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Finite Element Method (FEM)

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

﴿ وَعَلَّمَكَ مَا لَمْ تَكُنْ تَعْلَمُ وَكَانَ فَضْلُ اللهِ عَلَيْكَ عَظِيماً ﴾

صَدَقَ اللهُ العَلِيُّ العَظِيم

(النساء ١١٣)

الإهداء

بدانا بأكثر من يد وقاسينا أكثر من هم وعانينا الكثير من الصعوبات وها نحن اليوم والحمد لله نطوي سهر الليالي وتعب الإيام وخلاصة مشوارنا بين دفتي هذا العمل المتواضع .

إلى منارة العلم والامام المصطفي إلى الأمي الذي علم المتعلمين إلى سيد الخلق إلى رسولنا الكريم سيدنا محمد(ص)

إلى من أرضعتني الحب والحنان

إلى رمز الحب وبلسم الشفاء

إلى القلب الناصع بالبياض (والدتي الحبيبة)

إلى من جرع الكأس فارغاً ليسقيني قطرة حب

إلى من كلّت أنامله ليقدم لنا لحظة سعادة

إلى من حصد الأشواك عن دربي ليمهد لي طريق العلم

إلى القلب الكبير (والدي العزيز)

إلى القلوب الطاهرة الرقيقة والنفوس البريئة إلى رياحين حياتي (إخواني واخواتي)

الى من علمونا حروفا من ذهب وكلمات من درر وعبارات من أسمى وأجلى عبارات في العلم إلى من صاغوا لنا علمهم حروفا ومن فكرهم منارة تنير لنا سيرة العلم والنجاح إلى (أساتذتنا الكرام).

إلى الروح التي سكنت روحي

فالآن تفتح الأشرعة وترفع المرساة لتنطلق السفينة في عرض بحر واسع مظلم هو بحر الحياة وفي هذه الظلمة لا يضيء إلا قنديل الذكريات ذكريات الأخوة البعيدة إلى الذين أحببتهم وأحبوني (أصدقائي)

واخص الاخ سجاد محمد شكر

و الى الذين كانوا عونا لنا في بحثنا هذا ونورا يضيء الظلمة التي كانت تقف أحيانا في طريقنا .

مشرفتي الفاضلة م.م بشائر كاظم

Abstract

In the present study, we discussed the finite elements method, it is numerical techniques used to solve differential equations. The first chapter explains the mechanism of using the finite elements method for one and two-dimensional differential equations by means of two examples and then applying them in detail for both dimensions.

In the second chapter, we discussed the problem of differential equation (1-D) with boundary conditions. The results were compared between the analytic solution and numerical results and the results for the two methods were almost identical.

Contents

Introduction

Differential equation is a mathematical equation that relates some function with it is derivatives. In applications the function usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, differential equations play a prominent role in many disciplines including engineering, physics, economics and biology.

Generally, these equations can be divided into several types, Ordinary / Partial, Linear / Non-Linear, and Homogeneous / In homogeneous (For more details [2]).

The mathematical model is complete and for practical applications, the solution of these differential equations is very important. Therefore, many researchers concentrated on the solution of the differential equations.

The solutions are divided into two types: the first is analytical, which is used to find the exact solution for a specific problem and this includes separation of variables and the transformations such as a Laplace transform, Fourier transform … etc [2]. The other is numerically.

There will be times when solving the exact solution for the equation way be available or other times the solution it will be unavailable, since the complex nature of problem.

At the present time, there are many approaches that can be employed to solve the differential equations and found the approximation solution of these equation.

Basically, the main methods are like finite difference method (FDM), finite volume method (FVM) and finite element method (FEM). However, as presented in numerous paper of numerical method, the finite element method has emerged as available tool for the solution of differential equation.

The finite element method is numerical techniques to find the approximate solution of differential equations. Here, the principle idea of the finite element method is to divided the domain into a number of sub-domain so that the unknown function can be defined by shape (interpolation) functions with anonymous coefficients (see [3], [4], [5]).

Historically, this approach is suggested in an important work by Richard curant in (1943), but at that time unfortunately the important of this article is not recognized and the idea was forgotten. This method is discovered by engineers in the early (1950), but the mathematical analysis of finite element appeared much later, in (1960) and in (1968) the first significant result are due to Milo's Lama. From that time, the finite element approach has become most powerful tool to tread the differential equations numerically

Chapter 1

Finite Element Method

1.1 Introduction

The finite element method (FEM) is a numerical techniques for finding approximate solutions for differential equations. The basic idea in the finite element method is to find the solution of a complicated problem by replacing it by a simple one. Since the actual problem is replaced by a simpler one in finding the solution, we will be able to find only an approximate solution rather than the exact solution(see [1]). For some problem, the existing mathematical tools will not be sufficient to find the exact solution (and sometimes, even an approximate solution) of most of the practical problems. Thus in the absence of any other convenient method to find even the approximate solution of a problem, we prefer the finite element method.

1.2 Description of the finite element method

In this section, we introduce the finite element method, which include the following steps:

1. Discretization of the domain or solution region into sub-domains.
2. Selection of an interpolation model to represent the variation of the field variables.
3. Derivation of element characteristic matrices and vectors.
4. Assemblage of element characteristic matrices and vectors to obtain the system equations.
5. Solving the system equation to find the modal value of field variable.
6. Post processing of the results.

1.3 Basic Element Shapes

The discretization of the domain or solution region in to sub region is the first step in the finite element method. Shapes, sizes, numbers and configurations of the element have to be chosen carefully such that the original body or domain is simulated as closely as possible without increasing the computation effort needed for the solution. For any given physical body we have to use engineering in selecting appropriate elements for discretization. Mostly the choice of the type of element is dictated by the geometry of the body and the number of independent spatial coordinates necessary to describe the system. Some of the popularly used one-, two-, three-dimensional are shown in figures (2.1) , (2.2) and (2.3) respectively.

