$$\frac{dt}{dx} = \frac{n_1}{c}\frac{d}{dx}\sqrt{a^2 + x^2} + \frac{n_2}{c}\frac{d}{dx}\sqrt{b^2 + (d-x)^2}$$

$$= \frac{n_1}{c}\left(\tfrac{1}{2}\right)\frac{2x}{(a^2 + x^2)^{1/2}} + \frac{n_2}{c}\left(\tfrac{1}{2}\right)\frac{2(d-x)(-1)}{[b^2 + (d-x)^2]^{1/2}}$$

$$= \frac{n_1 x}{c(a^2 + x^2)^{1/2}} - \frac{n_2(d-x)}{c[b^2 + (d-x)^2]^{1/2}} = 0$$

Or

$$\frac{n_1 x}{(a^2 + x^2)^{1/2}} = \frac{n_2(d-x)}{[b^2 + (d-x)^2]^{1/2}} \quad \text{.......... (1)}$$

From Figure 3,

$$\sin\theta_1 = \frac{x}{(a^2 + x^2)^{1/2}} \qquad \sin\theta_2 = \frac{d-x}{[b^2 + (d-x)^2]^{1/2}}$$

Substituting these expressions into Equation 1, we find that

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

which is Snell's law of refraction

Optical Path

To derive one of the most fundamental principles in geometric optics, it is appropriate to define a quantity called the *optical path*. The path d of a ray of light in any medium is given by the product *velocity* times *time*:

$$d = vt$$

Since by definition $n = c/v$, which gives $v = c/n$, we can write

$$d = (c/n)\, t \quad \text{or} \quad nd = ct$$

The product nd is called the *optical path* Δ:

$$\Delta = nd$$

The optical path represents the distance light travels in a vacuum in the same time it travels a distance d in the medium. If a light ray travels through a series of optical media of thickness $d, d', d'', \ldots$ and refractive indices $n, n', n'', \ldots$, the total optical path is just the sum of the separate values:

$$A = nd + n'd' + n''d'' + \ldots..$$

A diagram illustrating the meaning of optical path is shown in Fig. E

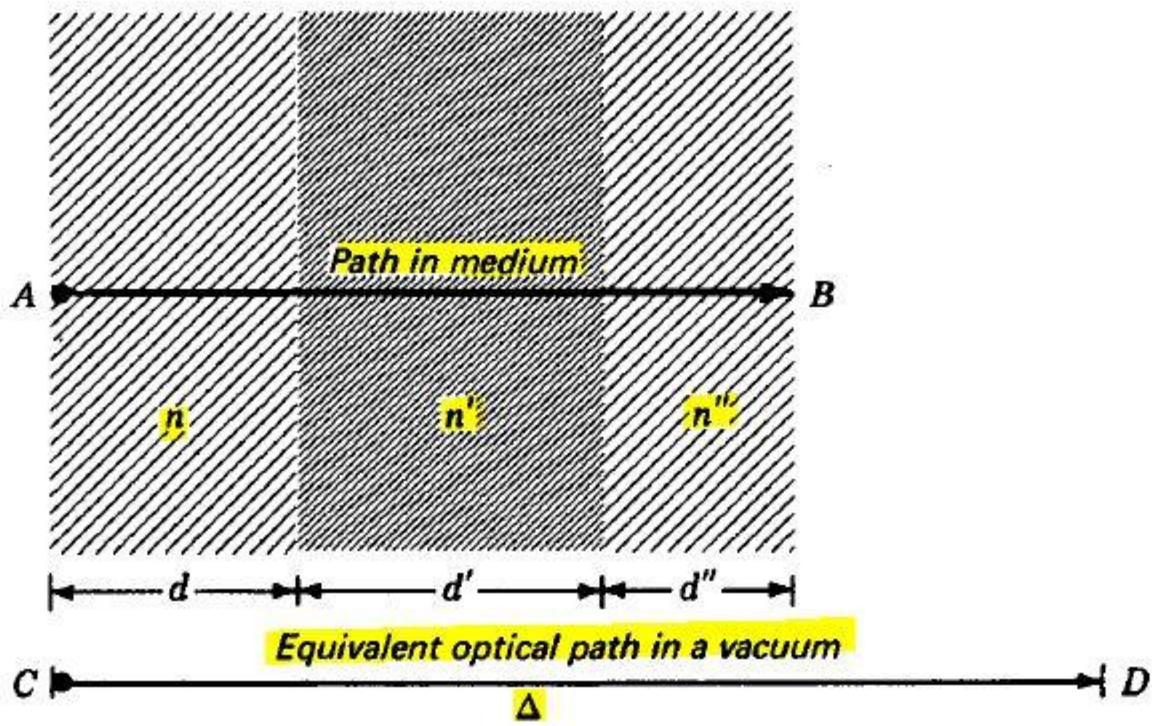


Fig .E. The optical path through a series of optical media.

