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Control Engineering Design

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Contents

Chapter One
"Design of the Control System in State Space"

1.1 Controllability and Observability Test:

1.1.1 Controllability Test:

A system is said to be controllable at $t = t_0$ if it is possible means of a control input $u(t)$ to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time. Consider the single input single output system:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{bu}$$

$$\boldsymbol{y} = \boldsymbol{cx}$$

where $A\ is\ n \times n\ matrix\ , b\ is\ n \times 1\ column\ vector\ and\ c\ is\ 1 \times n\ row\ vector$

It can be showed that if the matrix S forms as:

$$S = [b\ Ab\ A^2b\ \ldots\ A^{n-1}b]_{(n\times n)}$$

is of rank n (when n is the order of the system) then the system is completely state controllable. The rank of matrix S is n, if the matrix S is non-singular or $\det[S] \neq 0$.

<u>Example 1.1:</u> Given the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u$$

Is the system completely controllable?

Solution:

Since the order of the given system is *n=2*

$\therefore S = [b\ Ab]$

$$Ab = \begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

Therefore, $S = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \rightarrow det\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} = 0$

Since the $\det[S] = 0$, so the system is not completely controllable.

<u>Example 1.2</u>**<u>:</u>** Given the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

Is the system completely controllable?

Solution:

Since the order of the given system is *n=2*

$\therefore S = [b\ Ab]$

$$Ab = \begin{bmatrix}1 & 1\\ 2 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix} \rightarrow det\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix} = -1$$

Since the $\det[S] \neq 0$, so that the given system is completely controllable.

If the given system is Multi Input Multi Output (MIMO) system or Multi Input Signal Output (MISO) system:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx}$$

where $A\ is\ n \times n\ matrix\ , b\ is\ n \times m\ matrix\ and\ c\ is\ r \times n\ matrix$

The system is said completely controllable if:

$$det[SS^T] \neq 0$$

<u>Example 1.3:</u> Given the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

Is the system completely controllable?

Solution:

Since the order of the given system is *n=2*

$S = [B\ AB]$

$$AB = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$$

$$\therefore S = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & -2 & -3\end{bmatrix}$$

Because S is not square matrix we have to find the determinate of SS^T

$$SS^T = \begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & -2 & -3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & -2\\ 1 & -3\end{bmatrix} = \begin{bmatrix}2 & -3\\ -3 & 14\end{bmatrix}$$

$$det[SS^T] = det\begin{bmatrix}2 & -3\\ -3 & 14\end{bmatrix} = 19$$

Since the $\det[SS^T] \neq 0$, so that the given system is completely controllable.

1.1.2 Observability Test:

A system is said to be Observable if the state $x(t)$ can be determined from knowledge of the input $u(t)$ and the output $y(t)$ over a finite interval of time. Consider the single input single output system:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{bu}$$

$$\boldsymbol{y} = \boldsymbol{cx}$$

where $A\ is\ n \times n\ matrix\ , b\ is\ n \times 1\ column\ vector\ and\ c\ is\ 1 \times n\ row\ vector\ .$

to test the Observability, from the matrix Q that give as:

$$Q = \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix}$$

If the rank of Q is n then the system is completely observable. In other words, if $\det[Q] \neq 0$ then the given system is completely observable.

Example 1.4: Consider the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0\,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is the system completely observable?

Solution:

Since the order of the given system is *n=2*, then:

$$Q = \begin{bmatrix} c \\ cA \end{bmatrix}$$

$$cA = [1 \quad 0\,] \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = [1 \quad 1]$$

$$\therefore Q = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Since $\det[Q] \neq 0$, so that the given system is completely observable.

If the given system is multi input multi output system:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}$$

$$\boldsymbol{y} = \boldsymbol{Cx}$$

where $A\ is\ n \times n\ matrix\ , b\ is\ n \times m\ matrix\ and\ c\ is\ r \times n\ matrix$

The system is said completely observable if:

$$det[Q^T Q] \neq 0$$

Example 1.5: Given the following system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Is the system completely controllable?

Solution:

Since the order of the given system is *n=2*

$$Q = \begin{bmatrix}c\\ cA\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\\ -2 & -3\end{bmatrix}$$

Because S is not square matrix we have to find the determinate of Q^TQ

$$Q^TQ = \begin{bmatrix}1 & 0 & 0 & -2\\ 0 & 1 & 1 & -3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}5 & 6\\ 6 & 11\end{bmatrix}$$

$$det[Q^TQ] = det\begin{bmatrix}5 & 6\\ 6 & 11\end{bmatrix} = 19$$

Since the $\det[Q^TQ] \neq 0$, so that the given system is completely observable.

Home Work:

Test the Controllability and the Observability of the following systems:

1) $A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -3 & -2\end{bmatrix}$; $b = \begin{bmatrix}1\\ 3\\ 1\end{bmatrix}$; $c = [1 \quad -1 \quad 2]$
2) $A = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 3 & 1\end{bmatrix}$; $B = \begin{bmatrix}0 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}$; $C = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$
3) $\dddot{y} + 16\ddot{y} + 192\dot{y} + 640y = 160\dot{u} + 640u.$

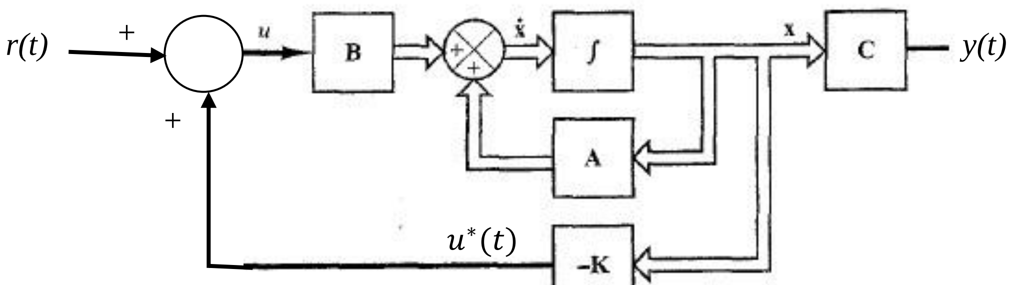
1.2 State Feedback Controller (Poles Placement)

Let the single input single output system be given by:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + bu(t) \qquad (1.1)$$

$$y(t) = c\boldsymbol{x}(t) \qquad (1.2)$$

if state feedback control $u^*(t) = -\boldsymbol{kx}$, then the control input to given system is given by $u(t) = r(t) + u^*(t)$, the system with state feedback system is given in Figure 1.1:



The state feedback controller is given by:

$$u(t) = r(t) - \boldsymbol{kx}(t) \qquad (1.3)$$

where $k = [k_1 \quad k_2 \quad .... \quad k_n]$ is $1 \times n$ feedback vector with constant elements.

$$\therefore \dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{b}(r - \boldsymbol{kx}) = [\boldsymbol{A} - \boldsymbol{bk}]\boldsymbol{x} + \boldsymbol{b}r \qquad (1.4)$$

Or: $\dot{\boldsymbol{x}} = [\boldsymbol{A} - \boldsymbol{bk}]\boldsymbol{x} + \boldsymbol{b}r$ (1.5)

It can be show that if the pair (***A,b***) is completely controllable, then the vector k exist and can give any arbitrary set of Eigen values (poles) of matrix $[\boldsymbol{A} - \boldsymbol{bk}]$ or roots of characteristic equation:

$|\boldsymbol{SI} - ([\boldsymbol{A} - \boldsymbol{bk}])| = 0$

Or: $|\boldsymbol{SI} - \boldsymbol{A} + \boldsymbol{bk}| = 0$ (1.6)

Example1.6: Given the system:

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find the constant gains vector ***k*** to move system poles to -4, -5?