Fig 2.1 One – dimensional finite elements

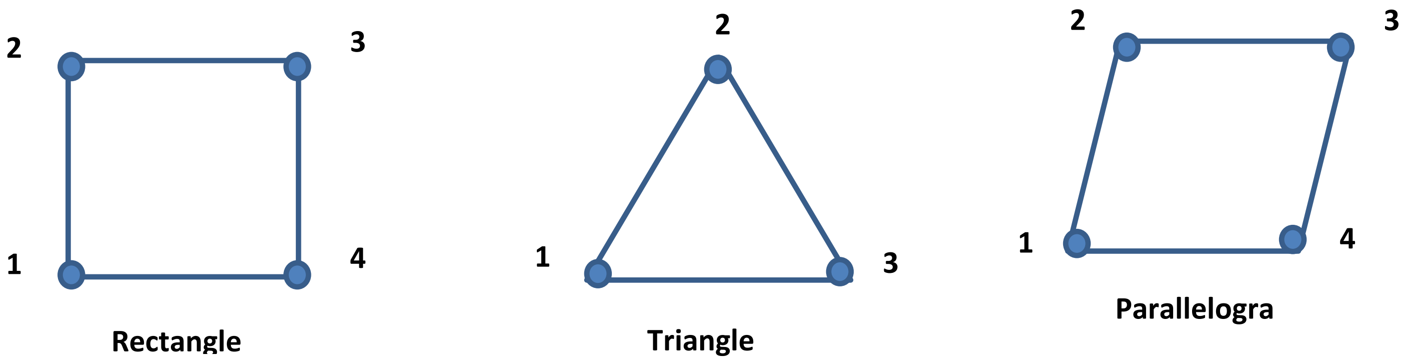


Fig 2.2 three – dimensional finite elements

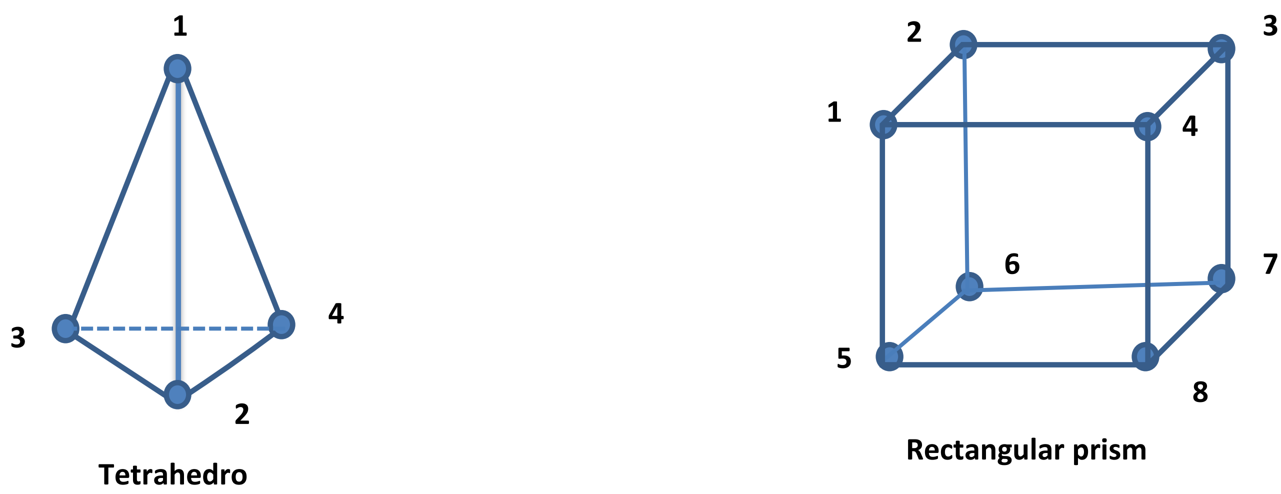


Fig 2.3 three – dimensional finite elements

1.4 Finite Element Method (FEM) One-Dimensional

The basic idea of FEM is to subdivided the region Ω in to sub regions, called finite element (atypical element is denoted Ω_e). In 1D case, these sub regions are intervals, the collection of finite elements in domain is called a finite element mesh of the domain.

Next, we choose in each sub regions a finite number of points, preferably on boundary of the sub regions. The points are called nodal points. Nodal points will be denoted by x_j, j=1,..,n.

Finally, we defined a basic function (shape function),

$\varphi : \Omega \to \mathcal{R}$ for each nodal points x_j in Ω. These function must satisfy the following properties:

1.

$$\varphi_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

2. φ_i has prescribed behavior on each sub region. For example, linear or quadratic depending on the number of nodal points on each sub region.

3. φ_i is continuous on Ω.

Using these shape functions φ_i an approximation function can be constructed as following:

$$\bar{u}(x)=\sum_{j=1}^{n+1} u_j \varphi_j(x)$$

where u_j are the values of the solution u(x) at the nodal x_j.

