are $\lambda_1 < 0$ and $\lambda_2 > 0$, with two corresponding eigenvectors $\vec{v}_1, \vec{v}_2$. To draw the phase portrait, we follow these guidelines:

- The general solution is

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$

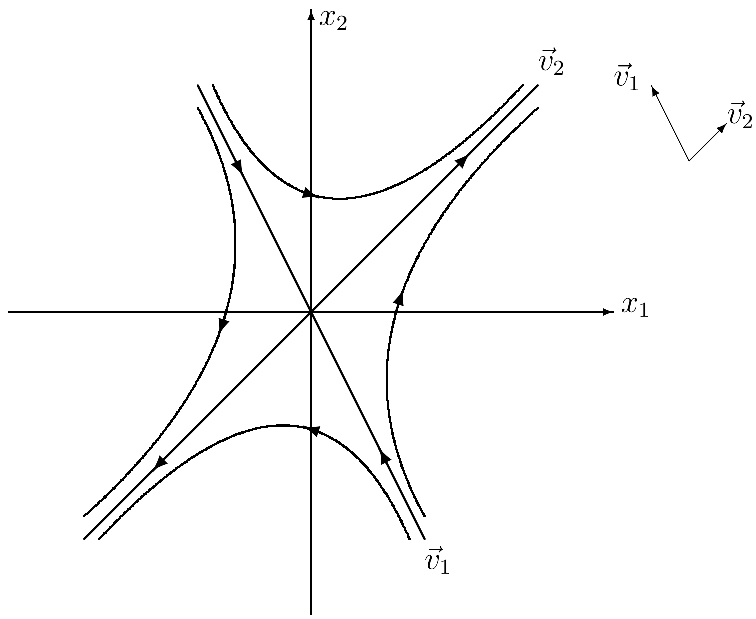
- If $c_1 = 0$, then the solution is $\vec{x} = c_2 e^{\lambda_2 t}\vec{v}_2$. We see that the solution vector is a scalar multiple of $\vec{v}_2$. This means a line parallel to $\vec{v}_2$ through the origin is a trajectory. Since $\lambda_2 > 0$, solutions $|\vec{x}| \to \infty$ along this line, so the arrows are pointing away from the origin.

- The similar other half: if $c_2 = 0$, then the solution is $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1$. We see that the solution vector is a scalar multiple of $\vec{v}_1$. This means a line parallel to $\vec{v}_1$ through the origin is a trajectory. Since $\lambda_1 < 0$, solutions approach 0 along this line, so the arrows are pointing toward the origin.

- Now these two lines cut the plane into 4 regions. We need to draw at least one trajectory in each region. In the region, we have the general case, i.e., $c_1 \neq 0$ and $c_2 \neq 0$. We need to know the asymptotic behavior. We have

$$\begin{aligned} t \to \infty, &\quad => \quad \vec{x} \to c_2 e^{\lambda_2 t}\vec{v}_2 \\ t \to -\infty, &\quad => \quad \vec{x} \to c_1 e^{\lambda_1 t}\vec{v}_1 \end{aligned}$$

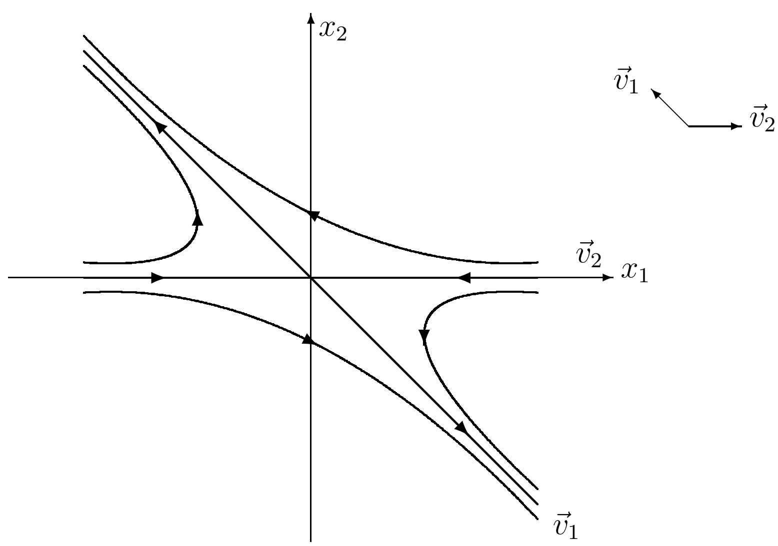
  We see these are exactly the two straight lines we just made. This means, all trajectories come from the direction of $\vec{v}_1$, and will approach $\vec{v}_2$ as t grows. See the plot below.



