Unit 9

9.1 Introduction to Transportation Problem
9.2 Mathematical Formulation
9.3 Tabular Representation
9.4 Some Basic Definitions
9.5 Methods for Initial Basic Feasible Solution
9.6 Examining the Initial Basic Feasible Solution for Non-Degeneracy
9.7 Transportation Algorithm for Minimization Problem

9.1 Introduction to Transportation Problem

The Transportation problem is to transport various amounts of a single homogeneous commodity that are initially stored at various origins, to different destinations in such a way that the total transportation cost is a minimum.

It can also be defined as to ship goods from various origins to various destinations in such a manner that the transportation cost is a minimum.

The availability as well as the requirements is finite. It is assumed that the cost of shipping is linear.

9.2 Mathematical Formulation

Let there be m origins, i^{th} origin possessing a_i units of a certain product

Let there be n destinations, with destination j requiring b_j units of a certain product

Let c_{ij} be the cost of shipping one unit from i^{th} source to j^{th} destination

Let x_{ij} be the amount to be shipped from i^{th} source to j^{th} destination

It is assumed that the total availabilities Σa_i satisfy the total requirements Σb_j i.e.

$\Sigma a_i = \Sigma b_j$ (i = 1, 2, 3 … m and j = 1, 2, 3 …n)

The problem now, is to determine non-negative of x_{ij} satisfying both the availability constraints

$$\sum_{j=1}^{n} x_{ij} = a_i \qquad \text{for } i = 1, 2, .., m$$

as well as requirement constraints

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad \text{for } j = 1, 2, .., n$$

and the minimizing cost of transportation (shipping)

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\, c_{ij} \qquad \text{(objective function)}$$

This special type of LPP is called as **Transportation Problem**.

9.3 Tabular Representation

Let 'm' denote number of factories ($F_1, F_2 \dots F_m$)

Let 'n' denote number of warehouse ($W_1, W_2 \dots W_n$)

W→ F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	c_{11}	c_{12}	…	c_{1n}	a_1
F_2	c_{21}	c_{22}	…	c_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
F_m	c_{m1}	c_{m2}	…	c_{mn}	a_m
Required	b_1	b_2	…	b_n	$\Sigma a_i = \Sigma b_j$
W→ F ↓	W_1	W_2	…	W_n	Capacities (Availability)
F_1	x_{11}	x_{12}	…	x_{1n}	a_1
F_2	x_{21}	x_{22}	…	x_{2n}	a_2
.	.	.	.	.	.
.	.	.	.	.	.
F_m	x_{m1}	x_{m2}	…	x_{mn}	a_m
Required	b_1	b_2	…	b_n	$\Sigma a_i = \Sigma b_j$

In general these two tables are combined by inserting each unit cost c_{ij} with the corresponding amount x_{ij} in the cell (i, j). The product c_{ij} x_{ij} gives the net cost of shipping units from the factory F_i to warehouse W_j.

9.4 Some Basic Definitions

- **Feasible Solution**

  A set of non-negative individual allocations ($x_{ij} \geq 0$) which simultaneously removes deficiencies is called as feasible solution.

- **Basic Feasible Solution**

  A feasible solution to 'm' origin, 'n' destination problem is said to be basic if the number of positive allocations are m+n-1. If the number of allocations is less than m+n-1 then it is called as **Degenerate Basic Feasible Solution**. Otherwise it is called as Non- Degenerate Basic Feasible Solution.

- **Optimum Solution**

A feasible solution is said to be optimal if it minimizes the total transportation cost.

9.5 Methods for Initial Basic Feasible Solution

Some simple methods to obtain the initial basic feasible solution are

1. North-West Corner Rule
2. Row Minima Method
3. Column Minima Method
4. Lowest Cost Entry Method (Matrix Minima Method)
5. Vogel's Approximation Method (Unit Cost Penalty Method)

9.5.1 North-West Corner Rule

Step 1

- The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the table.
- The maximum possible amount is allocated here i.e. $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1,1) of the transportation table.