Solution:

To find the poles of the open loop system (given system):

$$|sI - A| = 0$$

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right| = \left| \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \right| = 0$$

$$\therefore s(s+3) + 3 = s^2 + 3s + 2 = 0$$

$$(s+1)(s+2) = 0$$

Therefore, the poles of the open loop system are: $s = -1 \; and \; s = -2$.

Test the controllability:

Since the given system is second order n=2:

$S = [b \; Ab] = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \rightarrow det \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} = -1$

Therefore, the system is completely controllable.

To design the state feedback system (find the ***k*** vector), the new characteristic equation should be find.

Since the desired poles are $s = -4 \; and \; s = -5$, then the new char. Eq. is:

$$(s+4)(s+5) = 0$$

$$\therefore s^2 + 9s + 20 = 0 \qquad (I)$$

From eq. 1.6 $|s\boldsymbol{I}-\boldsymbol{A}+\boldsymbol{bk}|=0$, we obtain:

$$\left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}-\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \quad k_2]\right|=0$$

$$\left|\begin{bmatrix} s & -1 \\ 2 & s+ \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}\right|=0 \rightarrow \left|\begin{bmatrix} s & -1 \\ 2+k_1 & s+3+k_2 \end{bmatrix}\right|=0$$

$$\therefore s(s+3+k_2)+(2+k_1)=0$$

$$\therefore s^2+(3+k_2)s+(2+k_1)=0 \qquad (II)$$

Compare eq. (I) and eq. (II) we obtain:

$$3+k_2=9 \quad and \quad 2+k_1=20$$

$$\therefore k_1=18 \quad and \quad k_2=6$$

Example 1.7: for same system given in example 1.6, find the gain vector k if the required parameters of the new system are $\zeta=0.3\ and\ \omega_n=10\ rad/sec.$

Solution:

The standard char. Eq. of the second order system is given by:

$s^2+2\zeta\omega_n s+\omega_n^2=0$

$$\therefore s^2+6s+100=0 \qquad (I)$$

From example 1.6, the new char. Eq. is

$$\therefore s^2+(3+k_2)s+(2+k_1)=0 \qquad (II)$$

By comparing eq. (I) and eq. (II) we obtain:

$$3+k_2=6 \quad and \quad 2+k_1=100$$

$$\therefore k_1=98 \quad and \quad k_2=3$$

1.2.2 General Method to Determine the Matrix *k*:

Let us defined the following:

$\Delta_0(s)=\det(sI-A), open\ loop\ char\ eq.$

$\Delta_c(s)=\det(sI-A+bk)\ , closed\ loop\ char.eq.$

$\Delta(s)=1+k(sI-A)^{-1}b$

And, $F(s)=adj(sI-A)b$

Check the controllability condition for the system. If the system is completely state controllable, then the state feedback gain vector can be obtained from the following equation:

$kF(s)=\Delta_c(s)-\Delta_0(s)$ (1.7)

Example 1.8: Given the following system:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 1 \end{bmatrix} \; and \; b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Find the state feedback vector k to place the closed loop system poles at -2, -1 and -1?

Solution:

Find the controllability of the given system:

Since the given system is third order system then :

$$S = [b \quad Ab \quad A^2b]$$

$$Ab = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \; and \;\; A^2b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \rightarrow S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$det \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = -1$$

The system is controllable.

$$\Delta_c(s) = (s+2)(s+1)(s+1)$$

$$\Delta_0(s) = \det(sI - A) = s^3 - 2s^2 + 3s - 2$$

$$\Delta_c(s) - \Delta_0(s) = 6s^2 + 2s + 4$$

$$F(s) = adj(sI - A)b = \begin{bmatrix} s^2 - s + 2 \\ -(s-1) \\ 1 \end{bmatrix}$$

Since

$$kF(s) = \Delta_c(s) - \Delta_0(s)$$

$$[k_1 \quad k_2 \quad k_3] \begin{bmatrix} s^2 - s + 2 \\ -(s-1) \\ 1 \end{bmatrix} = 6s^2 + 2s + 4$$

$$k_1(s^2 + s + 2) - k_2(s-1) + k_3 = 6s^2 + 2s + 4$$

$$k_1 s^2 + (-k_1 - k_2)s + (2k_1 + k_2 + k_3) = 6s^2 + 2s + 4$$

$$\therefore k_1 = 6; \quad (-k_1 - k_2) = 2; \quad (2k_1 + k_2 + k_3) = 4$$

$$k_1 = 6; \quad k_2 = -8 \quad and \; k_3 = 0$$

To conform the solution is ok we have to find the following determinant of

$det(sI - A + bk) = 0$

1.2.3 Determination of Matrix *k* Using Ackermann's Formula:

To calculate the given vector ***k*** for *n*th order system:

If the system is controllable then

$$k = [k_1 \quad k_2 \quad \dots \quad k_n]$$

$$= [0 \quad 0 \quad \dots \quad 0 \quad 1][b \quad Ab \quad A^2b \quad \dots \quad A^{n-1}b]^{-1}\Delta_c(A) \qquad (1.8)$$

where $\Delta_c(A)$ represented the matrix polynomial formed with coefficient of the desired characteristic equation$\Delta_c(s)$.

Example 1.9: Given A system with ;

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find the values of gains ***k*** to locate the poles of closed loop system at: 1) -4,-4; 2) -4-j4 , -4+j4. Using Ackermann's formula?

Solution:

1) Test the controllability

$$S = |b\ Ab| \rightarrow S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$det(S) = det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

It is controllable.

$$[k_1 \quad k_2] = [0 \quad 1][b \quad Ab]^{-1}\Delta_c(A)$$

$$[b \quad Ab]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Delta_c(s) = (s+4)(s+4) = s^2 + 8s + 16$$

$$\Delta_c(A) = A^2 + 8A + 16I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 0 & 16 \end{bmatrix}$$

$$\therefore [k_1 \quad k_2] = [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 16 & 8 \\ 0 & 16 \end{bmatrix} = [16 \quad 8]$$

$$\therefore k_1 = 16; \quad k_2 = 8$$

2) For $s_1 = -4 + j4 \quad and \quad s_2 = -4 - j4$

$$\Delta_c(s) = (s + 4 - j4)(s + 4 + j4) = s^2 + 8s + 32$$

$$\Delta_c(A) = A^2 + 8A + 16I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix} = \begin{bmatrix} 32 & 8 \\ 0 & 32 \end{bmatrix}$$

$$\therefore [k_1 \quad k_2] = [0 \quad 1] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 32 & 8 \\ 0 & 32 \end{bmatrix} = [32 \quad 8]$$

$$\therefore k_1 = 32; \quad k_2 = 8$$

Example 1.10: Consider the regulator system. The plant is given by:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-1&-5&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