Example 2. Suppose we know the eigenvalues and eigenvectors of A:

$$\lambda_1 = 3, \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda_1 = -3, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then the phase portrait looks like this:



If the two real distinct eigenvalue have the same sign, the situation is quite different.

Example 3. Consider the homogeneous system

$$\vec{x}' = A\vec{x}, \qquad A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}.$$

Find the general solution and sketch the phase portrait.

Answer.

- Eigenvalues of A:

  $$\det(A-\lambda I) = \det \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix} = (-3-\lambda)(-2-\lambda)-2 = \lambda^2+5\lambda+4 = (\lambda+1)(\lambda+4) = 0,$$

  So $\lambda_1 = -1, \lambda_2 = -4$. (Two eigenvalues are both negative!)

- Find the eigenvector for λ_1. Call it $\vec{v}_1 = (a,b)^T$,

  $$(A-\lambda_1 I)\vec{v}_1 = \begin{pmatrix} -3+1 & 2 \\ 1 & -2+1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  This gives $a = b$. Choose it to be 1, we get $\vec{v}_1 = (1,1)^T$.

- Find the eigenvector for λ_2. Call it $\vec{v}_2 = (c,d)^T$,

  $$(A-\lambda_2 I)\vec{v}_1 = \begin{pmatrix} -3+4 & 2 \\ 1 & -2+4 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  This gives $c + 2d = 0$. Choose $d = 1$, then $c = -2$. So $\vec{v}_2 = (-2,1)^T$.

- General solution is

  $$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-4t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

  Write it out in components:

  $$\begin{cases} x_1(t) &= c_1 e^{-t} - 2c_2 e^{-4t} \\ x_2(t) &= c_1 e^{-t} + c_2 e^{-4t} \end{cases}.$$

Phase portrait:

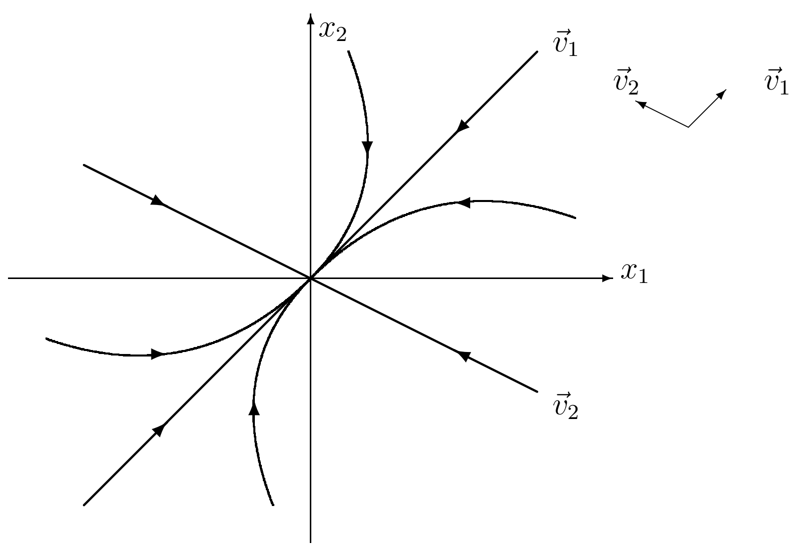
- If $c_1 = 0$, then $\vec{x} = c_2 e^{\lambda_2 t}\vec{v}_2$, so the straight line through the origin in the direction of $\vec{v}_2$ is a trajectory. Since $\lambda_2 < 0$, the arrows point toward the origin.

- If $c_2 = 0$, then $\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1$, so the straight line through the origin in the direction of $\vec{v}_1$ is a trajectory. Since $\lambda_1 < 0$, the arrows point toward the origin.

- For the general case, when $c_1 \neq 0$ and $c_2 \neq 0$, we have

$$\begin{array}{rccc} t \to -\infty, & => & \vec{x} \to 0, & \vec{x} \to c_2 e^{\lambda_2 t}\vec{v}_2 \\ t \to \infty, & => & |\vec{x}| \to \infty, & \vec{x} \to c_1 e^{\lambda_1 t}\vec{v}_1 \end{array}$$

  So all trajectories come into the picture in the direction of $\vec{v}_2$, and approach the origin in the direction of $\vec{v}_1$. See the plot below.