Step 2

i. If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).

ii. If $b_1 < a_1$, move horizontally right side to the second column and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

iii. If $b_1 = a_1$, there is tie for the second allocation. One can make a second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2)$ in the cell (1, 2) or $x_{21} = \min(a_2, b_1 - b_1)$ in the cell (2, 1)

Step 3

Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until all the requirements are satisfied.

Find the initial basic feasible solution by using North-West Corner Rule

W→ F↓	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Warehouse Requirement	5	8	7	14	34

Solution

	W_1	W_2	W_3	W_5	Availability
F_1	5 (19)	2 (30)			7 2 0
F_2		6 (30)	3 (40)		9 3 0
F_3			4 (70)	14 (20)	18 14 0
Requirement	5 0	8 6 0	7 4 0	14 0	

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = 1015

9.5.2 Row Minima Method

Step 1

- The smallest cost in the first row of the transportation table is determined.
- Allocate as much as possible amount x_{ij} = min (a_1, b_j) in the cell (1, j) so that the capacity of the origin or the destination is satisfied.

Step 2

- If $x_{1j} = a_1$, so that the availability at origin O_1 is completely exhausted, cross out the first row of the table and move to second row.
- If $x_{1j} = b_j$, so that the requirement at destination D_j is satisfied, cross out the j^{th} column and reconsider the first row with the remaining availability of origin O_1.
- If $x_{1j} = a_1 = b_j$, the origin capacity a_1 is completely exhausted as well as the requirement at destination D_j is satisfied. An arbitrary tie-breaking choice is made. Cross out the j^{th} column and make the second allocation $x_{1k} = 0$ in the cell (1, k) with c_{1k} being the new minimum cost in the first row. Cross out the first row and move to second row.

Step 3

Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied

Determine the initial basic feasible solution using Row Minima Method

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	80	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	5	8	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	8 (30)	(40)	(60)	1
F_3	(40)	(80)	(70)	(20)	18
	5	X	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	8 (30)	1 (40)	(60)	X
F_3	(40)	(80)	(70)	(20)	18
	5	X	6	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	8 (30)	1 (40)	(60)	X
F_3	5 (40)	(80)	6 (70)	7 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{14} = 7$, $x_{22} = 8$, $x_{23} = 1$, $x_{31} = 5$, $x_{33} = 6$, $x_{34} = 7$

The transportation cost is 7 (10) + 8 (30) + 1 (40) + 5 (40) + 6 (70) + 7 (20) = 1110

9.5.3 Column Minima Method

Step 1

Determine the smallest cost in the first column of the transportation table. Allocate $x_{i1} = \min(a_i, b_1)$ in the cell (i, 1).

Step 2

- If $x_{i1} = b_1$, cross out the first column of the table and move towards right to the second column
- If $x_{i1} = a_i$, cross out the i^{th} row of the table and reconsider the first column with the remaining demand.
- If $x_{i1} = b_1 = a_i$, cross out the i^{th} row and make the second allocation $x_{k1} = 0$ in the cell (k, 1) with c_{k1} being the new minimum cost in the first column, cross out the column and move towards right to the second column.

Step 3

Repeat steps 1 and 2 for the reduced transportation table until all the requirements are satisfied.

Use Column Minima method to determine an initial basic feasible solution

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	80	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	5 (19)	(30)	(50)	(10)	2
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	X	8	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	(80)	(70)	(20)	18
	X	6	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	(40)	(60)	3
F_3	(40)	(80)	(70)	(20)	18
	X	X	7	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X
F_3	(40)	(80)	(70)	(20)	18
	X	X	4	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X
F_3	(40)	(80)	4 (70)	(20)	14
	X	X	X	14	

	W_1	W_2	W_3	W_4	
F_1	5 (19)	2 (30)	(50)	(10)	X
F_2	(70)	6 (30)	3 (40)	(60)	X
F_3	(40)	(80)	4 (70)	14 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$