Find the gain vector $\boldsymbol{k}$ to locate the poles at:

$s_1=-2+j4;\qquad s_2=-2-j4;\quad s=-10.$

Solution:

Since, the given system is third order system, then the Ackermann's formal is written as:

$$[k_1\quad k_2\quad k_3]=[0\quad 0\quad 1][b\quad Ab\quad A^2b]^{-1}\Delta_c(A)$$

To test the controllability of the given system

$$\det(S)=det[b\quad Ab\quad A^2b]=-1$$

Therefore, the system is completely controllable.

$$\Delta_c(s)=(s+2-j4)(s+2+j4)(s+10)=s^3+14s^2+60s+200$$

$$\Delta_c(A)=A^3+14A^2+60A+200=$$

$A^3=\begin{bmatrix}-1&-5&-6\\6&29&31\\-31&-149&-157\end{bmatrix};\ 14A^2=\begin{bmatrix}0&0&14\\-14&-70&-84\\84&406&434\end{bmatrix};$

$60A=\begin{bmatrix}0&60&0\\0&0&60\\-60&-300&-360\end{bmatrix}$

$\therefore\ \Delta_c(A)=\begin{bmatrix}199&55&8\\-8&159&7\\-7&-43&117\end{bmatrix}$

$S=[b\quad Ab\quad A^2b]=\begin{bmatrix}0&0&1\\0&1&-6\\1&-6&31\end{bmatrix}$

$S^{-1}=\begin{bmatrix}5&6&1\\6&1&0\\1&0&0\end{bmatrix}$

$$[k_1\quad k_2\quad k_3]=[0\quad 0\quad 1]\begin{bmatrix}5&6&1\\6&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}199&55&8\\-8&159&7\\-7&-43&117\end{bmatrix}=[199\quad 55\quad 8]$$

1.2.4 Determination of Matrix *k* Using Transformation Matrix T:

Suppose that the system is defined by

$$\dot{\boldsymbol{x}}=\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}u$$

and the control signal is given by

$$u=-\boldsymbol{k}\boldsymbol{x}$$

The feedback gain matrix $\boldsymbol{k}$ that forces the eigenvalues of (**A – bk)** to be $\mu_1,\mu_2,\dots,\mu_n$ (desired values) can be determined by the following steps:

Step 1: Check the controllability condition for the system. If the system is completely state controllable, then use the following steps:

Step 2: From the characteristic polynomial for matrix **A,** that is,

$$|s\boldsymbol{I} - \boldsymbol{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

determine the values of $a_1, a_2, \ldots, a_n$.

Step 3: Determine the transformation matrix **T** that transforms the system state equation into the controllable canonical form. It is not necessary to write the state equation in the controllable canonical form. All we need here is to find the matrix T. The transformation matrix T is given by:

$$\boldsymbol{T} = \boldsymbol{SW}$$

Where the matrix W is given by:

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

And

$$\boldsymbol{S} = [\boldsymbol{b}\ \boldsymbol{Ab}\ \boldsymbol{A}^2\boldsymbol{b} \ldots \boldsymbol{A}^{n-1}\boldsymbol{b}]$$

Step 4: Using the desired eigenvalues (desired closed-loop poles), write the desired characteristic polynomial:

$$(s-\mu_1)(s-\mu_2)\ldots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

and determine the values of $\alpha_1, \alpha_2, \ldots, \alpha_n$

Step 5: The required state feedback gain matrix **k** can be determined from the following Equation

$$\boldsymbol{k} = [k_1 \quad k_2 \quad k_3 \quad \ldots \quad k_n]$$

$$= [\alpha_n - a_n \quad \alpha_{n-1} - a_{n-1} \quad \cdots \quad \alpha_2 - a_2 \quad \alpha_1 - a_1]\boldsymbol{T}^{-1} \qquad (1.9)$$

<u>Example 1.11:</u> For example 1.10, used eq. 1.9 to determine the matrix **k.**

Solution:

Find the char. Eq. of the open loop system.

$$|s\boldsymbol{I}-\boldsymbol{A}| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix} = s^3 + 6s^2 + 5s + 1$$

By comparing with $s^3 + a_1 s^2 + a_2 s + a_3$, we find that:

$$a_1 = 6;\ \ a_2 = 5;\ \ a_3 = 1$$

The desired characteristic equation is:

$$\Delta_c(s) = (s+2-j4)(s+2+j4)(s+10) = s^3 + 14s^2 + 60s + 200$$

By comparing with $s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$, we find that:

$$\alpha_1 = 14;\ \ \alpha_2 = 60;\ \ \alpha_3 = 200$$

To find the matrix $\boldsymbol{T} = \boldsymbol{SW}$

$$\boldsymbol{S} = [\boldsymbol{b} \quad \boldsymbol{Ab} \quad \boldsymbol{A^2b}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$W = \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}\begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{k} = [k_1 \quad k_2 \quad k_3] = [\alpha_3 - a_3 \quad \alpha_3 - a_3 \quad \alpha_1 - a_1]\boldsymbol{T^{-1}}$$

$$\boldsymbol{k} = [k_1 \quad k_2 \quad k_3] = [200-1 \quad 60-5 \quad 14-6] = [199 \quad 55 \quad 8]$$

H.W:

Determine the matrix $\boldsymbol{k}$ for the following systems using three different methods:

1) $A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; the desired poles of closed loop system at $s_1 = 2\omega_0$ and $s_2 = 2\omega_0$.
2) $G(s) = \frac{1}{s}$; the desired poles of closed loop system at $s_{1,2} = 1 \pm j$
3) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; the desired poles of closed loop system at $s_{1,2} = 1 \pm j\ and\ s_3 = -5$
4) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\zeta = 1\ and\ \omega_n = 4$ (for the closed loop system).
5) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$; $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $\zeta = 0.3\ and\ \omega_n = 10$ (for the closed loop system).

1.3 Design of Servo Systems:

In what follows we shall just discuss a problem of designing a type 1 servo system when the plant involves an integrator. Assume that SISO system is defined by:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{b}u$$

$$y = \boldsymbol{cx}$$

Figure 1.2 shows a general configuration of the type 1 servo system when the plant has an integrator. Here we assumed that $y = x_1$.

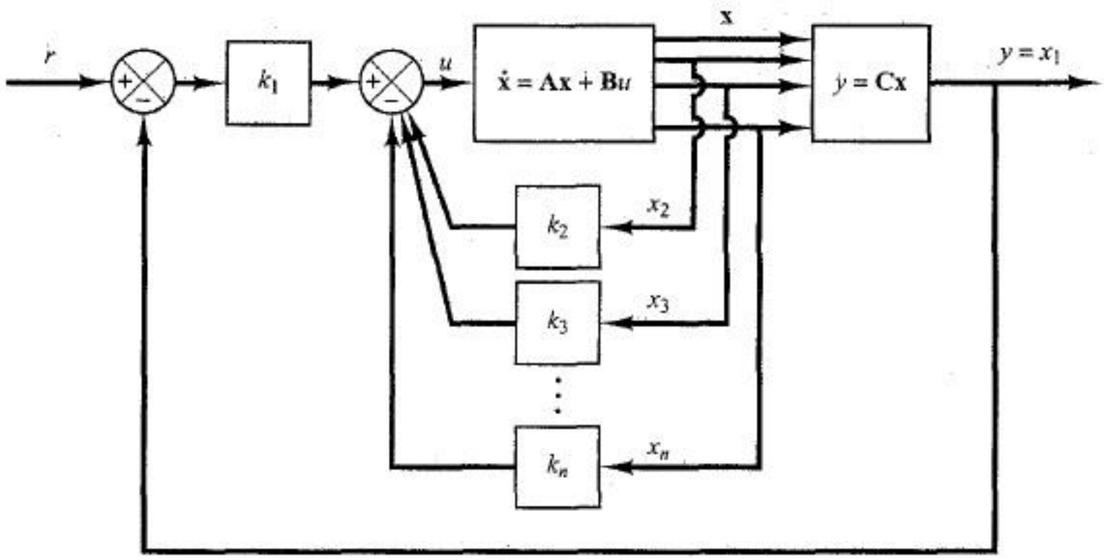


Figure 1.2 The servo system when the plant has an integrator

From Figure 1.2 the state feedback control signal is given by:

$$u = -[0 \quad k_2 \quad k_3 \quad \dots . \quad k_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + k_1(r - x_1)$$

or: $$u = -kx + k_1 r$$

where *r* is assumed step function applied at t=0 ($r(t) = R$).

Then, the system dynamics can be describe by:

$$\dot{x} = (A - bk)x + bk_1 r \qquad (1.10)$$

We shall design the type 1 servo system such that the closed-loop poles are located at desired positions. The designed system will be an asymptotically stable system, $y(\infty)$will approach the constant value *r*, and $u(\infty)$ will approach zero. (*r* is a step input).

Notice that at steady state we have:

$$\dot{x}(\infty) = (A - bk)x(\infty) + bk_1 r(\infty) \qquad (1.11)$$

where $r(\infty) = R$ (constant).

By subtracting Eq. (1.10) from Eq. (1.11) we obtain:

$$\dot{x}(t) - \dot{x}(\infty) = (A - bk)[x(t) - x(\infty)] \qquad (1.12)$$

If we define: $x(t) - x(\infty) = e(t)$ then eq. (1.12) becomes:

$$\dot{e} = (A - bk)e \qquad (1.13)$$

The steady-state values of x(t) and u(t) can be found as follows: At steady state (t = ∞), we have, from Equation (1.11),

$$\dot{x}(\infty) = (A - bk)x(\infty) + bk_1R = 0$$

$$\therefore x(\infty) = -(A - bk)^{-1}bk_1R \qquad (1.14)$$

Also, $u(\infty)$ can be obtained as:

$$u(\infty) = -kx(\infty) + k_1R = 0 \qquad (1.15)$$

<u>Example 1.12:</u> Design a type 1 servo system when the plant transfer function has an integrator. Assume that the plant transfer function is given by:

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+1)(s+2)}$$

The desired closed loops are $s_{1,2} = 2 \mp j2\sqrt{3}\ and\ s_3 = -10$. Assume that the reference input r is unit step function. Obtain the unit step response of the designed system.

Solution:

The state space representation of the system becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Using Ackerman's formula we can find the gains vector ***k.*** (H.W. Find ***k***). Or using the following MATLAB codes, we can find k.

```
A=[0 10;0 0 1;0 -2 -3];
B=[0;0;1];
J=[-2+j*2*sqrt(3) -2-j*2*sqrt(3) -10];
K=acker(A,B,J)
```

After run this program the values of k are:

$$k = [160 \quad 54 \quad 11]$$

The unit step response of the designed system can be obtained as follows:

$$A - bk = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [160 \quad 54 \quad 11] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}$$

The state equation of the given system with k matrix is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 160 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

By using the following MATLAB program, the unit step response can be ploted.

```
% Unit step response
clc;
clear all;
% Enter the state matrix, control matrix and output matrix of the given
% system
A=[0 1 0;0 0 1;0 -2 -3];
b=[0;0;1];
c=[1 0 0];
d=[0];
% Enter the state matrix, control matrix and output matrix of the designed
% system
AA=[0 1 0;0 0 1;-160 -54 -14];
bb=[0;0;160];
c=[1 0 0];
d=[0];
% Enter step command and plot command
t=0:0.1:5;
y1=step(A,b,c,d,1,t);
y2=step(AA,bb,c,d,1,t);
plot (t,y1,t,y2)
```

The resulting unit step response curve is shown in Figure 1.3.

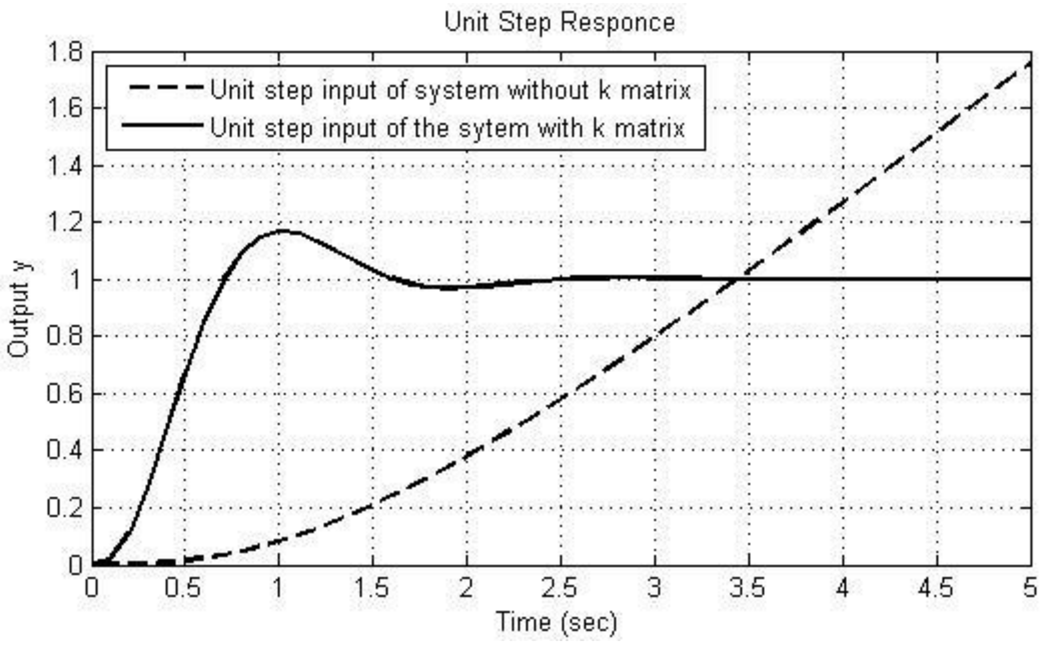


Figure 1.3 unit step response of given system in Example 1.12

1.4 State Observers:

In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables.

Estimation of immeasurable state variables is commonly called *observation.* A device (or a computer program) that estimates or observes the state variables is called a *state observer*, or simply an *observer.* If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called *a full-order state observer. There* are times when this will not be necessary, when we will need observation of only the immeasurable state variables, but not of those that are directly measurable as well. For example, since the output variables are observable and they are linearly related to the state variables, we need not observe all state variables, but observe only $n - m$ state variables, where n is the dimension of the state vector and m is the dimension of the output vector.