In the previous example, if $\lambda_1 > 0$, $\lambda_2 > 0$, say $\lambda_1 = 1$ and $\lambda_2 = 4$, and $\vec{v}_1, \vec{v}_2$ are the same, then the phase portrait will look the same, but with all arrows going away from 0.

Definition: If $\lambda_1 \neq \lambda_2$ are real with the same sign, the critical point $\vec{x} = 0$ is called a *node*.

If $\lambda_1 > 0$, $\lambda_2 > 0$, this node is called a *source*.

If $\lambda_1 < 0$, $\lambda_2 < 0$, this node is called a *sink*.

A sink is stable, and a source is unstable.

Example 4. (Source node) Suppose we know the eigenvalues and eigenvectors of A are

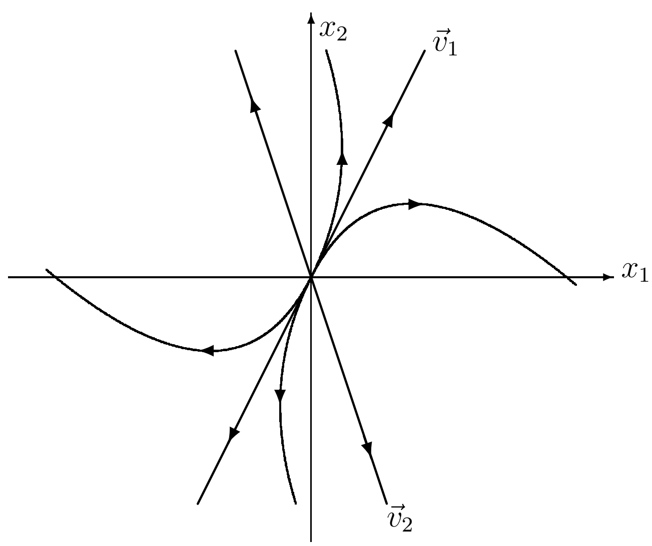
$$\lambda_1 = 3, \quad \lambda_2 = 4, \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(1) Find the general solution for $\vec{x}' = A\vec{x}$, (2) Sketch the phase portrait.

Answer. (1) The general solution is simple, just use the formula

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

(2) Phase portrait: Since $\lambda_2 > \lambda_1$, then the solution approach $\vec{v}_2$ as time grows. As $t \to -\infty$, $\vec{x} \to c_1 e^{\lambda_1 t}\vec{v}_1$. See the plot below.



Summary:

(1). If λ_1 and λ_2 are real and with opposite sign: the origin is a saddle point,

and it's unstable;
(2). If λ_1 and λ_2 are real and with same sign: the origin is a node.
If $\lambda_1, \lambda_2 > 0$, it's a source node, and it's unstable;
If $\lambda_1, \lambda_2 < 0$, it's a sink node, and it's stable;

7.6: Complex eigenvalues

If A has two complex eigenvalues, they will be a pair of complex conjugate numbers, say $\lambda_{1,2} = \alpha \pm i\beta$, $\beta \neq 0$.

The two corresponding eigenvectors will also be complex conjugate, i.e,

$$\vec{v}_1 = \bar{\vec{v}}_2.$$

We have two solutions

$$\vec{z}_1 = e^{\lambda_1 t}\vec{v}_1, \qquad \vec{z}_2 = e^{\lambda_2 t}\vec{v}_2.$$

They are complex-valued functions, and they also are complex conjugate. We seek real-valued solutions. By the principle of superposition,

$$\vec{y}_1 = \frac{1}{2}(\vec{z}_1 + \vec{z}_2) = \mathrm{Re}(\vec{z}_1), \qquad \vec{y}_2 = \frac{1}{2i}(\vec{z}_1 - \vec{z}_2) = \mathrm{Im}(\vec{z}_1)$$

are also two solutions, and they are real-valued.

One can show that they are linearly independent, so they form a set of fundamental solutions. The general solution is then $\vec{x} = c_1\vec{y}_1 + c_2\vec{y}_2$.

