Unit 8: Assignment Problem

8.1 Introduction to Assignment Problem

In assignment problems, the objective is to assign a number of jobs to the equal number of persons at a minimum cost of maximum profit.

Suppose there are ‘n’ jobs to be performed and ‘n’ persons are available for doing these jobs. Assume each person can do each job at a time with a varying degree of efficiency. Let c_{ij} be the cost of i^{th} person assigned to j^{th} job. Then the problem is to find an assignment so that the total cost for performing all jobs is minimum. Such problems are known as **assignment problems.**

These problems may consist of assigning men to offices, classes to the rooms or problems to the research team etc.

Mathematical formulation

Cost matrix: $c_{ij}=$

$$\begin{matrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ . \\ . \\ . \\ c_{n1} & c_{n2} & c_{n3} & \dots & c_{nn} \end{matrix}$$

$$\text{Minimize cost : } z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij} \qquad i = 1, 2, .., n \quad j = 1, 2, .., n$$

Subject to restrictions of the form

$$x_{ij} = \begin{cases} 1 \text{ if ith person is assigned jth job} \\ 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \text{(one job is done by the ith person, } i = 1,2,\ldots n)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \text{(only one person should be assigned the jth job, } j = 1,2,\ldots n)$$

Where x_{ij} denotes that j^{th} job is to be assigned to the i^{th} person.

This special structure of assignment problem allows a more convenient method of solution in comparison to simplex method.

8.2 Algorithm for Assignment Problem (Hungarian Method)

Step 1

Subtract the minimum of each row of the effectiveness matrix, from all the elements of the respective rows (Row reduced matrix).

Step 2

Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus first modified matrix is obtained.

Step 3

Draw the minimum number of horizontal and vertical lines to cover all the zeroes in the resulting matrix. Let the minimum number of lines be N. Now there may be two possibilities

- If N = n, the number of rows (columns) of the given matrix then an optimal assignment can be made. So make the zero assignment to get the required solution.
- If N < n then proceed to step 4

Step 4

Determine the smallest element in the matrix, not covered by N lines. Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus the second modified matrix is obtained.

Step 5

Repeat step 3 and step 4 until minimum number of lines become equal to number of rows (columns) of the given matrix i.e. N = n.

Step 6

To make zero assignment - examine the rows successively until a row-wise exactly single zero is found; mark this zero by '□'to make the assignment. Then, mark a 'X' over all zeroes if lying in the column of the marked zero, showing that they cannot be considered for further assignment. Continue in this manner until all the rows have been examined. Repeat the same procedure for the columns also.

Step 7

Repeat the step 6 successively until one of the following situations arise

- If no unmarked zero is left, then process ends
- If there lies more than one of the unmarked zeroes in any column or row, then mark '□'one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its row and column. Repeat the process until no unmarked zero is left in the matrix.

Step 8

Exactly one marked zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked zeroes will give the optimal assignment.

8.3 Worked Examples

Example 1

A department head has four subordinates and four tasks have to be performed. Subordinates differ in efficiency and tasks differ in their intrinsic difficulty. Time each man would take to perform each task is given in the effectiveness matrix. How the tasks should be allocated to each person so as to minimize the total man-hours?

Subordinates

		I	II	III	IV
Tasks	A	8	26	17	11
	B	13	28	4	26
	C	38	19	18	15
	D	19	26	24	10

Solution

Row Reduced Matrix

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

I Modified Matrix

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

N = 4, n = 4

Since N = n, we move on to zero assignment

Zero assignment

[0]	14	9	3
9	20	[0]	22
23	[0]	3	~~0~~
9	12	14	[0]

Optimal assignment	A – I	B – III	C – II	D – IV
Man-hours	8	4	19	10

Total man-hours = 8 + 4 + 19 + 10 = 41 hours

Example 2

Solve the assignment problem whose effectiveness matrix is given in the table

	1	2	3	4
A	49	60	45	61
B	55	63	45	69
C	52	62	49	68
D	55	64	48	66

Solution

Row-Reduced Matrix

4	15	0	16
10	18	0	24
3	13	0	19
7	16	0	18

I Modified Matrix

1	2	0	0
7	5	0	8
0	0	0	3
4	3	0	2

N < n i.e 3 < 4, so II modified matrix

II Modified Matrix

1	2	2	0
5	3	0	6
0	0	2	3
2	1	0	0

N < n i.e 3 < 4

III Modified matrix

0	1	2	0
4	2	0	6
0	0	3	4
1	0	0	0

Since N = n, we move on to zero assignment

Zero assignment

Multiple optimal assignments exists

Solution - I

0	1	2	0
4	2	0	6
0	0	3	4
1	0	0	0

Optimal assignment A – 1 B – 3 C – 2 D - 4
Value 49 45 62 66

Total cost = 49 + 45 + 62 + 66 = 222 units

Solution – II

0	1	2	0
4	2	0	6
0	0	3	4
1	0	0	0

Optimal assignment A – 4 B – 3 C – 1 D – 2
Value 61 45 52 64

Minimum cost = 61 + 45 + 52 + 64 = 222 units

8.4 Unbalanced Assignment Problems

If the number of rows and columns are not equal then such type of problems are called as unbalanced assignment problems.