An observer that estimates fewer than n state variables, where n is the dimension of the state vector, is called a *reduced-order state observer* or, simply, a *reduced-order observer.* If the order of the reduced-order state observer is the minimum possible, the observer is called a *minimum-order state observer* or *minimum-order observer.* In this section, we shall discuss both the full-order state observer and the minimum-order state observer.

1.4.1 State Observer:

In the following discussions of state observers, we shall use the notation $\tilde{x}$ to designate the observed state vector. In many practical cases, the observed state vector $\tilde{x}$ is used in the state feedback to generate the desired control vector.

Consider the plant defined by

$$\dot{x} = Ax + bu \qquad (1.16)$$

$$y = cx \qquad (1.17)$$

The observer is a subsystem to reconstruct the state vector of the plant. The mathematical model of the observer is basically the same as that of the plant, except that we include an additional term that includes the estimation error to compensate for inaccuracies in matrices ***A*** and ***b*** and the lack of the initial error. The estimation error or observation error is the difference between the measured output and the estimated output. The initial error is the difference between the initial state and the initial estimated state. Figure 1.4 shows the block diagram of the system and the full-order state observer.

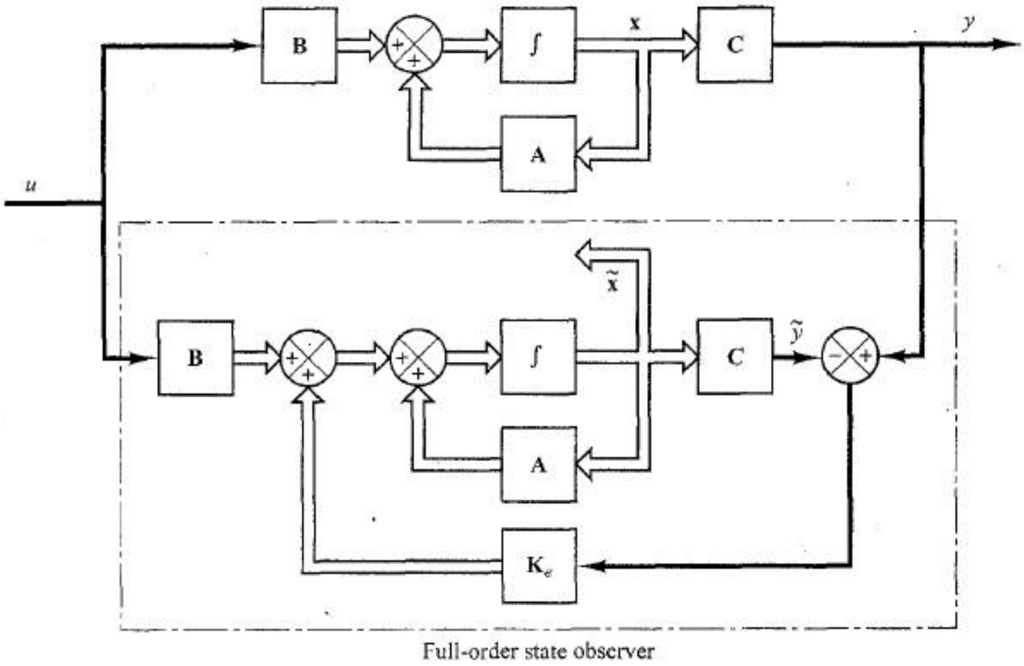


Figure 1.4 Block diagram of systems and full-order state observer

Thus, we define the mathematical model of the observer to be

$$\frac{d\tilde{\boldsymbol{x}}}{dt} = A\tilde{x} + bu + k_e(y - c\tilde{x})$$

$$= (A - k_e c)\tilde{x} + bu + k_e y \qquad (1.18)$$

where $\tilde{x}$ is the estimated state and $\tilde{y} = \boldsymbol{c}\tilde{x}$ is the estimated output. The inputs to the observer are the output *y* and the control input *u*. Matrix k_e**,** which is called the observer gain matrix, is a weighting matrix to the correction term involving the difference between the measured output *y* and the estimated output $\tilde{y} = \boldsymbol{c}\tilde{x}$**.** This term continuously corrects the model output and improves the performance of the observer

1.4.2 Full-Order State Observer:

To obtain the observer error equation, let us subtract Equation (1.16) from Equation (1.18):

$$\dot{\boldsymbol{x}} - \frac{d\tilde{\boldsymbol{x}}}{dt} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\tilde{\boldsymbol{x}} - \boldsymbol{k_e}(\boldsymbol{c}\boldsymbol{x} - \boldsymbol{c}\tilde{\boldsymbol{x}}) = (\boldsymbol{A} - \boldsymbol{k_e}\boldsymbol{c})(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \qquad (1.19)$$

Define the difference between $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ as the error vector **e,** or

$$\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$$

Then equation 1.19 can be written as:

$$\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{k_e c})\, \boldsymbol{e} \qquad (1.20)$$

From Equation (1.20), we see that the dynamic behavior of the error vector is determined by the eigenvalues of matrix $\boldsymbol{A} - \boldsymbol{k_e c}$**.** If matrix $\boldsymbol{A} - \boldsymbol{k_e c}$ is a stable matrix, the error vector will converge to zero for any initial error vector **e(0).** That is $\tilde{x}(t)$ will converge to $x(t)$ regardless of the values of $x(0)$ and $\tilde{x}(0)$. If the eigenvalues of matrix $\boldsymbol{A} - \boldsymbol{k_e c}$ are chosen in such a way that the dynamic behavior of the error vector is asymptotically stable and is adequately fast, then any error vector will tend to zero (the origin) with an adequate speed.

If the plant is completely observable, then it can be proved that it is possible to choose matrix$\boldsymbol{k_e}$**,** such that $\boldsymbol{A} - \boldsymbol{k_e c}$ has arbitrarily desired eigenvalues.

There are three ways by which the state observer gain matrix $\boldsymbol{k_e}$ can be determined:

- ***Transformation Approach to obtain state observer gain matrix*** $\boldsymbol{k_e}$***:***

To obtain the state observer gain matrix, follow the steps below:

Step 1: Check the Observability condition for the system. If the system is completely state observable, then use the following step:

Step 2: use the following equation to find k_e;

$$\boldsymbol{k}_e = \boldsymbol{N}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = (\boldsymbol{W}\boldsymbol{V}^{\boldsymbol{T}})^{-1}\begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} \qquad (1.21)$$

Where $\boldsymbol{k}_e$ is an $n \times 1$ matrix ($\boldsymbol{k}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ \vdots \\ k_{e(n-1)} \\ k_{en} \end{bmatrix}$), and

$$\boldsymbol{V} = [\boldsymbol{c^T} \;\; \boldsymbol{A^T c^T} \;\; (\boldsymbol{A^T})^2\boldsymbol{c^T} \ldots (\boldsymbol{A^T})^{\boldsymbol{n-1}}\boldsymbol{c^T}]$$

And

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

- ***Direct Substitution Approach to obtain state observer gain matrix*** $\boldsymbol{k_e}$***:***

Similar to the case of pole placement, if the system is of low order, then direct substitution of matrix k_e into the desired characteristic polynomial may be simpler. For example, if $\boldsymbol{x}$ is a 3×3, then write the observer gain matrix k_e as:

$$\boldsymbol{k}_e = \begin{bmatrix} k_{e1} \\ k_{e2} \\ k_{e3} \end{bmatrix}$$

Check the Observability condition for the system. If the system is completely state observable, then substitute the $\boldsymbol{k_e}$ matrix into the desired characteristic polynomial:

$$|s\boldsymbol{I} - (\boldsymbol{A} - \boldsymbol{k_e c})| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

By equating the coefficients of the like powers of s on both sides of this last equation, we can determine the values of k_{e1}, k_{e2}, and k_{e3}. This approach is convenient if n = 1, 2, or 3, where n is the dimension of the state vector x.