Example:-

$$-\frac{d}{dx}\left(\lambda\left(\frac{d}{dx}u\right)\right) = f(x) \qquad for\ \ 0 < x < 1 \qquad (1.1)$$

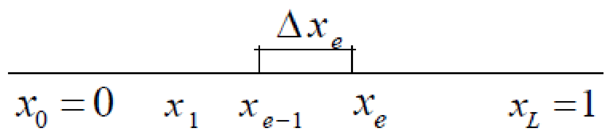
Strong form

With Boundary Condition

$u(0) = u_0$, $\lambda\left(\frac{d}{dx}u\right)|x = 1 = q$

<u>Step 1</u>

Discretization of the domain into a set of finite elements (mesh generation)



$$-\frac{d}{dx}\left(\lambda\left(\frac{d}{dx}u\right)\right)-f(x)=0 \qquad (1.2)$$

Step 2

To set up a weak formulation of the differential equation

i- Multiply the equation by a weight function $W(x)$ and integral the equation over the domain Ω^e

$$\int_{\Omega^e} w\left(-\frac{d}{dx}\left(\lambda\left(\frac{d}{dx}u\right)\right)-f\right)dx=0 \qquad (1.3)$$

ii- Move the differentiation to the weight function by doing integration by parts

$$\int_{\Omega^e} -\lambda\frac{dw}{dx}\left(\frac{du^e}{dx}\right)dx-\int_{\Omega^e} wf\,dx+\left[w\lambda\frac{du^e}{dx}\right]_{x_{e-1}}^{x_e}=0 \qquad (1.4)$$

Weak form

suppose $q_1=\lambda\frac{d}{dx}u^e|x_{e-1}$, $q_2=\lambda\frac{d}{dx}u^e|x_e$

$wq_2 = w(x_e)q_2$, $wq_1 = w(x_{e-1})q_1$.

We will get that

$$w\lambda\frac{du^e}{dx}\Big]_{x_{e-1}}^{x_e}=w(x_e)q_2-w(x_{e-1})q_1.$$

Step 3

The polynomial approximation of the solution within a typical finite element Ω^e is assumed to be of the form

$$u^e=\sum_{j=1}^{2}\varphi_j^e\,u_j^e$$

Where $\Delta x_e=x_e-x_{e-1}$

$$\varphi_1^e=\frac{x_e-x}{\Delta x_e} \quad , \quad \varphi_2^e=\frac{x-x_{e-1}}{\Delta x_e}.$$

$$\frac{d}{dx}\varphi_1^e=\frac{-1}{\Delta x_e} \quad , \quad \frac{d}{dx}\varphi_2^e=\frac{1}{\Delta x_e}.$$

From the weak form

$$-\int_{x_{e-1}}^{x_e}\lambda\frac{dw}{dx}\left(\sum_{j=1}^{2}\frac{d}{dx}\varphi_j^e u_j^e\right)dx-\int_{x_{e-1}}^{x_e} wf\,dx+\left(w(x_e)\,q_e-w(x_{e-1})q_{e-1}\right)=0 \qquad (1.5)$$

The choice $w = \varphi_i$

$$-\int_{x_{e-1}}^{x_e} \lambda \frac{d\varphi_i^e}{dx}\left(\sum_{j=1}^{2} \frac{d}{dx}\varphi_j^e u_j^e\right) dx - \int_{x_{e-1}}^{x_e} \varphi_i^e f dx + (\varphi_i^e(x_e)\, q_2 - \varphi_i^e(x_{e-1}) q_1 = 0. \quad (1.6)$$

Such that $M = \varphi_i^e(x_e) q_2 - \varphi_i^e(x_{e-1}) q_1 \quad , i = 1,2,\dots,n$

$$\Rightarrow\!\!> \; -\int_{x_{e-1}}^{x_e} \lambda \frac{d\varphi_i^e}{dx}\left(\sum_{j=1}^{2} \frac{d}{dx}\varphi_j^e u_j^e\right) dx - \int_{x_{e-1}}^{x_e} \varphi_i^e f dx + M = 0, \quad (1.7)$$

which takes the form $k_{ij}^e u_j^e = f_i^e + q_i^e$

$$k_{ij}^e = -\int_{x_{e-1}}^{x_e} \lambda (\frac{d}{dx}\varphi_i^e)(\frac{d}{dx}\varphi_j^e) dx \quad , f_i^e = -\int_{x_{e-1}}^{x_e} \varphi_i^e f dx$$

Therefore (i=1, j=1)

$$\Rightarrow\!\!> -\int_{x_{e-1}}^{x_e} \lambda \frac{d}{dx}\varphi_1 \left(\frac{d}{dx}(\varphi_1)\right) dx\, u_1 - \int_{x_{e-1}}^{x_e} \varphi_1 f\, dx + M = 0$$

$$\Rightarrow\!\!> -\int_{x_{e-1}}^{x_e} \frac{\lambda}{(\Delta x_e)^2} dx\, u_1 - \int_{x_{e-1}}^{x_e} f (\frac{x_e - x}{\Delta x_e}) dx + M = 0$$

$$\Rightarrow\!\!> -\frac{\lambda}{(\Delta x_e)^2}[x]_{x_{e-1}}^{x_e} u_1 + \frac{1}{2\Delta x_e} f [(x_e - x)^2]_{x_{e-1}}^{x_e} + M = 0$$

$$\Rightarrow\!\!> -\frac{\lambda}{(\Delta x_e)} u_1 - \left(\frac{\Delta x_e}{2}\right) f + M = 0$$

$$k_{11} = -\frac{\lambda}{(\Delta x_e)} \quad , f_1^1 = -\left(\frac{\Delta x_e}{2}\right) f + M$$

