Unit 4

4.1 Introduction to Linear Programming

A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2 + ... + a_nx_n$, where $a_1, a_2, ..., a_n$ are constants and $x_1, x_2 ... x_n$ are variables. The term Programming refers to the process of determining a particular program or plan of action. So Linear Programming (LP) is one of the most important optimization (maximization / minimization) techniques developed in the field of Operations Research (OR).

The methods applied for solving a linear programming problem are basically simple problems; a solution can be obtained by a set of simultaneous equations. However a unique solution for a set of simultaneous equations in n-variables ($x_1, x_2 ... x_n$), at least one of them is non-zero, can be obtained if there are exactly *n* relations. When the number of relations is greater than or less than *n*, a unique solution does not exist but a number of trial solutions can be found.

In various practical situations, the problems are seen in which the number of relations is not equal to the number of the number of variables and many of the relations are in the form of inequalities ($\leq$ or $\geq$) to maximize or minimize a linear function of the variables subject to such conditions. Such problems are known as Linear Programming Problem (LPP).

Definition – The general LPP calls for optimizing (maximizing / minimizing) a linear function of variables called the '**Objective function**' subject to a set of linear equations and / or inequalities called the '**Constraints**' or '**Restrictions'**.

4.2 General form of LPP

We formulate a mathematical model for general problem of allocating resources to activities. In particular, this model is to select the values for $x_1, x_2 \ldots x_n$ so as to maximize or minimize

$Z = c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_nx_n$

subject to restrictions

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n$ ($\leq$ or $\geq$) b_1

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n$ ($\leq$ or $\geq$) b_2

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$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots + a_{mn}x_n$ ($\leq$ or $\geq$) b_m

and

$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$

Where

Z = value of overall measure of performance

x_j = level of activity (for j = 1, 2, ..., n)

c_j = increase in Z that would result from each unit increase in level of activity j

b_i = amount of resource i that is available for allocation to activities (for i = 1,2, …, m)

a_{ij} = amount of resource i consumed by each unit of activity j

Resource	Resource usage per unit of activity	Amount of resource available
	Activity	
	1 2 …………………….. n	
1	a_{11} a_{12} …………………….a_{1n}	b_1
2	a_{21} a_{22} …………………….a_{2n}	b_2
.	.	.
.	.	.
.	.	.
m	a_{m1} a_{m2} …………………….a_{mn}	b_m
Contribution to Z per unit of activity	c_1 c_2 ……………………….c_n	

Data needed for LP model

- The level of activities x_1, x_2……..x_n are called **decision variables**.
- The values of the c_j, b_i, a_{ij} (for i=1, 2 … m and j=1, 2 … n) are the **input constants** for the model. They are called as **parameters** of the model.
- The function being maximized or minimized $Z = c_1x_1 + c_2x_2 +.... +c_nx_n$ is called **objective function**.
- The restrictions are normally called as **constraints**. The constraint $a_{i1}x_1 + a_{i2}x_2$ … $a_{in}x_n$ are sometimes called as **functional constraint** (L.H.S (left hand side) constraint). $x_j \geq 0$ restrictions are called **non-negativity constraint**.

4.3 Assumptions in LPP

1. **Proportionality**

The contribution of each variable in the objective function or its usage of the resources is directly proportional to the value of the variable i.e. if resource availability increases by some percentage, then the output shall also increase by same percentage.

2. **Additivity**

Sum of the resources used by different activities must be equal to the total quantity of resources used by each activity for all resources individually or collectively.

3. **Divisibility**

The variables are not restricted to integer values

4. **Deterministic**

Coefficients in the objective function and constraints are completely known and do not change during the period under study in all the problems considered.

5. **Finiteness**

Variables and constraints are finite in number.

6. **Optimality**

In LPP, we determine the decision variables so as to optimize the objective function of the LPP.

7. The problem involves only one objective, profit maximization or cost minimization.

4.4 Applications of Linear Programming

1. Personnel Assignment Problem
2. Transportation Problem
3. Efficiency on Operation of system of Dams
4. Optimum Estimation of Executive Compensation
5. Agriculture Applications
6. Military Applications
7. Production Management
8. Marketing Management
9. Manpower Management
10. Physical distribution

4.5 Advantages of Linear Programming Techniques

1. It helps us in making the optimum utilization of productive resources.
2. The quality of decisions may also be improved by linear programming techniques.
3. Provides practically solutions.
4. In production processes, high lighting of bottlenecks is the most significant advantage of this technique.

4.6 Limitations of Linear Programming

Some limitations are associated with linear programming techniques

1. In some problems, objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.

2. There is no guarantee of getting integer valued solutions. For example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems.

3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

4. Sometimes large scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

5. Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.

6. Linear programming deals with only single objective, whereas in real life situation problems come across with multi objectives. Goal programming and multi-objective programming deals with such problems.

4.7 Formulation of LP Problems

Example 1

A firm manufactures two types of products A and B and sells them at a profit of ID 2 on type A and ID 3 on type B. Each product is processed on two machines G and H. Type A requires 1 minute of processing time on G and 2 minutes on H; type B requires 1 minute on G and 1 minute on H. The machine G is available for not more than 6 hours 40 minutes while machine H is available for 10 hours during any working day. Formulate the problem as a linear programming problem.

Solution

Let

x_1 be the number of products of type A

x_2 be the number of products of type B

After understanding the problem, the given information can be systematically arranged in the form of the following table.