Example 1

A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table

		Machines			
		W	X	Y	Z
Jobs	A	18	24	28	32
	B	8	13	17	19
	C	10	15	19	22

Solution

18	24	28	32
8	13	17	19
10	15	19	22
0	0	0	0

Row Reduced matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

I Modified Matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

N < n i.e. 2 < 4

II Modified Matrix

0	1	5	9
0	0	4	6
0	0	4	7
5	0	0	0

N < n i.e. 3 < 4

III Modified Matrix

0	1	1	5
0	0	0	2
0	0	0	3
9	4	0	0

N = n

Zero assignment

Multiple assignments exists

Solution -I

[0]	1	1	5
⊠	[0]	⊠	2
⊠	⊠	[0]	3
9	4	⊠	[0]

Optimal assignment W – A X – B Y – C
Cost 18 13 19

Minimum cost = 18 + 13 + 19 = 50 ID

Solution -II

[0]	1	1	5
⊠	⊠	[0]	2
⊠	[0]	⊠	3
9	4	⊠	[0]

Optimal assignment W – A X – C Y – B
Cost 18 17 15

Minimum cost = 18 + 17 + 15 = 50 ID

8.5 Maximal Assignment Problem

Example 1

A company has 5 jobs to be done. The following matrix shows the return in terms of Dinars on assigning i^{th} (i = 1, 2, 3, 4, 5) machine to the j^{th} job (j = A, B, C, D, E). Assign the five jobs to the five machines so as to maximize the total expected profit.

		Jobs				
		A	B	C	D	E
Machines	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution

Subtract all the elements from the highest element

Highest element = 14

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

Row Reduced matrix

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	7

I Modified Matrix

3	1	2	0	7
0	2	0	3	0
7	2	9	0	7
4	0	10	3	6
1	3	4	0	6

N < n i.e. 3 < 5

II Modified Matrix

2	0	1	0	6
0	2	0	4	0
6	1	8	0	6
4	0	10	4	6
0	2	3	0	5

N < n i.e. 4 < 5

III Modified Matrix

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

N = n

Zero assignment

1	0	0	0	5
0	3	0	5	0
5	1	7	0	5
3	0	9	4	5
0	3	3	1	5

Optimal assignment 1 – C 2 – E 3 – D 4 – B 5 – A

Maximum profit = 10 + 5 + 14 + 14 + 7 = 50 ID.

Exercise

1- What is assignment problem? Give any two areas of its applications.

2- Certain equipment needs 5 repair jobs which have to be assigned to 5 machines. The estimated time (in hours) that a mechanic requires to complete the repair job is given in the table. Assuming that each mechanic can be assigned only one job, determine the minimum time assignment.

	J1	J2	J3	J4	J5
M1	7	5	9	8	11
M2	9	12	7	11	10
M3	8	5	4	6	9
M4	7	3	6	9	5
M5	4	6	7	5	11

Ans.

Optimal assignment	M1 – J2	M2 – J3	M3 – J4	M4 – J5	M5 – J1
Hours	5	7	6	5	4

Minimum time = 5 + 7 + 6 + 5 + 4 = 27 hours.

3- Solve the assignment problem whose effectiveness matrix is given in the table

	R1	R2	R3	R4
C1	9	14	19	15
C2	7	17	20	19
C3	9	18	21	18
C4	10	12	18	19
C5	10	15	21	16

Ans.

Optimal assignment	C1 – R3	C2 – R1	C4 – R2	C5 – R4
Units	19	7	12	16

Minimum cost = 19 + 7 + 12 + 16 = 54 units

4- The jobs A, B, C are to be assigned to three machines X, Y, Z. The processing costs (ID.) are as given in the matrix below. Find the allocation which will minimize the overall processing cost.

	X	Y	Z
A	19	28	31
B	11	17	16
C	12	15	13

[Ans. A – X, B - Y, C – Z]