Now let's derive the formula for the general solution. We have two eigenvalues: λ and $\bar{\lambda}$, two eigenvectors: $\vec{v}$ and $\bar{\vec{v}}$, which we can write

$$\lambda = \alpha + i\beta, \qquad \vec{v} = \vec{v}_r + i\vec{v}_i.$$

One solution can be written

$$\begin{aligned} \vec{z} &= e^{\lambda t}\vec{v} \;=\; e^{(\alpha+i\beta)t}(\vec{v}_r + i\vec{v}_i)e^{\alpha t}(\cos\beta t + i\sin\beta t)\cdot(\vec{v}_r + i\vec{v}_i) \\ &= e^{\alpha t}\left(\cos\beta t\cdot\vec{v}_r - \sin\beta t\cdot\vec{v}_i + i(\sin\beta t\cdot\vec{v}_r + \cos\beta t\cdot\vec{v}_i)\right). \end{aligned}$$

The general solution is

$$\vec{x} = c_1 e^{\alpha t}\left(\cos\beta t\cdot\vec{v}_r - \sin\beta t\cdot\vec{v}_i\right) + c_2 e^{\alpha t}\left(\sin\beta t\cdot\vec{v}_r + \cos\beta t\cdot\vec{v}_i\right).$$

Notice now if $\alpha = 0$, i.e., we have pure imaginary eigenvalues. The $\vec{x}$ is a harmonic oscillation, which is a periodic function. This means in the phase portrait all trajectories are closed curves.

Example 1. (pure imaginary eigenvalues.) Find the general solution and sketch the phase portrait of the system:

$$\vec{x}\,' = A\vec{x}, \qquad A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}.$$

Answer. First find the eigenvalues of A:

$$\det(A - \lambda I) = \lambda^2 + 4 = 0, \qquad \lambda_{1,2} = \pm 2i.$$

Eigenvectors: need to find one $\vec{v} = (a, b)^T$ for $\lambda = 2i$:

$$(A - \lambda I)\vec{v} = 0, \qquad \begin{pmatrix} -2i & -4 \\ 1 & -2i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$a - 2ib = 0, \qquad \text{choose } b = 1, \text{ then } a = 2i,$$

then

$$\vec{v} = \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

The general solution is

$$\vec{x} = c_1 \left[\cos 2t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin 2t \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right] + c_2 \left[\sin 2t \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cos 2t \cdot \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right].$$

Write out the components, we get

$$\begin{aligned} x_1(t) &= -2c_1 \sin 2t + 2c_2 \cos 2t \\ x_2(t) &= c_1 \cos 2t + c_2 \sin 2t. \end{aligned}$$

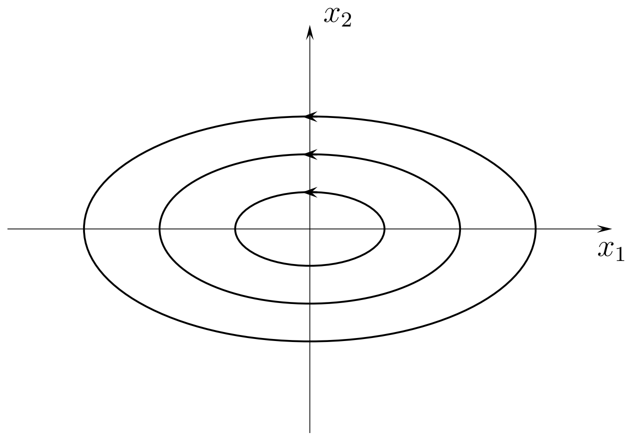
Phase portrait:

- $\vec{x}$ is a periodic function, so all trajectories are closed curves around the origin.
- They do not intersect with each other. This follows from the uniqueness of the solution.
- They are ellipses. Because we have the relation:

$$(x_1/2)^2 + (x_2)^2 = \text{ constant}.$$

- The arrows are pointing either clockwise or counter clockwise, determined by A. In this example, take $\vec{x} = (1, 0)^T$, a point on the x_1-axis. By the differential equations, we get $\vec{x}\,' = A\vec{x} = (0, 1)^T$, which is a vector pointing upward. So the arrows are counter-clockwise.

See plot below.



Definition. The origin in this case is called a *center*. A center is stable (b/c solutions don't blow up), but is not asymptotically stable (b/c solutions don't approach the origin as time goes).

If the complex eigenvalues have non-zero real part, the situation is still different.

Example 2. Consider the system

$$\vec{x}\,' = A\vec{x}, \qquad A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}.$$

First, we compute the eigenvalues:

$$\det(A - \lambda I) = (3 - \lambda)(-1 - \lambda) + 8 = \lambda^2 - 2\lambda + 5 = 0,$$

$$\lambda_{1,2} = 1 \pm 2i, \quad \Rightarrow \quad \alpha = 1, \quad \beta = 2.$$

Eigenvectors: need to compute only one $\vec{v} = (a, b)^T$. Take $\lambda = 1 + 2i$,

$$(A - \lambda I)\vec{v} = \begin{pmatrix} 2 - 2i & -2 \\ 4 & -2 - 2i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$(2-2i)a - 2b = 0.$$