Therefore (i=1 , j=2)

$$\Rightarrow\!\!> -\int_{x_{e-1}}^{x_e} \lambda \frac{d}{dx}\varphi_1 \left(\frac{d}{dx}(\varphi_2)\right) dx\, u_2 - \int_{x_{e-1}}^{x_e} f\varphi_1\, dx + M = 0$$

$$\Rightarrow\!\!> \int_{x_{e-1}}^{x_e} \frac{\lambda}{(\Delta x_e)^2} dx\, u_2 - f \int_{x_{e-1}}^{x_e} (\frac{x_e - x}{\Delta x_e}) dx + M = 0$$

$$\Rightarrow\!\!> \frac{\lambda}{(\Delta x_e)^2}[x]_{x_{e-1}}^{x_e} u_2 + \frac{1}{2\Delta x_e} f [(x_e - x)^2]_{x_{e-1}}^{x_e} + M = 0$$

$$=\gg \frac{\lambda}{(\Delta x_e)}u_2 - \left(\frac{\Delta \mathrm{x}_e}{2}\right)f + M = 0$$

$$k_{12} = u_2\frac{\lambda}{(\Delta x_e)} \quad ,f_1^1 = -\left(\frac{\Delta \mathrm{x}_e}{2}\right)f + M$$

From the equation above we have $(k_{12} = k_{21})$

Therefore (i=2 , j=2)

$$=\gg -\int_{\mathrm{x}_{e-1}}^{x_e} \lambda \frac{d}{dx}\varphi_2\left(\frac{d}{dx}(\varphi_2)\right)dx\, u_2 - \int_{\mathrm{x}_{e-1}}^{\mathrm{x_e}} \mathrm{f}\varphi_2\, dx + M = 0$$

$$=\gg -\int_{x_{e-1}}^{x_e}\frac{\lambda}{(\Delta x_e)^2}dx u_2 - f\int_{x_{e-1}}^{x_e}\left(\frac{x - x_{e-1}}{\Delta x_e}\right)dx + M = 0$$

$$=\gg -\frac{\lambda}{(\Delta x_e)^2}[x]_{x_{e-1}}^{x_e}u_2 - \frac{1}{2\Delta x_e}f[(x - x_{e-1})^2]_{x_{e-1}}^{x_e}$$

$$=\gg -\frac{\lambda}{(\Delta x_e)}u_2 - \left(\frac{\Delta \mathrm{x}_e}{2}\right)f + M = 0$$

$$k_{22} = -u_2\frac{\lambda}{(\Delta x_e)} \quad ,f_2^1 = -\left(\frac{\Delta \mathrm{x}_e}{2}\right)f + M$$

Solving this system, we obtain the approximation solution for differential equation.

$$k_{ij} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{pmatrix} = \begin{pmatrix} -\frac{\lambda}{(\Delta x_e)} & \frac{\lambda}{(\Delta x_e)} \\ \frac{\lambda}{(\Delta x_e)} & -\frac{\lambda}{(\Delta x_e)}\end{pmatrix} = \frac{\lambda}{(\Delta x_e)}\begin{pmatrix} -1 & 1 \\ 1 & -1\end{pmatrix}$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -\left(\frac{\Delta \mathrm{x}_e}{2}\right)f + M \\ -\left(\frac{\Delta \mathrm{x}_e}{2}\right)f + M \end{pmatrix}$$

Two element and three nodes

In this case, stiffness matrix can be written as

$$k = \begin{pmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{pmatrix}$$

Three element and four nodes

In this case, stiffness matrix can be written as

$$k = \begin{pmatrix} k_{11}^{1} & k_{12}^{1} & 0 & 0 \\ k_{21}^{1} & k_{22}^{1} + k_{11}^{2} & k_{12}^{2} & 0 \\ 0 & k_{21}^{2} & k_{22}^{2} + k_{11}^{3} & k_{12}^{3} \\ 0 & 0 & k_{21}^{3} & k_{22}^{3} \end{pmatrix}$$

Generalization

N element and (N+1) nodes

$$K = \begin{pmatrix} K^{1}_{11} & k^{1}_{12} & & & & & & \\ K^{1}_{11} & k^{2}_{22} + k^{2}_{11} & k^{2}_{12} & & & 0 & & \\ & k^{2}_{21} & k^{2}_{22} + k^{3}_{11} & & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0 & & & & & k^{n}_{22} + k^{n+1}_{11} & k^{n+1}_{12} \\ & & & & & & k^{n+1}_{21} & k^{n+1}_{22} \end{pmatrix}$$

1.5 The Finite Element Method – Two Dimensional

The finite element analysis of two-dimensional problems involves the same basic steps as those described for one-dimensional problems in section(1.4). The analysis is same that complicate by the fact that two-dimensional problems are described by partial differential equations over geometrically complex regions. Therefore, the finite element mesh consists of a simply two-dimensional element, such as triangles, rectangles and quadrilaterals, that allow unique derivation of the interpolation functions.

Example:

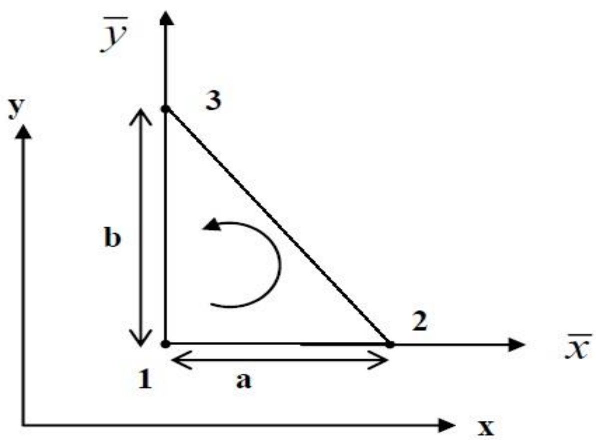
$$-\left(a_{11}\frac{\partial^2 u}{\partial x^2}+a_{22}\frac{\partial^2 u}{\partial y^2}\right)-f=0 \qquad (1.8)$$