	Type of products (minutes)		
Machine	Type A (x_1 units)	Type B (x_2 units)	Available time (mins)
G	1	1	400
H	2	1	600
Profit per unit	ID 2	ID 3	

Since the profit on type A is ID 2 per product, $2x_1$ will be the profit on selling x_1 units of type A. similarly, $3x_2$ will be the profit on selling x_2 units of type B. Therefore, total profit on selling x_1 units of A and x_2 units of type B is given by

Maximize $Z = 2x_1 + 3x_2$ (**objective function**)

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total number of minutes required on machine G is given by $x_1 + x_2$.

Similarly, the total number of minutes required on machine H is given by $2x_1 + x_2$.

But, machine G is not available for more than 6 hours 40 minutes (400 minutes). Therefore,

$x_1 + x_2 \leq 400$ (**first constraint**)

Also, the machine H is available for 10 hours (600 minutes) only, therefore,

$2x_1 + x_2 \leq 600$ (**second constraint**)

Since it is not possible to produce negative quantities

$x_1 \geq 0$ and $x_2 \geq 0$ (**non-negative restrictions**)

Hence

Maximize $Z = 2\ x_1 + 3\ x_2$

Subject to restrictions

$$x_1 + x_2 \leq 400$$

$$2x_1 + x_2 \leq 600$$

and non-negativity constraints

$x_1 \geq 0$, $x_2 \geq 0$

Example 2

A firm manufactures 3 products A, B and C. The profits are ID 3, ID 2 and ID 4 respectively. The firm has 2 machines and below is given the required processing time in minutes for each machine on each product.

	Products		
Machine	A	B	C
X	4	3	5
Y	2	2	4

Machine X and Y have 2000 and 2500 machine minutes. The firm must manufacture 100 A's, 200 B's and 50 C's type, but not more than 150 A's.

Solution

Let

x_1 be the number of units of product A

x_2 be the number of units of product B

x_3 be the number of units of product C

	Products			
Machine	A	B	C	Availability
X	4	3	5	2000
Y	2	2	4	2500
Profit	3	2	4	

Max $Z = 3x_1 + 2x_2 + 4x_3$

Subject to

$4x_1 + 3x_2 + 5x_3 \leq 2000$

$2x_1 + 2x_2 + 4x_3 \leq 2500$

$100 \leq x_1 \leq 150$

$x_2 \geq 200$

$x_3 \geq 50$

Example 3

ABC Company produces both interior and exterior paints from 2 raw materials M1 and M2. The following table produces basic data of problem.

	Exterior paint	Interior paint	Availability
M1	6	4	24
M2	1	2	6
Profit per ton	5	4	

A market survey indicates that daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also maximum daily demand for interior paint is 2 tons. Formulate LPP to determine the best product mix of interior and exterior paints that maximizes the daily total profit.

Solution

Let x_1 be the number of units of exterior paint

x_2 be the number of units of interior paint

Maximize $Z = 5x_1 + 4x_2$

Subject to

$6x_1 + 4x_2 \leq 24$

$x_1 + 2x_2 \leq 6$

$x_2 - x_1 \leq 1$

$x_2 \leq 2$

$x_1 \geq 0,\ x_2 \geq 0$

b) The maximum daily demand for exterior paint is at most 2.5 tons

$x_1 \leq 2.5$

c) Daily demand for interior paint is at least 2 tons

$x_2 \geq 2$

d) Daily demand for interior paint is exactly 1 ton higher than that for exterior paint.

$x_2 > x_1 + 1$

Example 4

A company produces 2 types of hats. Each hat of the I type requires twice as much as labour time as the II type. The company can produce a total of 500 hats a day. The market limits daily sales of I and II types to 150 and 250 hats. Assuming that the profit per hat are ID 8 for type A and ID 5 for type B. Formulate a LPP models in order to determine the number of hats to be produced of each type so as to maximize the profit.

Solution

Let x_1 be the number of hats produced by type A

Let x_2 be the number of hats produced by type B

Maximize $Z = 8x_1 + 5x_2$

Subject to

$2x_1 + x_2 \leq 500$ (labour time)

$x_1 \leq 150$

$x_2 \leq 250$

$x_1 \geq 0,\ x_2 \geq 0$

Exercise

1. Define the terms used in LPP.
2. Mention the advantages of LPP.
3. What are the assumptions and limitations of LPP?
4. A company produces two products A and B which possess raw materials 400 kilograms and 450 labour hours. It is known that 1 unit of product A requires 5 kilograms of raw materials and 10 man hours and yields a profit of ID 45. Product B requires 20 kilograms of raw materials, 15 man hours and yields a profit of ID 80. Formulate the LPP.
5. ABC Company produces both interior and exterior paints from 2 raw materials M1 and M2. The following table produces basic data of problem.

	Exterior paint	Interior paint	Availability
M1	6	4	24
M2	1	2	6
Profit per ton	5	4	

A market survey indicates that daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also maximum daily demand for interior paint is 2 tons. Formulate LPP to determine the best product mix of interior and exterior paints that maximizes the daily total profit.

6. A company has 3 operational departments weaving, processing and packing with the capacity to produce 3 different types of clothes that are suiting, shirting and woolen yielding with the profit of ID 2, ID 4 and ID 3 per meters respectively. 1m suiting requires 3mins in weaving 2 mins in processing and 1 min in packing. Similarly 1m of shirting requires 4 mins in weaving 1 min in processing and 3 mins in packing while 1m of woolen requires 3 mins in each department. In a week total run time of each department is 60, 40 and 80 hours for weaving, processing and packing department respectively. Formulate a LPP to find the product to maximize the profit.