Choosing $a = 1$, then $b = 1 - i$, so

$$\vec{v} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

So the general solution is:

$$\begin{aligned} \vec{x} &= c_1 e^t \left[\cos 2t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \sin 2t \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] + c_2 e^t \left[\sin 2t \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \cos 2t \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right] \\ &= c_1 e^t \begin{pmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ \sin 2t - \cos 2t \end{pmatrix}. \end{aligned}$$

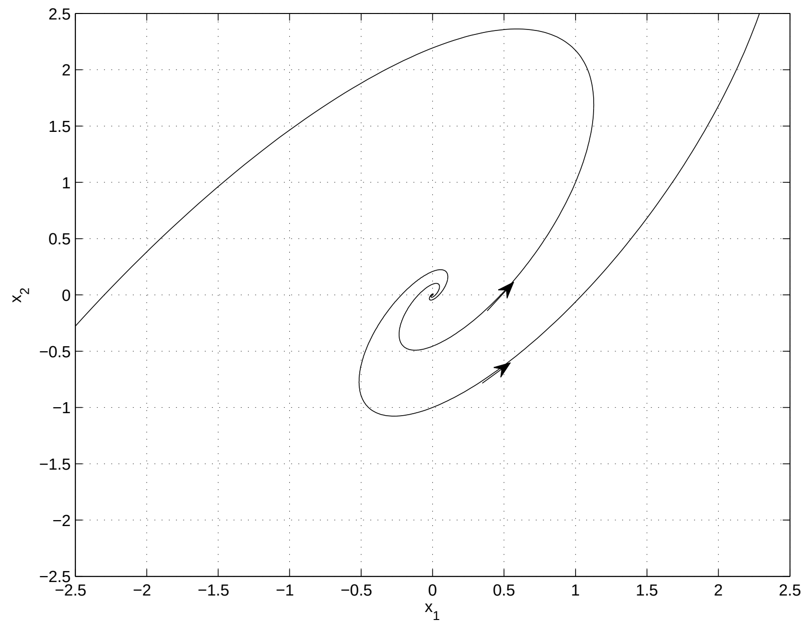
Phase portrait. Solution is growing oscillation due to the e^t. If this term is not present, (i.e., the eigenvalues would be pure imaginary), then the solutions are perfect oscillations, whose trajectory would be closed curves around origin, as the center. But with the e^t term, we will get spiral curves. Since $\alpha = 1 > 0$, all arrows are pointing away from the origin.

To determine the direction of rotation, we need to go back to the original equation and take a look at the directional field.

Consider the point $(x_1 = 1, x_2 = 0)$, then $\vec{x}' = A\vec{x} = (3,4)^T$. The arrow should point up with slope $4/3$.

At the point $\vec{x} = (0,1)^T$, we have $\vec{x}' = (-2,-1)^T$.

Therefore, the spirals are rotating counter clockwise. We don't stress on the exact shape of the spirals. See plot below.



In this case, the origin (the critical point) is called the **spiral point**. The origin in this example is an unstable critical point since $\alpha > 0$.

Remark: If $\alpha < 0$, then all arrows will go towards the origin. The origin will be a stable critical point. An example is provided in the text book. We will go through it here.

Example 3. Consider

$$\vec{x}\,' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \vec{x}.$$

The eigenvalues and eigenvectors are:

$$\lambda_{1,2} = -\frac{1}{2} \pm i, \qquad \vec{v} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since the formula for the general solution is not so "friendly" to memorize, we use a different approach.

We know that one solution is

$$\vec{z} = e^{\lambda_1 t}\vec{v}_1 = e^{-(\frac{1}{2}+i)t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 01 \end{pmatrix}\right].$$

This is a complex values function. We know the real part and the imaginary part are both solutions, so work them out:

$$\vec{z} = e^{-\frac{1}{2}t}\left[\cos t\begin{pmatrix}1\\0\end{pmatrix} - \sin t\begin{pmatrix}0\\1\end{pmatrix} + i\sin t\begin{pmatrix}1\\0\end{pmatrix} + i\cos t\begin{pmatrix}0\\1\end{pmatrix}\right].$$

The general solution is:

$$\vec{x} = c_1 e^{-\frac{1}{2}t}\left[\cos t\begin{pmatrix}1\\0\end{pmatrix} - \sin t\begin{pmatrix}0\\1\end{pmatrix}\right] + c_2 e^{-\frac{1}{2}t}\left[\sin t\begin{pmatrix}1\\0\end{pmatrix} + \cos t\begin{pmatrix}0\\1\end{pmatrix}\right],$$

and we can write out each component

$$\begin{aligned} x_1(t) &= e^{-\frac{1}{2}t}(c_1\cos t + c_2\sin t) \\ x_2(t) &= e^{-\frac{1}{2}t}(-c_1\sin t + c_2\cos t) \end{aligned}$$