- ***Ackermann's formula to obtain state observer gain matrix $\boldsymbol{k_e}$:***

Check the Observability condition for the system. If the system is completely state observable, then use the following equation to obtain state observer gain matrix $\boldsymbol{k_e}$:

$$\boldsymbol{k_e} = \boldsymbol{\Delta}_c(\boldsymbol{A}) \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{cA} \\ \boldsymbol{cA}^2 \\ \vdots \\ \boldsymbol{cA}^{n-2} \\ \boldsymbol{cA}^{n} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (1.22)$$

Example 1.13: Consider the system

$$A = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}; \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad c = [0 \quad 1]$$

We use the observed state feedback such that $u = -k\tilde{x}$. Design a full-order state observer, assuming that the desired eigenvalues of the observer matrix are:

$$s_1 = \mu_1 = -10; \quad s_2 = \mu_2 = -10$$

***Solution*:**

The design of the state observer reduces to the determination of an appropriate observer gain matrix $\boldsymbol{k_e}$.

Let us test the Observability of the given system

$$Q = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \det(Q) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

The system is completely observable.

Method 1: We shall determine the observer gain matrix by use of Equation (1.21)

Since the characteristic equation of the given system is:

$$|s\boldsymbol{I}-\boldsymbol{A}| = \begin{vmatrix} s & -20.6 \\ -1 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1 s + a_2$$

$$\therefore a_1 = 0; \qquad and \qquad a_2 = -20.6$$

The desired char. Eq. is:

$(s+10)^2 = s^2 + 20s + 100 = s^2 + \alpha_1 s + \alpha_2$

Hence:

$$\alpha_1 = 20; \qquad \alpha_2 = 100$$

$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$N = (WV^T)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$k_e = (WV^T)^{-1} \begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 100 + 20.6 \\ 20 - 0 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

Method 2:

$$|sI - A + k_e c| = 0$$

$$\left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} s & -20.6 + k_{e1} \\ -1 & s + k_{e2} \end{vmatrix} = s^2 + k_{e2}s - 20.6 + k_{e1} = 0$$

Since the desired char. Eq. is:

$$s^2 + 20s + 100 = 0$$

$$\therefore k_{e1} = 120.6; \qquad k_{e2} = 20$$

Method 3: We shall use Ackermann's formula given by Equation (1.22):

$$\boldsymbol{k_e} = \boldsymbol{\Delta_c}(\boldsymbol{A}) \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{cA} \end{bmatrix}^{\boldsymbol{-1}} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where:

$$\Delta_c(s) = (s - \mu_1)(s - \mu_2) = s^2 + 20s + 100$$

Thus,

$$\Delta_c(A) = A^2 + 20A + 100I = \begin{bmatrix} 120.6 & 412 \\ 20 & 120.6 \end{bmatrix}$$

$$k_e = \begin{bmatrix} 120.6 & 412 \\ 20 & 120.6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

1.4.2.1 Effects of the Addition of the Observer on a closed-loop System:

In the pole-placement design process, we assumed that the actual state $x(t)$ was available for feedback. In practice, however, the actual state $x(t)$may not be measurable, so we will need to design an observer and use the observed state $\tilde{x}(t)$ for feedback as shown in Figure 1.5. The design process, therefore, becomes a two-stage process, the first stage being the determination of the feedback gain matrix k to yield the desired characteristic equation and the second stage being the determination of the observer gain matrix k_e to yield the desired observer characteristic equation.

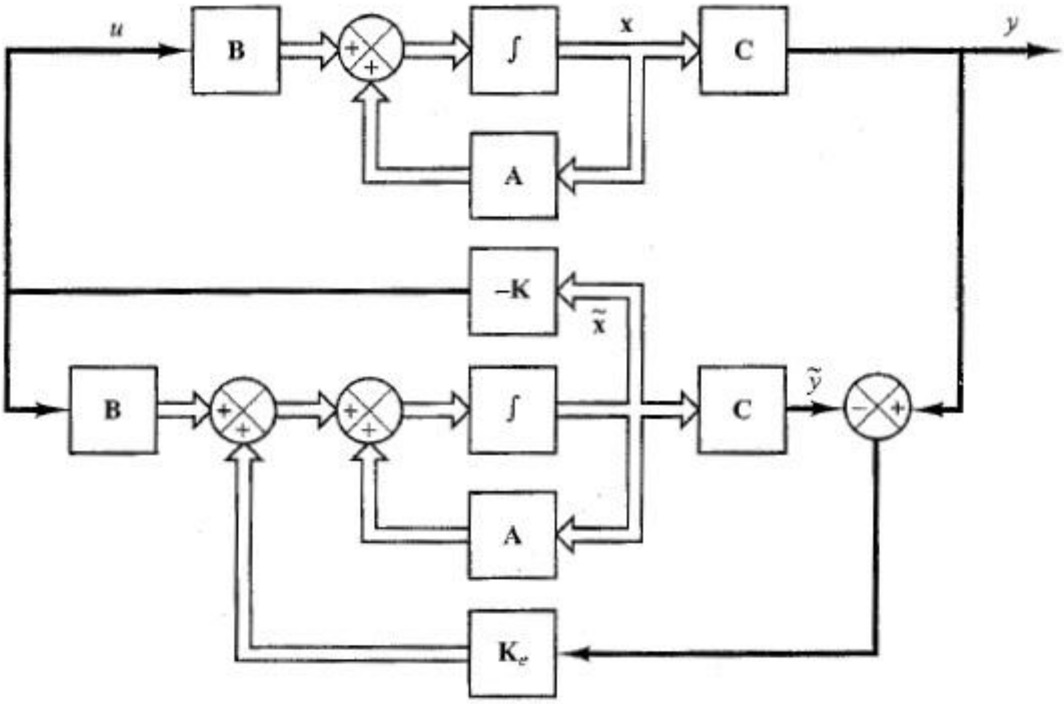


Figure 1.5 Observer state feedback control

Consider the completely state controllable and completely observable system defined by the equation:

$$\dot{x} = Ax + bu$$

$$y = cx$$

For the state feedback control based on the observed state $\tilde{x}(t)$

$$u = -k\tilde{x}$$

With this control, the state equation becomes:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} - \boldsymbol{bk}\tilde{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{bk})\boldsymbol{x} + \boldsymbol{bk}(\boldsymbol{x} - \tilde{\boldsymbol{x}}) \quad (1.23)$$

The difference between the actual state $x(t)$ and the observed state $\tilde{x}(t)$has been defined as the error $e(t)$**:**

$$\boldsymbol{e}(\boldsymbol{t}) = \boldsymbol{x}(\boldsymbol{t}) - \tilde{\boldsymbol{x}}(\boldsymbol{t})$$

So that, the equation 1.23 can be rewritten as:

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{bk})\boldsymbol{x} + \boldsymbol{bke} \quad (1.24)$$

Not that the observer error equation was given by Equation 1.20:

$$\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{k_e c})\boldsymbol{e}$$

So, we obtain that:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - bk & bk \\ 0 & A - k_e c \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \qquad (1.25)$$

Eq. 1.25 describes the dynamics of the observed-state feedback control system.