Strong form

$$\int_{\Omega^e} -w\left(a_{11}\frac{\partial^2 u}{\partial x^2}+a_{22}\frac{\partial^2 u}{\partial y^2}\right)dx\,dy-\int_{\Omega^e} w\,f\,dx\,dy=0 \qquad (1.9)$$

Weak form

$$a_{11}\int_{\Omega^e}\frac{\partial w}{\partial x}\frac{\partial u}{\partial x}+a_{22}\int_{\Omega^e}\frac{\partial w}{\partial x}\frac{\partial u}{\partial y}-\int_{\Omega^e} w\,f\,dx\,dy$$
$$-\left[\int_{\Gamma_n} n_x\,w\frac{\partial u}{\partial x}+\int_{\Gamma_n} n_y\,w\,\frac{\partial u}{\partial y}\right]=0 \qquad (1.10)$$



Connectivity			
Element (E)	**Local node number (LNN)**		
1	**1**	**2**	**3**

Fig.1.4 The triangular element.

We define the approximation of u

$$u(x,y)=\sum_{j=1}^{n}\psi_j u_j \qquad .$$

Suppose that

$$q=n_x a_{11}\frac{\partial u}{\partial x}+n_y a_{22}\frac{\partial u}{\partial y}$$

$$\int_{\Omega_e} \frac{\partial w}{\partial x}\left(a_{11}\sum_{j=1}^{3}\frac{\partial \psi_j}{\partial x}u_j\;dx\;dy\right)+\frac{\partial w}{\partial y}\left(a_{22}\sum_{j=1}^{3}\frac{\partial \psi_j}{\partial x}u_j\;dx\;dy\right) - \int_{\Omega_e} wf\;dx\;dy\;-\oint wq\;ds=0$$

$$\int_{\Omega_e}\frac{\partial \psi_i}{\partial x}\left(a_{11}\sum_{j=1}^{3}\frac{\partial \psi_j}{\partial x}u_j\;dx\;dy\right)+a_{22}\frac{\partial \psi_i}{\partial y}\left(a_{22}\sum_{j=1}^{3}\frac{\partial \psi_j}{\partial x}u_j\;dx\;dy\right)- \int_{\Omega_e}\psi_i f\;dx\;dy\;-\oint \psi_i q\;ds=0 \quad (1.10)$$

We will choice

$i=1\,,j=1$

$$a_{11}\int_{\Omega_e}\frac{\partial \psi_1}{\partial x}\,\frac{\partial \psi_1}{\partial x}\;dx\;dy\;u_1+a_{22}\int_{\Omega_e}\;\frac{\partial \psi_1}{\partial y}\,\frac{\partial \psi_1}{\partial y}\;dx\;dyu_1 - \int_{\Omega_e}\psi_1 f\;dx\;dy\;-\oint \psi_1 q\;ds=0$$

$i=1\,,j=2$

$$a_{11}\int_{\Omega_e}\frac{\partial \psi_1}{\partial x}\,\frac{\partial \psi_2}{\partial x}\;dx\;dy\;u_2+a_{22}\int_{\Omega_e}\;\frac{\partial \psi_1}{\partial y}\,\frac{\partial \psi_2}{\partial y}\;dx\;dyu_2 - \int_{\Omega_e}\psi_1 f\;dx\;dy\;-\oint \psi_1 q\;ds=0$$

$i=1\,,j=3$

$$a_{11}\int_{\Omega_e}\frac{\partial \psi_1}{\partial x}\,\frac{\partial \psi_3}{\partial x}\;dx\;dyu_3\;+a_{22}\int_{\Omega_e}\;\frac{\partial \psi_1}{\partial y}\,\frac{\partial \psi_3}{\partial y}\;dx\;dyu_3 - \int_{\Omega_e}\psi_1 f\;dx\;dy\;-\oint \psi_1 q\;ds=0$$

$i=2\,,j=2$

$$a_{11}\int_{\Omega_e}\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_2}{\partial x}\ dx\,dy\,u_2+a_{22}\int_{\Omega_e}\ \frac{\partial\psi_2}{\partial y}\frac{\partial\psi_2}{\partial y}\,dx\,dyu_2$$
$$-\int_{\Omega_e}\psi_2 f\ dx\,dy\ -\oint\psi_2 q\ ds=0$$

$i=2\,,j=3$

$$a_{11}\int_{\Omega_e}\frac{\partial\psi_2}{\partial x}\frac{\partial\psi_3}{\partial x}\ dx\,dy\,u_3\ +a_{22}\int_{\Omega_e}\ \frac{\partial\psi_2}{\partial y}\frac{\partial\psi_3}{\partial y}\,dx\,dy\,u_3$$
$$-\int_{\Omega_e}\psi_2 f\ dx\,dy\ -\oint\psi_2 q\ ds=0$$

$i=3\,,j=3$

$$a_{11}\int_{\Omega_e}\frac{\partial\psi_3}{\partial x}\frac{\partial\psi_3}{\partial x}\ dx\,dy\,u_3+a_{22}\int_{\Omega_e}\ \frac{\partial\psi_3}{\partial y}\frac{\partial\psi_3}{\partial y}\,dx\,dyu_3$$
$$-\int_{\Omega_e}\psi_3 f\ dx\,dy\ -\oint\psi_3 q\ ds=0$$

So that, the three shape functions ψ_1, ψ_2 and ψ_3 are defined as

$$\psi_i=\tfrac{1}{2A}(\alpha_i+\beta_i x+\lambda_i y),$$

Where

$$\alpha_i=x_j y_k-x_k y_j\ ,\ \ \alpha_1=ab\,,\alpha_2=0\ \,,\alpha_3=0$$

$$\beta_i=y_j-y_k\ ,\ \ \beta_1=-b\,,\beta_2=b\ \,,\beta_3=0$$

$$\lambda_i=-(x_j-x_k)\ ,\ \ \lambda_1=-a\,,\lambda_2=0\,,\lambda_3=a$$

$$2A=\alpha_1+\alpha_2+\alpha_3\ \Rightarrow\ 2A=ab\ \ \Rightarrow A=\frac{ab}{2}$$

Where, **A** is the area of the element's triangular and α_i, β_i and λ_i are coefficients.