Phase portrait: If $c_1 = 0$, we have

$$x_1^2 + x_2^2 = (e^{-\frac{1}{2}t})^2 c_2^2(\sin^2 t + \cos^2 t) = (e^{-\frac{1}{2}t})^2 c_2^2.$$

If $c_2 = 0$, we have

$$x_1^2 + x_2^2 = (e^{-\frac{1}{2}t})^2 c_1^2.$$

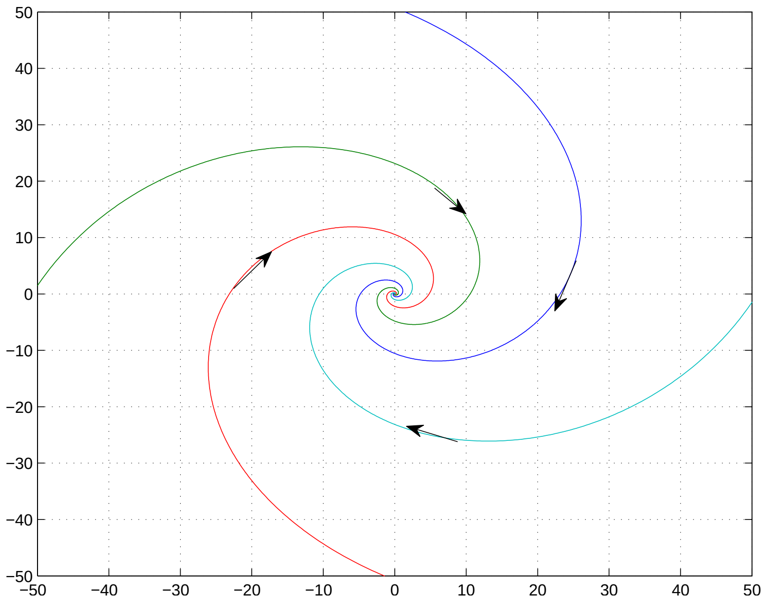
In general, if $c_1 \neq 0$ and $c_2 \neq 0$, we can show:

$$x_1^2 + x_2^2 = (e^{-\frac{1}{2}t})^2(c_1^2 + c_1^2).$$

The trajectories will be spirals, with arrows pointing toward the origin. To determine with direction they rotate, we check a point on the x_1 axis:

$$\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}, \quad \vec{x}' = A\vec{x} = \begin{pmatrix}-\frac{1}{2}\\-1\end{pmatrix}.$$

So the spirals rotate clockwise. And the origin is a stable equilibrium point. See the picture below.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20
10
0
−10
−20
−30
−40
−50
−50
−40
−30
−20
−10
0
10
20
30
40
50

7.8: Repeated eigenvalues

Here we study the case where the two eigenvalues are the same, say $\lambda_1 = \lambda_2 = \lambda$. This can happen, as we will see through our first example.

Example 1. Let
$$A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$
Then
$$\det(A-\lambda I) = \det \begin{pmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda)+1 = \lambda^2-4\lambda+3+1 = (\lambda-2)^2 = 0,$$
so $\lambda_1 = \lambda_2 = 2$. And we can find only one eigenvector $\vec{v} = (a,b)^T$
$$(A-\lambda I)\vec{v} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0, \quad a+b=0.$$
Choosing $a = 1$, then $b = -1$, and we find $\vec{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Then, one solution is:
$$\vec{z}_1 = e^{\lambda t}\vec{v} = e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
We need to find a second solution. Let's try $\vec{z}_2 = te^{\lambda t}\vec{v}$. We have
$$\vec{z}' = e^{\lambda t}\vec{v} + \lambda t e^{\lambda t}\vec{v} = (1+\lambda t)e^{\lambda t}\vec{v}$$
$$A\vec{z}_2 = Ate^{\lambda t}\vec{v} = te^{\lambda t}(A\vec{v}) = te^{\lambda t}\lambda\vec{v} = \lambda te^{\lambda t}\vec{v}$$
If $\vec{z}_2$ is a solution, we must have
$$\vec{z}' = A\vec{z} \quad \to \quad 1+\lambda t = \lambda t$$
which doesn't work.