The Principle Of Reversibility

We know that

$$\textit{angle of incidence} = \textit{angle of reflection}$$

$$\frac{\sin\phi}{\sin\phi'} = \text{const}$$

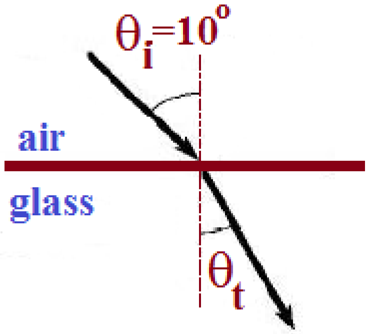
$$\frac{\sin\phi}{\sin\phi'} = \frac{n'}{n}$$

$$n \ \sin\phi = n' \ \sin\phi'$$

The symmetry of these equations with respect to the symbols used shows at once that if a reflected or refracted ray is reversed in direction, it will retrace its original path. For any given pair of media with indices n and n' any one value of ϕ is correlated with a corresponding value of n'. This will be equally true when the ray is reversed and ϕ' becomes the angle of incidence in the medium of n'; the angle of refraction will then be ϕ. Since reversibility holds at each reflecting and refracting surface, it holds also for even the most complicated light paths.

Example: A ray of light in air is incident on the polished surface of a block of glass at an angle of 10°. *(a)* If the refractive index of the glass is 1.525, find the angle of refraction to four significant figures. *(b)* Assuming the sines of the angles in Snell's law can be replaced by the angles themselves, what would be the angle of refraction? *(c)* Find the percentage error.

Solution (a):



$$n_1 \sin\theta_i = n_2 \sin\theta_t \qquad 1\sin 10^\circ = 1.525\sin\theta_t$$

$$\theta_t = 6.534^\circ$$

Solution (b) :

$$n_1 \sin\theta_i = n_2 \sin\theta_t'$$

$$\theta_t' = \frac{1\times 10^\circ}{1.525} = 6.554$$

$$\Delta = \frac{\theta_t' - \theta_t}{\theta_t} = 0.003061\%$$

Example: A ray of light in air is incident at an angle of 54° on the smooth surface of a piece of glass. If the refractive index is 1.5152, find the angle of refraction to four significant figures.

Solution:

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$1\times\sin 54 = 1.5152\sin\theta_t$$

$$\theta_t = 32.27^\circ$$

Example: In studying the refraction of light Kepler arrived at a refraction formula

$$\phi = \frac{\phi'}{1 - k\sec\phi'} \qquad \text{where} \qquad k = \frac{n'-1}{n'}$$

n' being the relative index of refraction. Calculate the angle of incidence ,ϕ for a piece of glass for which $n' = 1.732$ and the angle of refraction , $\phi' = 32°$ according to (a) Kepler's formula and (b) Snell's law. Note that $\sec\phi' = 1/(\cos\phi')$.

$$\phi' = 32^\circ, n' = 1.732$$

Solution(a) : $k = \frac{n'-1}{n'} = k = \frac{1.732-1}{1.732} = 0.42263$

$$\phi = \frac{32}{1-0.42263\times(1/\cos 32)} = 63.4°$$

Solution(b) : $n_1 \sin\phi = n_2 \sin\phi'$

$$\sin\phi = \frac{n_2}{n_1}\sin\phi' = n'\sin\phi'$$

$= 1.732 \sin 32°$

$\phi = 66.49$

Example: (a) When the light illustrated in Figure 4 passes through the glass block, it is shifted laterally by the distance d. Taking $n = 1.5$, find the value of d

(b) Find the time interval required for the light to pass through the glass block described in the previous problem.