Such that

$$\frac{\partial\psi_1}{\partial x}=\frac{\beta_1}{2A} \quad , \quad \frac{\partial\psi_2}{\partial x}=\frac{\beta_2}{2A} \quad , \frac{\partial\psi_3}{\partial x}=\frac{\beta_3}{2A}$$
$$\frac{\partial\psi_1}{\partial y}=\frac{\lambda_1}{2A} \quad , \quad \frac{\partial\psi_2}{\partial y}=\frac{\lambda_2}{2A} \quad , \frac{\partial\psi_3}{\partial y}=\frac{\lambda_3}{2A}$$

Thus,

$$k_{11}=\frac{\beta_1^2}{4A^2}a_{11}\iint dA\, u_1+\frac{\lambda_1^2}{4A^2}a_{22}\iint dA\, u_1$$
$$k_{11}=a_{11}\frac{\beta_1^2}{4A^2}\mathrm{A}u_1+a_{22}\frac{\lambda_1^2}{4A^2}Au_1 \;\Rightarrow\; k_{11}=a_{11}\frac{\beta_1^2}{4A}u_1+a_{22}\frac{\lambda_1^2}{4A}u_1$$

$$k_{12}=a_{11}\frac{\beta_1\beta_2}{4A}u_2+a_{22}\frac{\lambda_1\lambda_2}{4A}\,u_2 \;\Rightarrow\; k_{12}=k_{21}$$
$$k_{13}=a_{11}\frac{\beta_1\beta_3}{4A}u_3+a_{22}\frac{\lambda_1\lambda_3}{4A}\,u_3 \;\Rightarrow\; k_{13}=k_{31}$$
$$k_{22}=a_{11}\frac{\beta_2^2}{4A}u_2+a_{22}\frac{\lambda_2^2}{4A}u_2$$
$$k_{23}=a_{11}\frac{\beta_2\beta_3}{4A}u_3+a_{22}\frac{\lambda_2\lambda_3}{4A}u_3 \;\Rightarrow\; k_{23}=k_{32}$$
$$k_{33}=a_{11}\frac{\beta_3^2}{4A}u_3+a_{22}\frac{\lambda_3^2}{4A}u_3$$

Thus

$$[k^e]=\frac{1}{4A}\begin{bmatrix} a_{11}\beta_1^2+a_{22}\lambda_1^2 & a_{11}\beta_1\beta_2+a_{22}\lambda_1\lambda_2 & a_{11}\beta_1\beta_3+a_{22}\lambda_1\lambda_3 \\ a_{11}\beta_2\beta_1+a_{22}\lambda_2\lambda_1 & a_{11}\beta_2^2+a_{22}\lambda_2^2 & a_{11}\beta_2\beta_3+a_{22}\lambda_2\lambda_3 \\ a_{11}\beta_3\beta_1+a_{22}\lambda_3\lambda_1 & a_{11}\beta_3\beta_2+a_{22}\lambda_3\lambda_2 & a_{11}\beta_3^2+a_{22}\lambda_3^2 \end{bmatrix}$$

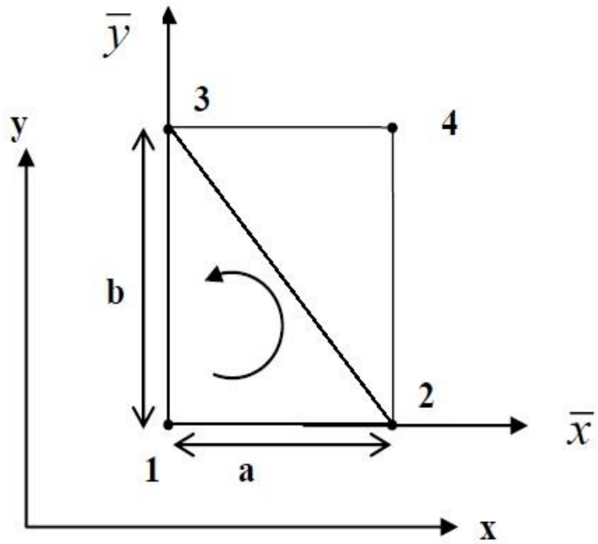
Then we have

$$[k^e]=\frac{a_{11}}{2ab}\begin{bmatrix} b^2 & -b^2 & 0 \\ -b^2 & b^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}+\frac{a_{22}}{2ab}\begin{bmatrix} a^2 & 0 & -a^2 \\ 0 & 0 & 0 \\ -a^2 & 0 & a^2 \end{bmatrix}$$

Also

$$f_i^e=\frac{1}{3}Af_e \;\Rightarrow\; f_1^e=f_2^e=f_3^e=\frac{ab}{6}f_e \;\Rightarrow\; \{f^e\}=\frac{ab}{6}f_e\begin{Bmatrix}1\\1\\1\end{Bmatrix}$$

If we want to extend the results for 2-elements, as shown in Figure (1.5).