Try something else: $\vec{z}_2 = te^{\lambda t}\vec{v} + \vec{\eta}e^{\lambda t}$. (here $\vec{\eta}$ is a constant vector to be determined later). Then
$$\vec{z}_2' = (1+\lambda t)e^{\lambda t}\vec{v} + \lambda\vec{\eta}e^{\lambda t} = \lambda te^{\lambda t}\vec{v} + e^{\lambda t}(\vec{v} + \lambda\vec{\eta})$$
$$A\vec{z}_2 = \lambda te^{\lambda t}\vec{v} + A\vec{\eta}e^{\lambda t}.$$

Since $\vec{z}_2$ is a solution, we must have $\vec{z}' = A\vec{z}$. Comparing terms, we see we must have

$$\vec{v} + \lambda\vec{\eta} = A\vec{\eta}, \qquad (A - \lambda I)\vec{\eta} = \vec{v}.$$

This is what one uses to solve for $\vec{\eta}$. Such an $\vec{\eta}$ is called a *generalized eigenvector* corresponding to the eigenvalue λ.

Back to the original problem, to compute this $\vec{\eta}$, we plug in A and λ, and get

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \eta_1 + \eta_2 = -1.$$

We can choose $\eta_1 = 0$, then $\eta_2 = -1$, and so $\vec{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

So the general solution is

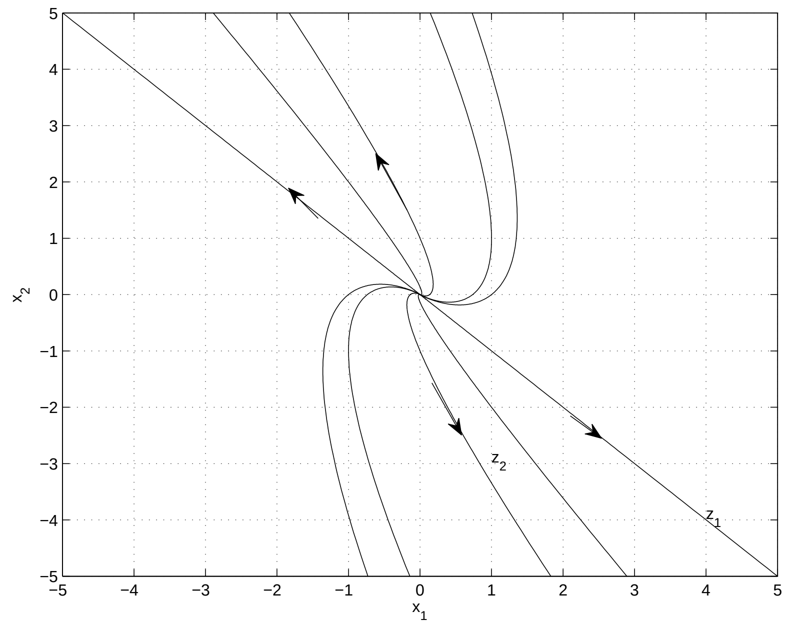
$$\begin{aligned} \vec{x} &= c_1\vec{z}_1 + c_2\vec{z}_2 \;=\; c_1 e^{\lambda t}\vec{v} + c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{\eta}) \\ &= c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left[te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right]. \end{aligned}$$

Phase portrait:

- As $t \to \infty$, we have $|\vec{x}| \to \infty$ unbounded.
- As $t \to -\infty$, we have $\vec{x} \to 0$.
- If $c_2 = 0$, then $\vec{x} = c_1 e^{\lambda t}\vec{v}$, so the line through the origin in the direction of $\vec{v}$ is a trajectory. Since $\lambda > 0$, the arrows point away from the origin.
- If $c_1 = 0$, then $\vec{x} = c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{\eta})$. For this solution, as $t \to \infty$, the dominant term in $\vec{x}$ is $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. On the other hand, as $t \to -\infty$, the dominant term in $\vec{x}$ is still $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. But, due to the change of sign of t, the $\vec{x}$ will change direction and point toward the opposite direction as when $t \to \infty$.

  How does it turn? We need to go back to the system and check the directional field. At $\vec{x} = (1, 0)$, we have $\vec{x}' = (1, 1)^T$, and at $\vec{x} = (0, 1)$, we have $\vec{x}' = (-1, 3)^T$. There it turns kind of counter clockwise. See figure below.

