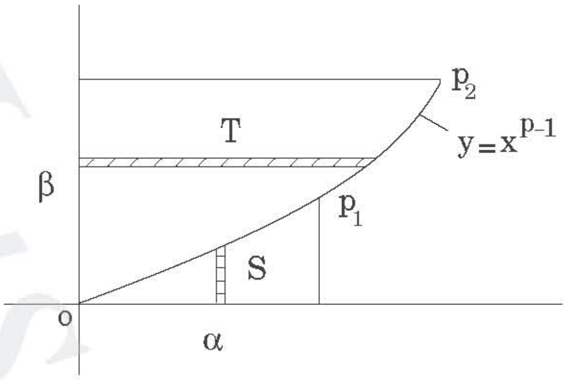
*Proof.* Let us study the curve $op_1p_2$.




$$y = x^{p-1} \quad \Rightarrow \quad x = y^{\frac{1}{p-1}}.$$

$$\left.\begin{array}{l} \frac{1}{p} + \frac{1}{q} = 1. \quad \Rightarrow \quad \frac{q}{p} + 1 = q. \quad \Rightarrow \quad \frac{q}{p} = q - 1. \\ \frac{1}{p} + \frac{1}{q} = 1. \quad \Rightarrow \quad 1 + \frac{p}{q} = p. \quad \Rightarrow \quad \frac{q}{p} = \frac{1}{p-1}. \end{array}\right\} \quad \Rightarrow \quad \frac{1}{p-1} = q - 1.$$

$$A_S = \int_0^{\alpha} x^{p-1} dx = \frac{x^p}{p}\Big|_0^{\alpha} = \frac{\alpha^p}{p}.$$

$$A_T = \int_0^{\beta} y^{q-1} dy = \frac{y^q}{q}\Big|_0^{\beta} = \frac{\beta^q}{q}.$$

Note that

$$\begin{aligned} \alpha\beta &\leqslant A_S + A_T \\ &\leqslant \frac{\alpha^p}{p} + \frac{\beta^q}{q}. \end{aligned} \tag{1.1}$$

Assume that $\alpha = \frac{|A(x)|}{\|A\|_p}$, $\beta = \frac{|B(x)|}{\|B\|_q}$ and substitute in (1.1) to get

$$\frac{|A(x)|}{\|A\|_p} \frac{|B(x)|}{\|B\|_q} \leqslant \frac{|A(x)|^p}{p\|A\|_p^p} + \frac{|B(x)|^q}{q\|B\|_q^q}.$$

$$\begin{aligned} \frac{1}{\|A\|_p \|B\|_q} \int_a^b |A(x)B(x)| dx &\leqslant \frac{1}{p\|A\|_p^p} \int_a^b |A(x)|^p dx + \frac{1}{q\|B\|_q^q} \int_a^b |B(x)|^q dx \\ &\leqslant \frac{1}{p\|A\|_p^p} \|A\|_p^p + \frac{1}{q\|B\|_q^q} \|B\|_q^q = \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

Hence

$$\int_a^b |A(x)B(x)| dx \leqslant \|A\|_p \|B\|_q = \left[\int_a^b |A(x)|^p dx\right]^{\frac{1}{p}} \cdot \left[\int_a^b |B(x)|^q dx\right]^{\frac{1}{q}}.$$

□

**Remark 1.2.** *When $p = q = 2$ the Holder inequality becomes*

$$\int_a^b |A(x)B(x)|dx \leqslant \left[\int_a^b |A(x)|^2 dx\right]^{\frac{1}{2}} \cdot \left[\int_a^b |B(x)|^2 dx\right]^{\frac{1}{2}}.$$

*The above inequality called Cauchy-Schwartz inequality.*

**Theorem 1.2.** (Minkowski inequality) *If $p \geqslant 1$ and $A, B \in C[a,b]$, then*

$$\left[\int_a^b \big[|A(x) + B(x)|\big]^p dx\right]^{\frac{1}{p}} \leqslant \left[\int_a^b |A(x)|^p dx\right]^{\frac{1}{p}} + \left[\int_a^b |B(x)|^p dx\right]^{\frac{1}{p}}.$$

*Proof.*

$$\begin{aligned}\big[|A(x) + B(x)|\big]^p &= \big[|A(x) + B(x)|\big] \cdot \big[|A(x) + B(x)|\big]^{p-1} \\ &\leqslant |A(x)|\big[|A(x) + B(x)|\big]^{p-1} + |B(x)|\big[|A(x) + B(x)|\big]^{p-1}. \qquad (1.2)\end{aligned}$$

Applying Holder inequality to every term on the right hand side of (1.2)

$$\begin{aligned}\int_a^b \big[|A(x) + B(x)|\big]^p dx &\leqslant \|A\|_p \left[\int_a^b \big[|A(x) + B(x)|\big]^{(p-1)q} dx\right]^{\frac{1}{q}} \\ &\quad + \|B\|_p \left[\int_a^b \big[|A(x) + B(x)|\big]^{(p-1)q} dx\right]^{\frac{1}{q}} \\ &\leqslant \left[\int_a^b \big[|A(x) + B(x)|\big]^{(p-1)q} dx\right]^{\frac{1}{q}} \Big[\|A\|_p + \|B\|_p\Big].\end{aligned}$$