The transportation cost is 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = 1015

9.5.4 Lowest Cost Entry Method (Matrix Minima Method)

Step 1

Determine the smallest cost in the cost matrix of the transportation table. Allocate x_{ij} = min (a_i, b_j) in the cell (i, j)

Step 2

- If $x_{ij} = a_i$, cross out the i^{th} row of the table and decrease b_j by a_i. Go to step 3.
- If $x_{ij} = b_j$, cross out the j^{th} column of the table and decrease a_i by b_j. Go to step 3.
- If $x_{ij} = a_i = b_j$, cross out the i^{th} row or j^{th} column but not both.

Step 3

Repeat steps 1 and 2 for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Find the initial basic feasible solution using Matrix Minima method

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	(10)	7
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	14	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	(20)	10
	5	X	7	7	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	(40)	8 (8)	(70)	7 (20)	3
	5	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	(70)	(30)	(40)	(60)	9
F_3	3 (40)	8 (8)	(70)	7 (20)	X
	2	X	7	X	

	W_1	W_2	W_3	W_4	
F_1	(19)	(30)	(50)	7 (10)	X
F_2	2 (70)	(30)	7 (40)	(60)	X
F_3	3 (40)	8 (8)	(70)	7 (20)	X
	X	X	X	X	

Initial Basic Feasible Solution

$x_{14} = 7$, $x_{21} = 2$, $x_{23} = 7$, $x_{31} = 3$, $x_{32} = 8$, $x_{34} = 7$

The transportation cost is 7 (10) + 2 (70) + 7 (40) + 3 (40) + 8 (8) + 7 (20) = 814

9.5.5 Vogel's Approximation Method (Unit Cost Penalty Method)

Step1

For each row of the table, identify the **smallest** and the **next to smallest cost**. Determine the difference between them for each row. These are called **penalties**. Put them aside by enclosing them in the parenthesis against the respective rows. Similarly compute penalties for each column.

Step 2

Identify the row or column with the largest penalty. If a tie occurs then use an arbitrary choice. Let the largest penalty corresponding to the i^{th} row have the cost c_{ij}. Allocate the largest possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross out either i^{th} row or j^{th} column in the usual manner.

Step 3

Again compute the row and column penalties for the reduced table and then go to step 2. Repeat the procedure until all the requirements are satisfied.

Find the initial basic feasible solution using vogel's approximation method

	W_1	W_2	W_3	W_4	Availability
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	

Solution

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	19	30	50	10	7	19-10=9
F_2	70	30	40	60	9	40-30=10
F_3	40	8	70	20	18	20-8=12
Requirement	5	8	7	14		
Penalty	40-19=21	30-8=22	50-40=10	20-10=10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	(19)	(30)	(50)	(10)	7	9
F_2	(70)	(30)	(40)	(60)	9	10
F_3	(40)	**8**(8)	(70)	(20)	18/10	12
Requirement	5	8/0	7	14		
Penalty	21	22	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	9
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	(20)	18/10	20
Requirement	5/0	X	7	14		
Penalty	21	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	(10)	7/2	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	18/10/0	50
Requirement	X	X	7	14/4		
Penalty	X	X	10	10		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	7/2/0	40
F_2	(70)	(30)	(40)	(60)	9	20
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	7	14/4/2		
Penalty	X	X	10	50		

	W_1	W_2	W_3	W_4	Availability	Penalty
F_1	**5**(19)	(30)	(50)	**2**(10)	X	X
F_2	(70)	(30)	**7**(40)	**2**(60)	X	X
F_3	(40)	**8**(8)	(70)	**10**(20)	X	X
Requirement	X	X	X	X		
Penalty	X	X	X	X		

Initial Basic Feasible Solution

$x_{11} = 5, x_{14} = 2, x_{23} = 7, x_{24} = 2, x_{32} = 8, x_{34} = 10$

The transportation cost is 5 (19) + 2 (10) + 7 (40) + 2 (60) + 8 (8) + 10 (20) = 779