- For the general case, with $c_1 \neq 0$ and $c_2 \neq 0$, a similar thing happens. As $t \to \infty$, the dominant term in $\vec{x}$ is $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. As $t \to -\infty$, the dominant term in $\vec{x}$ is still $te^{\lambda t}\vec{v}$. This means the solution approach the direction of $\vec{v}$. But, due to the change of sign of t, the $\vec{x}$ will change direction and point toward the opposite direction as when $t \to \infty$. See plot below.



Remark: If $\lambda < 0$, the phase portrait looks the same except with reversed arrows.

Definition. If A has repeated eigenvalues, the origin is called a *improper node*. It is stable if $\lambda < 0$, and unstable if $\lambda > 0$.

Example 2. Find the general solution to the system $\vec{x}' = \begin{pmatrix} -2 & 2 \\ -0.5 & -4 \end{pmatrix} \vec{x}$.

We start with finding the eigenvalues:

$$\det(A-\lambda I) = (-2-\lambda)(-4-\lambda)+1 = \lambda^2+6\lambda+8+1 = (\lambda+3)^2 = 0, \qquad \lambda_1 = \lambda_2 = \lambda = -3$$

We see we have double eigenvalue. The corresponding eigenvector $\vec{v} = (a, b)^T$

$$(A - \lambda I)\vec{v} = \begin{pmatrix} -2+3 & 2 \\ -0.5 & -4+3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -0.5 & -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

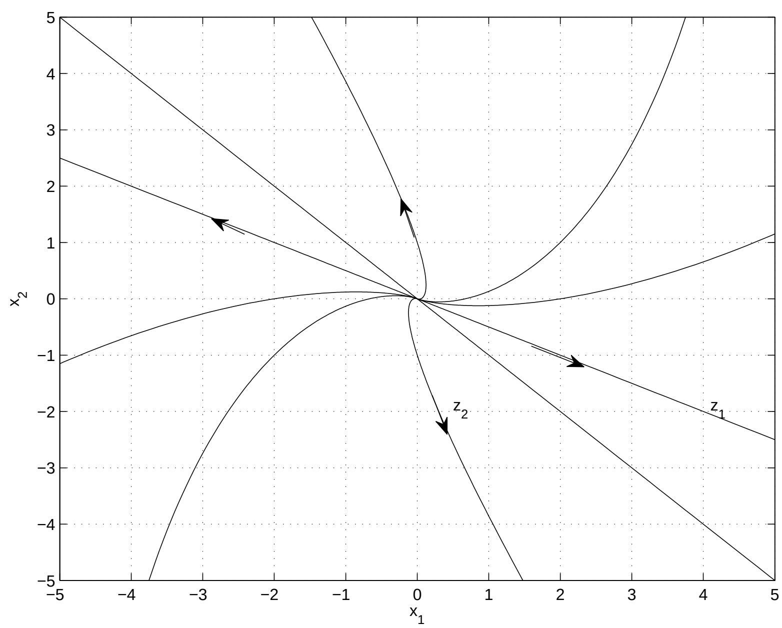
So we must have $a + 2b = 0$. Choose $a = 2$, then $b = -1$, and we get $\vec{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$. To find the generalized eigenvector $\vec{\eta}$, we solve

$$(A - \lambda I)\vec{\eta} = \vec{v}, \qquad \begin{pmatrix} 1 & 2 \\ -0.5 & -1 \end{pmatrix} \cdot \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

This gives us one relation $\eta_1 + 2\eta_2 = 2$. Choose $\eta_1 = 0$, then we have $\eta_2 = 1$, and so $\vec{\eta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The general solution is

$$\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2(te^{\lambda t}\vec{v} + e^{\lambda t}\vec{\eta}) = c_1 e^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2 \left[te^{3t} \begin{pmatrix} 2 \\ -1 \end{pmatrix} + e^{3t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

Just for fun, I include the phase portrait below.



The origin is an improper node which is unstable.

Summary of the chapter:

$\lambda_{1,2}$	eigenvalues	type of origin	stability
real	$\lambda_1 \cdot \lambda_2 < 0$	saddle point	unstable
real	$\lambda_1 > 0, \lambda_2 > 0, \lambda_1 \neq \lambda_2$	node (source)	unstable
real	$\lambda_1 < 0, \lambda_2 < 0, \lambda_1 \neq \lambda_2$	node (sink)	stable
real	$\lambda_1 = \lambda_2 = \lambda$	improper node	stable if $\lambda < 0$, unstable if $\lambda > 0$
complex	$\lambda_{1,2} = i \pm \beta$	center	stable but not asymptotically
complex	$\lambda_{1,2} = \alpha \pm i\beta$	spiral point	stable if $\alpha < 0$, unstable if $\alpha > 0$