Connectivity	
Element (E)	Local node number (LNN)
1	1 2 3
2	2 4 3

Fig.1.5 The triangular elements

$$
\begin{array}{cc|c}
 & & \begin{array}{ccc} 2 & 4 & 3 \\ \hline 1 & 2 & 3 \end{array} \\
\end{array}
$$

$$
[K] = \frac{1}{2ab} \quad \begin{array}{c:c} 2 & 1 \\ 4 & 2 \\ 3 & 3 \end{array} \begin{bmatrix} a_{11}^{e} b^{2} + a_{22}^{e} a^{2} & -a_{11}^{e} b^{2} & -a_{22}^{e} a^{2} \\ -a_{11}^{e} b^{2} & a_{11}^{e} b^{2} & 0 \\ -a_{22}^{e} a^{2} & 0 & a_{22}^{e} a^{2} \end{bmatrix}
$$

$$
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}
$$

$$
[K] = \frac{1}{2ab} \begin{bmatrix} a_{11}^{e} b^{2} + a_{22}^{e} a^{2} & -a_{11}^{e} b^{2} & -a_{22}^{e} a^{2} & 0 \\ -a_{11}^{e} b^{2} & 2a_{11}^{e} b^{2} + a_{22}^{e} a^{2} & -a_{22}^{e} a^{2} & -a_{11}^{e} b^{2} \\ -a_{22}^{e} a^{2} & -a_{22}^{e} a^{2} & 2a_{22}^{e} a^{2} & 0 \\ 0 & -a_{22}^{e} b^{2} & 0 & a_{22}^{e} b^{2} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
$$

Chapter 2

2.1 Introduction

The finite element methods are useful to obtain approximate solution to differential governing equation. In order to explain the finite element method and comparison with exact solution, we consider the following sample problem

$$\frac{d^2}{dx^2}u - u = -x\,.\ \ 0 < x < 1 \tag{2.1}$$

With Boundary Condition

$$u(0) = 0 \ \ and \ \ u(1) = 0 \ \ \ at\ x = 0.5$$

2.2 Exact Solution of problem

$$\frac{d^2}{dx^2}u - u = -x\,.\ \ 0 < x < 1$$

With Boundary Condition

$$u(0) = 0 \ \ and \ \ u(1) = 0 \ \ \ at\ x = 0.5$$

solution :- $\mathrm{u = u_P + u_H}$

$u_H = (D^2 - 1)u = 0$

$D^2 - 1 = 0 \ \ \Rightarrow \ D = \mp 1$

$u_H = c_1 \mathrm{e}^{\mathrm{x}} + c_2 e^{-x}$

$u_P = \dfrac{1}{D^2 - 1}(-x) \ \ \Rightarrow \ u_P = -\dfrac{1}{1 - D^2}(-x)$

$\Rightarrow \ u_P = -[-x] = x$

Then $\quad u = u_P + u_H = x + c_1 e^x + c_2 e^{-x}$

$u(0) = 0 \ \ \Rightarrow \ \ c_1 + c_2 = 0 \ \ \Rightarrow \ \ c_1 = -c_2$

$u(1) = 0 \ \ \Rightarrow \ \ c_1 e + c_2 \mathrm{e}^{-1} + 1 = 0$

$$\Rightarrow\gg \ c_1 = \frac{-1}{e - e^{-1}}$$

$$\Rightarrow\gg \ c_2 = \frac{1}{e - e^{-1}}$$

$$u = \frac{-1}{e - e^{-1}} \ e^{x} + \frac{1}{e - e^{-1}} \ e^{-x} + x$$

$$u(0.5) = \frac{-1}{e - e^{-1}} \ (e^{0.5} + \ e^{-0.5} + 0.5)$$

$$u(0.5) = \frac{1}{2.782 - 0.367}(-1.648 + 0.606 + 0.5) = 0.566$$

2.3 Numerical Solution of problem

$$\frac{d^2}{dx^2}u - u = -x$$

$$u(0) = 0 \ \ and \ \ u(1) = 0$$

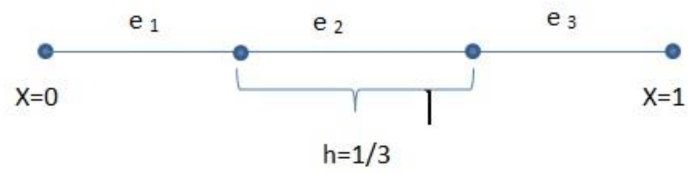
$$\frac{d^2}{dx^2}u - u + x = 0 \tag{2.2}$$

$$\int_\Omega w \frac{d^2}{dx^2} u \, d\Omega - \int_\Omega wu \, d\Omega + \int_\Omega wx \, d\Omega = 0 \tag{2.3}$$

$$-\int_\Omega \frac{dw}{dx} \ \frac{du}{dx} + [\mathrm{w}\frac{du}{dx}]_0^1 - \int_\Omega wu \, d\Omega + \int_\Omega wx \, d\Omega = 0 \tag{2.4}$$

$$[w\frac{du}{dx}]_0^1 = w(1)\frac{du}{dx} - w(0)\frac{du}{dx} = (0)\frac{du}{dx} - (0)\frac{du}{dx} = 0$$

$$\Rightarrow\gg -\int_\Omega \frac{dw}{dx} \ \frac{du}{dx} d\Omega \ - \int_\Omega wu \, d\Omega + \int_\Omega wx \, d\Omega = 0 \tag{2.5}$$



$$u_j = \sum_{j=1}^{2} \varphi_j \, u_j$$

$$\varphi_1 = \frac{x_b - x}{h} \;\; . \;\; \varphi_2 = \frac{x - x_a}{h} \quad such\ that\ h = \frac{1}{3}$$