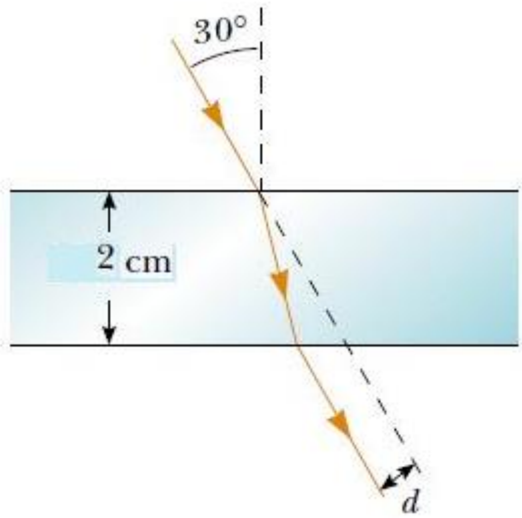
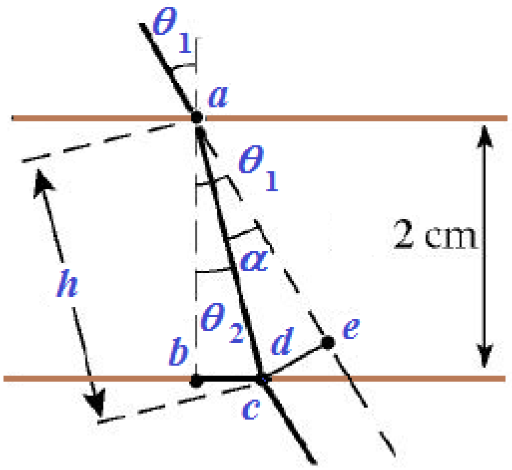


Fig.4

Solution (a):



At entry, $n_1 \sin\theta_1 = n_2 \sin\theta_2$

or $1 \quad \sin 30° = 1.5 \quad \sin\theta_2$

$\theta_2 = 19.5°$.

The distance h the light travels in the medium is given by

$$\cos\theta_2 = \frac{2 \text{ cm}}{h}$$

or $$h = \frac{2 \text{ cm}}{\cos 19.5°} = 2.12 \text{ cm}.$$

The angle of deviation upon entry is $\alpha = \theta_1 - \theta_2 = 30° - 19.5° = 10.5°$.

The offset distance comes from $\sin\alpha = \frac{d}{h}$: $d = (2.21 \text{ cm}) \sin 10.5° = \boxed{0.388 \text{ cm}}$.

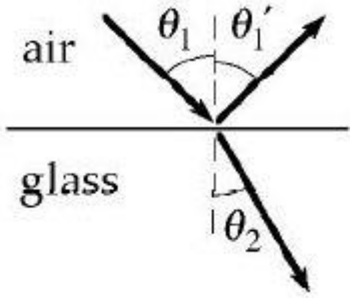
The distance, h, traveled by the light is $h = \frac{2 \text{ cm}}{\cos 19.5°} = 2.12 \text{ cm}$

The speed of light in the material is $v = \frac{c}{h} = \frac{3 \times 10^8 \text{ m/s}}{1.50} = 2 \times 10^8 \text{ m/s}$

Therefore, $t = \frac{h}{v} = \frac{2.12 \times 10^{-2} \text{ m}}{2 \times 10^8 \text{ m/s}} = 1.06 \times 10^{-10} \text{ s} = 10^6 \text{ ps}$

Example: (a) Consider a horizontal interface between air above and glass of index 1.55 below. Draw a light ray incident from the air at angle of incidence 30°. Determine the angles of the reflected and refracted rays and show them on the diagram. (b) **What If?** Suppose instead that the light ray is incident from the glass at angle of incidence 30°. Determine the angles of the reflected and refracted rays and show all three rays on a new diagram.

Solution (a):

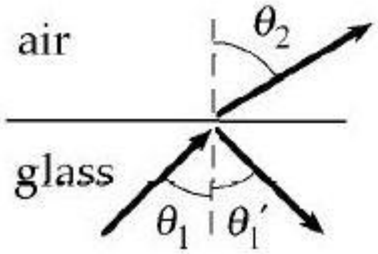


$$\theta_1' = \theta_1 = \boxed{30^\circ} \qquad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$$1 \ \sin 30^\circ = 1.55 \sin\theta_2$$