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \Rightarrow \quad 1 + \frac{p}{q} = p \quad \Rightarrow \quad \frac{p}{q} = p - 1 \quad \Rightarrow \quad p = (p-1)q.$$

$$\int_a^b \big[|A(x) + B(x)|\big]^p dx \leqslant \left[\int_a^b \big[|A(x) + B(x)|\big]^p dx\right]^{\frac{1}{q}} \Big[\|A\|_p + \|B\|_p\Big].$$

Divide by $\left[\int_a^b \big[|A(x)+B(x)|\big]^p dx\right]^{\frac{1}{q}}$ and use the fact that $1 - \frac{1}{q} = \frac{1}{p}$ to get the required result. □

**Remark 1.3.** *It will be noted that this method yields also the Holder inequality and Minkowski inequality for series. i.e., we have*

(1) *If* $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\sum_{k=1}^{N} |a_k b_k| \leqslant \left[\sum_{k=1}^{N} |a_k|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{k=1}^{N} |b_k|^q\right]^{\frac{1}{q}},$$

*where* $a_k, b_k \in \mathbb{R}$ *for* $k = 1, 2, \ldots, N$.

(2) *If* $p \geqslant 1$ *and* $a_k, b_k \in \mathbb{R}$ *for* $k = 1, 2, \ldots, N$, *then*

$$\left[\sum_{k=1}^{N} \left[|a_k + b_k|\right]^p\right]^{\frac{1}{p}} \leqslant \left[\sum_{k=1}^{N} |a_k|^p\right]^{\frac{1}{p}} + \left[\sum_{k=1}^{N} |b_k|^p\right]^{\frac{1}{p}}.$$

*Proof.* H.W. □

**Examples of Normed Linear Spaces**

**Example 1.1.** $C[a, b]$ *with the p-norm*

$$\|f\|_p = \left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}} \quad 1 \leqslant p < \infty$$

*is a normed linear space over a field* $\mathbb{R}$ *with respect to operations addition and standard multiplication which is defined as follows:*

(1) $(f + g)(x) = f(x) + g(x)$ *for all* $f, g \in C[a, b]$.

(2) $(r \cdot f)(x) = r \cdot f(x)$ *for all* $r \in \mathbb{R}$ *and for all* $f \in C[a, b]$.

*Proof.*

i. $C[a, b]$ is a linear space.

(1) $(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x)$.

(2) $(f + (g + h))(x) = f(x) + (g + h)(x) = f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x) = (f + g)(x) + h(x) = ((f + g) + h)(x)$.

(3) $(f+O)(x) = f(x). \quad \Rightarrow \quad f(x)+O(x) = f(x). \quad \Rightarrow \quad O(x) = 0 \quad \forall\, x \in [a,b]$.
i.e., the identity element is the function $O : [a,b] \to \mathbb{R}$ which is defined by $O(x) = 0 \quad \forall\, x \in [a,b]$.

(4) $(f + (-f))(x) = O(x). \quad \Rightarrow \quad (-f)(x) = -f(x)$. i.e., the inverse element is the function $-f : [a,b] \to \mathbb{R}$

(5) $(r \cdot (f+g))(x) = r \cdot (f+g)(x) = r \cdot (f(x) + g(x)) = r \cdot f(x) + r \cdot g(x) = (r \cdot f)(x) + (r \cdot g)(x)$.

(6) $((r+s) \cdot f)(x) = (r+s) \cdot f(x) = r \cdot f(x) + s \cdot f(x) = (r \cdot f)(x) + (s \cdot f)(x) = (r \cdot f + s \cdot f)(x)$.

(7) $((r \cdot s) \cdot f)(x) = (r \cdot s) \cdot f(x) = r \cdot (s \cdot f(x)) = r \cdot (s \cdot f)(x) = (r \cdot (s \cdot f))(x)$.

(8) $(1 \cdot f)(x) = 1 \cdot f(x) = f(x)$.

ii. The $p$-norm, $1 \leqslant p < \infty$, defines a norm on $C[a,b]$.

(1) $\|f\|_p > 0$ unless $f = 0$.

$$\|f\|_p = \left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}}.$$

if $f(x) = 0 \quad \Rightarrow \quad \|f\|_p = 0$.

if $f(x) \neq 0 \quad \Rightarrow \quad \|f\|_p > 0$.

(2) $\|rf\|_p = |r|\|f\|_p$ where $r$ is scaler.

$$\|rf\|_p = \left[\int_a^b |rf(x)|^p dx\right]^{\frac{1}{p}} = |r|\left[\int_a^b |f(x)|^p dx\right]^{\frac{1}{p}} = |r|\|f\|_p.$$

(3) By Minkowski inequality we get $\|f+g\|_p \leqslant \|f\|_p + \|g\|_p$.

□

**Example 1.2.** $\mathbb{R}^N$ *with the p-norm*

$$\|f\|_p = \left[\sum_{i=1}^{N} |f(x_i)|^p\right]^{\frac{1}{p}} \quad 1 \leqslant p < \infty$$