$$\varphi_1 = 3(x_b - x) \;\; . \;\; \varphi_2 = 3(x - x_a)$$

$$\frac{d\varphi_1}{dx} = -3 \;\; . \quad \frac{d\varphi_2}{dx} = 3$$

$$-\int_\Omega \frac{d\varphi_i}{dx}\,\frac{d\varphi_j}{dx}\; u_j \, d\Omega - \int_\Omega \varphi_j \varphi_i \, d\Omega \, u_j + \int_\Omega \varphi_i \, x \, d\Omega = 0 \qquad (2.6)$$

$$i = 1 \;\; . \; j = 1$$

$$-\int_\Omega \frac{d\varphi_1}{dx}\,\frac{d\varphi_1}{dx}\; d\Omega \, u_1 - \int_\Omega \varphi_1 \varphi_1 \, d\Omega \, u_1 + \int_\Omega \varphi_1 \, x \, d\Omega = 0$$

$$i = 1 \;\; . \;\; j = 2$$

$$-\int_\Omega \frac{d\varphi_1}{dx}\,\frac{d\varphi_2}{dx}\; d\Omega \, u_2 - \int_\Omega \varphi_1 \varphi_2 \, d\Omega \, u_2 + \int_\Omega \varphi_1 \, x \, d\Omega = 0$$

$$k_{12} = k_{21}$$

$$i = 2 \;\; . \;\; j = 2$$

$$-\int_\Omega \frac{d\varphi_2}{dx}\,\frac{d\varphi_2}{dx}\; d\Omega \, u_2 - \int_\Omega \varphi_2 \varphi_2 \, d\Omega \, u_2 + \int_\Omega \varphi_2 \, x \, d\Omega = 0$$

$$k_{11} = -\int_\Omega \left(\frac{d\varphi_1}{dx}\right)^2 \; d\Omega \, u_1 - \int_\Omega (\varphi_1)^2 \, d\Omega \, u_1 \;\; , \quad f_1 = \int_\Omega \varphi_1 \, x \, d\Omega$$

$$k_{12} = -\int_\Omega \frac{d\varphi_1}{dx}\,\frac{d\varphi_2}{dx}\; d\Omega \, u_2 - \int_\Omega \varphi_1 \varphi_2 \, d\Omega \, u_2 = k_{21}$$

$$k_{22} = -\int_\Omega (\frac{d\varphi_2}{dx})^2 \quad d\Omega\, u_2 - \int_\Omega (\varphi_2)^2\, d\Omega\, u_2 \quad , \; f_2 = \int_\Omega \varphi_2\, x\, d\Omega$$

$$f_1 = \int_\Omega \varphi_1\, x\, dx \quad . \quad f_2 = \int_\Omega \varphi_2\, x\, dx$$

$Element$ (1):

$$x_a = 0 \quad . \quad x_b = \frac{1}{3}$$

$\varphi_1 = 1 - 3x \quad . \quad \varphi_2 = 3x$

Then we will have

$$k_{11} = \frac{-28}{9} u_1$$

$$f_1 = \frac{9}{486}$$

$$k_{12} = k_{21} = \frac{53}{18}$$

$$k_{22} = \frac{-28}{9} u_2$$

$$f_2 = \frac{1}{27}$$

can by written as

$$\begin{bmatrix} -3.111 & 2.9444 \\ 2.9444 & -3.111 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0.0185 \\ 0.037 \end{bmatrix}$$

$Element$ (2):

$$x_a = \frac{1}{3} \quad . \quad x_b = \frac{2}{3}$$

$\varphi_1 = 2 - 3x \quad . \quad \varphi_2 = 3x - 1$

Then we will have

$$k_{11} = \frac{-28}{9} u_2$$

$$f_1 = \frac{2}{27}$$

$$k_{12} = k_{21} = 53/18$$

$$k_{22} = \frac{-28}{9} u_3$$

$$f_2 = \frac{45}{486}$$

can by written as

$$\begin{bmatrix} -3.111 & 2.9444 \\ 2.9444 & -3.111 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0.0741 \\ 0.0926 \end{bmatrix}$$

$Element$ (3):

$$x_a = \frac{2}{3} \quad . \quad x_b = 1$$

$\varphi_1 = 3 - 3x \quad , \quad \varphi_2 = 3x - 2$

Then we will have

$$k_{11} = \frac{-28}{9} u_3$$

$$f_1 = \frac{21}{162}$$

$$k_{12} = k_{21} = \frac{53}{18}$$

$$k_{22} = \frac{-28}{9} u_4$$

$$f_2 = \frac{4}{27}$$

can by written as

$$\begin{bmatrix} -3.111 & 2.9444 \\ 2.9444 & -3.111 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0.1296 \\ 0.1481 \end{bmatrix}$$

Assemble the element equations to obtain the global system

$$\begin{bmatrix} -3.1111 & 2.9444 & 0 & 0 \\ 2.9444 & -6.2222 & 2.9444 & 0 \\ 0 & 2.9444 & -6.2222 & 2.9444 \\ 0 & 0 & 2.9444 & -3.1111 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0.0184 \\ 0.1111 \\ 0.2222 \\ 0.1481 \end{bmatrix} = 0$$

Imposition of boundary conditions

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2.9444 & -6.2222 & 2.9444 & 0 \\ 0 & 2.9444 & -6.2222 & 2.9444 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -0.1111 \\ -0.2222 \\ 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2.9444 & -6.2222 & 2.9444 & 0 \\ 0 & 2.9444 & -6.2222 & 2.9444 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -0.1111 \\ -0.2222 \\ 0 \end{bmatrix}$$

Solution gives $u_1 = 0$, $u_2 = 0.04448$, $u_3 = 0.0569$ and $u_4 = 0$.

The comparison in results between the analytic solution and numerical results is illustrated in Table 1.

Table 1: Comparison of Solution to Equation at $x = 0.5$

Exact Solution	Numerical Solution
0.0566	0.0569

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