$$\theta_2 = \boxed{18.8^\circ}$$

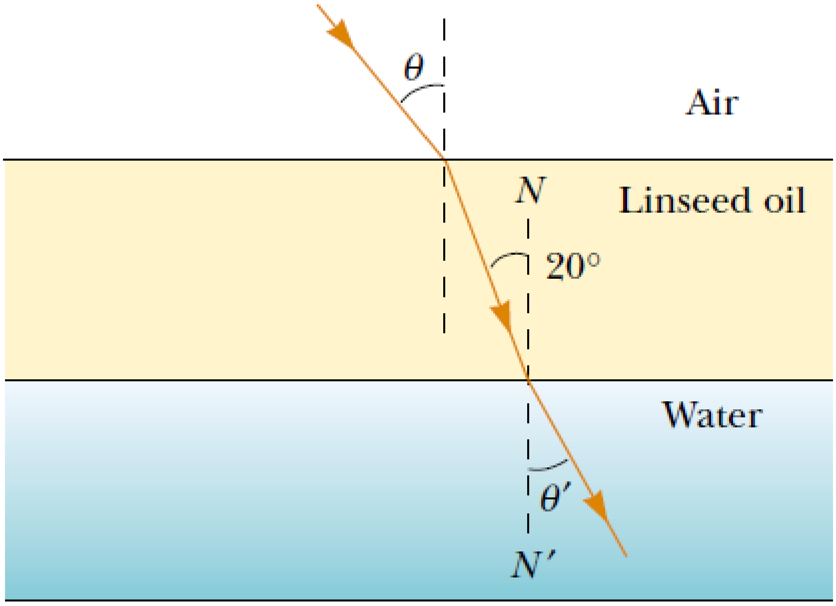
Solution (b):



$$\theta_1' = \theta_1 = \boxed{30^\circ} \qquad \theta_2 = \sin^{-1}\left(\frac{n_1 \sin\theta_1}{n_2}\right)$$

$$= \sin^{-1}\left(\frac{1.55 \sin 30^\circ}{1}\right) = \boxed{50.8^\circ}$$

Example: The light beam shown in Figure below makes an angle of 20° with the normal line NN' in the linseed oil. Determine the angles θ and θ'. (The index of refraction of linseed oil is 1.48).



Solution:

Applying Snell's law at the air-oil interface,

$$n_{air} \sin\theta = n_{oil} \sin 20^\circ$$

yields $\boxed{\theta = 30.4^\circ}$.

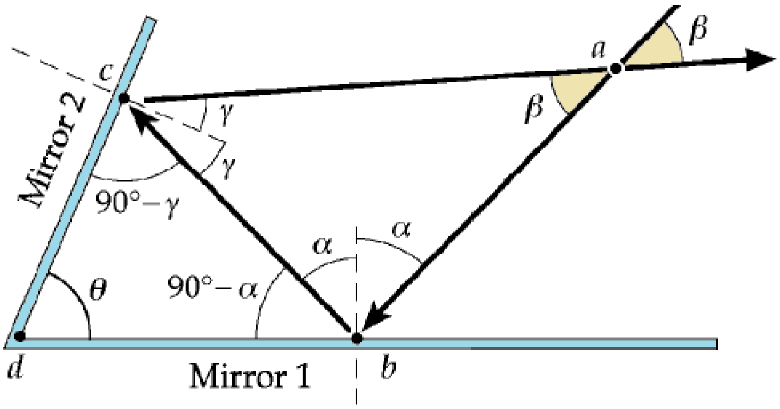
Applying Snell's law at the oil-water interface

$$n_w \sin\theta' = n_{oil} \sin 20^\circ$$

yields $\boxed{\theta' = 22.3^\circ}$.

Example: The reflecting surfaces of two intersecting flat mirrors are at an angle θ (0° < θ < 90°) , as shown in Figure below . For a light ray that strikes the horizontal mirror, show that the emerging ray will intersect the incident ray at an angle β=180° - 2θ.

Solution :



light ray. α and γ are angles of incidence at mirrors 1 and 2.

For triangle *abca*,

$$2\alpha + 2\gamma + \beta = 180°$$

or $\quad \beta = 180° - 2(\alpha + \gamma)$. (1)

Now for triangle *bcdb*,

$$(90° - \alpha) + (90° - \gamma) + \theta = 180°$$

or $\quad \theta = \alpha + \gamma$. (2)

Substituting Equation (2) into Equation (1) gives $\boxed{\beta = 180° - 2\theta}$