The characteristic equation for the system is:

$$\begin{vmatrix} sI - A + bk & -bk \\ 0 & sI - A + k_e c \end{vmatrix} = 0$$

or

$$|sI - A + bk||sI - A + k_e c| = 0 \qquad (1.26)$$

Notice that the closed-loop poles of the observed-state feedback control system consist of the poles due to the pole-placement design alone and the poles due to the observer design alone. This means that the pole-placement design and the observer design are independent of each other. They can be designed separately and combined to form the observed-state feedback control system. Note that, if the order of the plant is *n*, then the observer is also of nth order (if the full-order state observer is used), and the resulting characteristic equation for the entire closed-loop system becomes of order *2n***.**

1.4.2.2 Transfer Function of the observer based controller:

Assume that the plant is completely observable. The equations for the observer are given by:

$$\frac{d\tilde{x}}{dt} = (A - k_e c - bk)\tilde{x} + k_e y \qquad (1.27)$$

$$u = -k\tilde{x} \qquad (1.28)$$

By taking the Laplace transform of Equation (1.27), assuming a zero initial condition, and solving for $\tilde{X}(s)$:

$$\tilde{X}(s) = (sI - A + k_e c + bk)^{-1} k_e Y(s)$$

By substituting this $\tilde{X}(s)$ into the Laplace transform of Eq. 1.28, we obtain:

$$U(s) = -k(sI - A + k_e c + bk)^{-1} k_e Y(s)$$

Then the transfer function is given by:

$$\frac{U(s)}{Y(s)} = -k(sI - A + k_e c + bk)^{-1} k_e \qquad (1.29)$$

Figure 1.6 show the block diagram represented for the system. The transfer function ($k(sI - A + k_e c + bk)^{-1}$) acts as controller for the system.

The transfer function in eq. 1.29 is called the observer-controller transfer function.

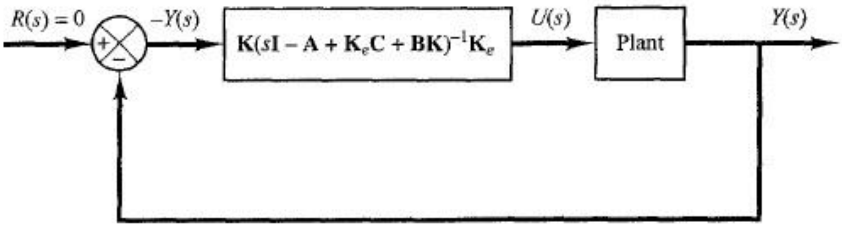


Figure 1.6 the system with controller observer

<u>Example 1.14</u> Consider the design of a regulator system for the following plant:

$$\dot{x} = Ax + bu$$

$$y = cx$$

$$A = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix}; \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Suppose that we use the pole-placement approach to the design of the system and that the desired closed-loop poles for this system are at $\mu_1 = -1.8 + j2.4$ and $\mu_2 = -1.8 - j2.4$. Design the observer controller.

Solution:

The state feedback gain matrix k for this case can be obtained as follows:

$$k = \begin{bmatrix} 29.6 & 3.6 \end{bmatrix}$$

Using this state-feedback gain matrix k, the control signal u is given by:

$$u = -kx = -\begin{bmatrix} 29.6 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Suppose that we use the observed-state feedback control instead of the actual-state feedback control, or:

$$u = -kx = -\begin{bmatrix} 29.6 & 3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

where we choose the observer poles to be at:

$$s = -8, \quad s = -8$$

Obtain the observer gain matrix k_e and draw a block diagram for the observed-state feedback control system. Then obtain the transfer function $\frac{U(s)}{-Y(s)}$ for the observer controller, and draw another block diagram with the observer controller as a series controller in the feed forward path. Finally, obtain the response of the system to the following initial condition:

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \qquad e(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

The characteristic polynomial is

$$|sI - A| = \begin{vmatrix} s & -1 \\ -20.6 & s \end{vmatrix} = s^2 - 20.6 = s^2 + a_1 s + a_2$$

$$a_1 = 0; \qquad and \qquad a_2 = -20.6$$

The desired characteristic polynomial for the observer is

$$(s+8)(s+8) = s^2 + 16s + 64 = s^2 + \alpha_1 s + \alpha_2$$

Hence,

$$\alpha_1 = 16; \qquad and \qquad \alpha_2 = 64$$

The observer gain matrix is given as:

$$k_e = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}^{-1} \begin{bmatrix} 64 + 20.6 \\ 16 - 0 \end{bmatrix} = \begin{bmatrix} 16 \\ 84.6 \end{bmatrix}$$

Since;

$$\frac{d\tilde{x}}{dt} = (A - k_e c - bk)\tilde{x} + k_e y$$

$$\begin{bmatrix} \frac{d\tilde{x}_1}{dt} \\ \frac{d\tilde{x}_2}{dt} \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} - \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} [1 \quad 0] - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [29.6 \quad 3.6] \right\} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y$$

$$= \begin{bmatrix} -16 & 1 \\ -93.6 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y; \quad \tilde{y} = [1 \quad 0] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The block diagram of the system with observed-state feedback is shown in Figure 1.7:

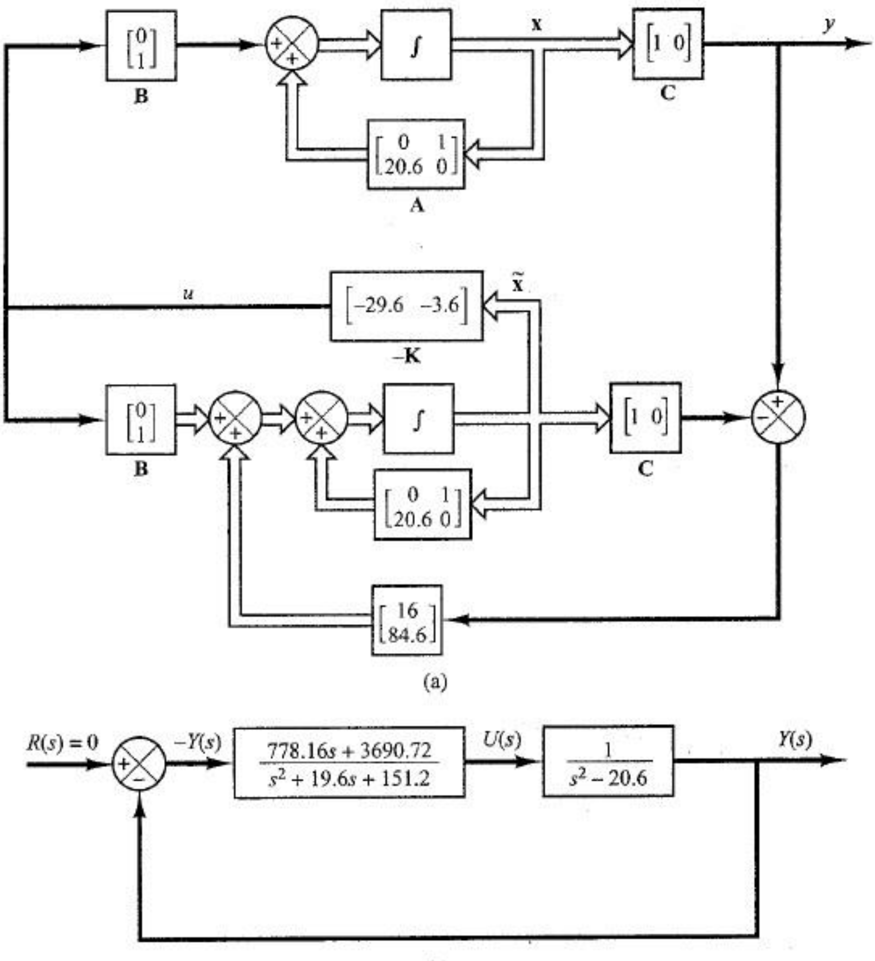


Figure 1.7 (a) block diagram of system with observed state feedback.

(b) block diagram of transfer function system.

$$\frac{U(s)}{-Y(s)} = k(sI - A + k_e c + bk)^{-1} k_e$$

$$= [29.6 \;\; 3.6] \begin{bmatrix} s+16 & -1 \\ 93.6 & s+3.6 \end{bmatrix}^{-1} \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} = \frac{778.16s + 3690.72}{s^2 + 19.6s + 151.2}$$

The dynamics of the observed-state feedback control system just designed can be described by the following equations: For the plant,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 20.6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the observer:

$$\begin{bmatrix} \frac{d\tilde{x}_1}{dt} \\ \frac{d\tilde{x}_2}{dt} \end{bmatrix} = \begin{bmatrix} -16 & 1 \\ -93.6 & -3.6 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} 16 \\ 84.6 \end{bmatrix} y$$

$$u = -[29.6 \quad 3.6] \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

The system, as a whole, is of fourth order. The characteristic equation for the system is:

$$|sI - A + bk||sI - A + k_e c| = 0$$

$$(s^2 + 3.6s + 9)(s^2 + 16s + 64) = s^4 + 19.6s^3 + 130.6s^2 + 374.4s + 576 = 0$$

The characteristic equation can also be obtained from the block diagram for the system shown in Figure 1.7b. Since the closed-loop transfer function is:

$$\frac{Y(s)}{U(s)} = \frac{778.16s + 3690.72}{(s^2 + 19.6s + 151.2)(s^2 - 20.6) + 778.16s + 3690.72}$$

1.4.3 Minimum-Order Observer:

The observers discussed thus far are designed to reconstruct all the state variables. In practice, some of the state variables may be accurately measured. Such accurately measurable state variables need not be estimated.

Suppose that the state vector $\boldsymbol{x}(t)$ is an *n*-vector and the output vector $y(t)$ is an *m*-vector that can be measured. Since *m* output variables are linear combinations of the state variables, *m* state variables need not be estimated. We need to estimate only $n - m$ state variables. Then the reduced-order observer becomes an $(n - m)^{th}$ order observer. Such an $(n - m)^{th}$ order observer is the minimum-order observer. Figure 1.8 shows the block diagram of a system with a minimum-order observer.

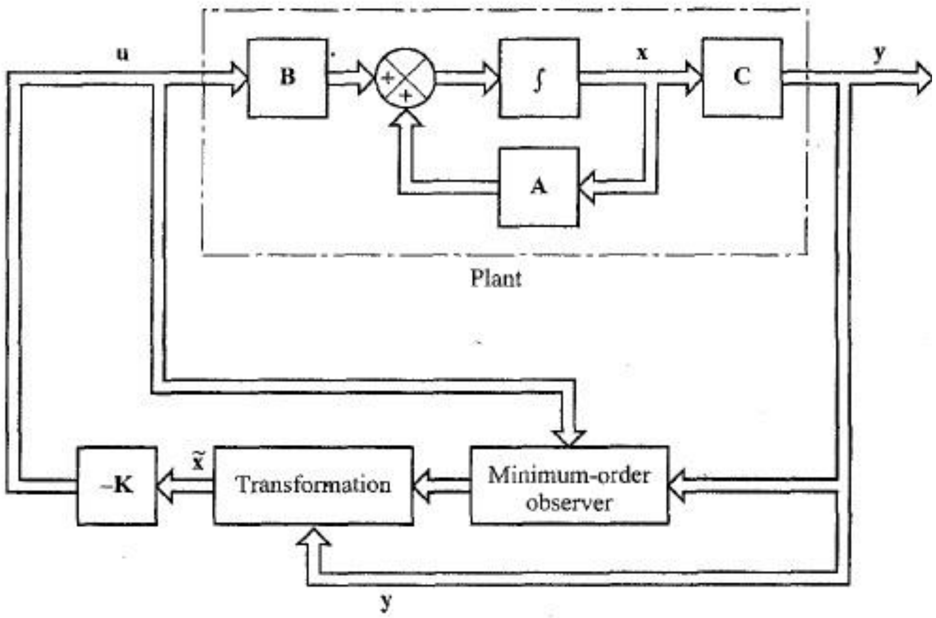


Figure 1.8 Observer state feedback control system with a minimum order observer

It is important to note, however, that if the measurement of output variables involves significant noises and is relatively inaccurate, then the use of the full-order observer may result in a better system performance.

To present the basic idea of the minimum-order observer, without undue mathematical complications, we shall present the case where the output is a scalar (that is, $m = 1$) and derive the state equation for the minimum-order observer. Consider the system:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{bx}$$

$$\boldsymbol{y} = \boldsymbol{cx}$$

where the state vector $\boldsymbol{x}$ can be partitioned into two parts x_a (a scalar) and $\boldsymbol{x_b}$ [an $(n-1)$-vector]. Here the state variable x_a is equal to the output y and thus can be directly measured, and $\boldsymbol{x_b}$ is the unmeasurable portion of the state vector. Then the partitioned state and output equations become:

$$\begin{bmatrix} \dot{x}_a \\ \dot{\boldsymbol{x}}_{\boldsymbol{b}} \end{bmatrix} = \begin{bmatrix} A_{aa} & \boldsymbol{A_{ab}} \\ \boldsymbol{A_{ba}} & \boldsymbol{A_{bb}} \end{bmatrix} \begin{bmatrix} x_a \\ \boldsymbol{x_b} \end{bmatrix} + \begin{bmatrix} B_a \\ \boldsymbol{B_b} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} x_a \\ \boldsymbol{x_b} \end{bmatrix}$$

where: A_{aa} is scalar;

$\boldsymbol{A_{ab}}$ is $1 \times (n-1)$ matrix ;

$\boldsymbol{A_{ba}}$ is $(n-1) \times 1$ matrix ;

$\boldsymbol{A_{bb}}$ is $(n-1) \times (n-1)$ matrix;

B_a is scalar;

and $\boldsymbol{B_b}$ is $(n-1)\times 1$ matrix.

Figure 1.9 shows the block diagram of the observed state feedback control system with the minimum order observer.

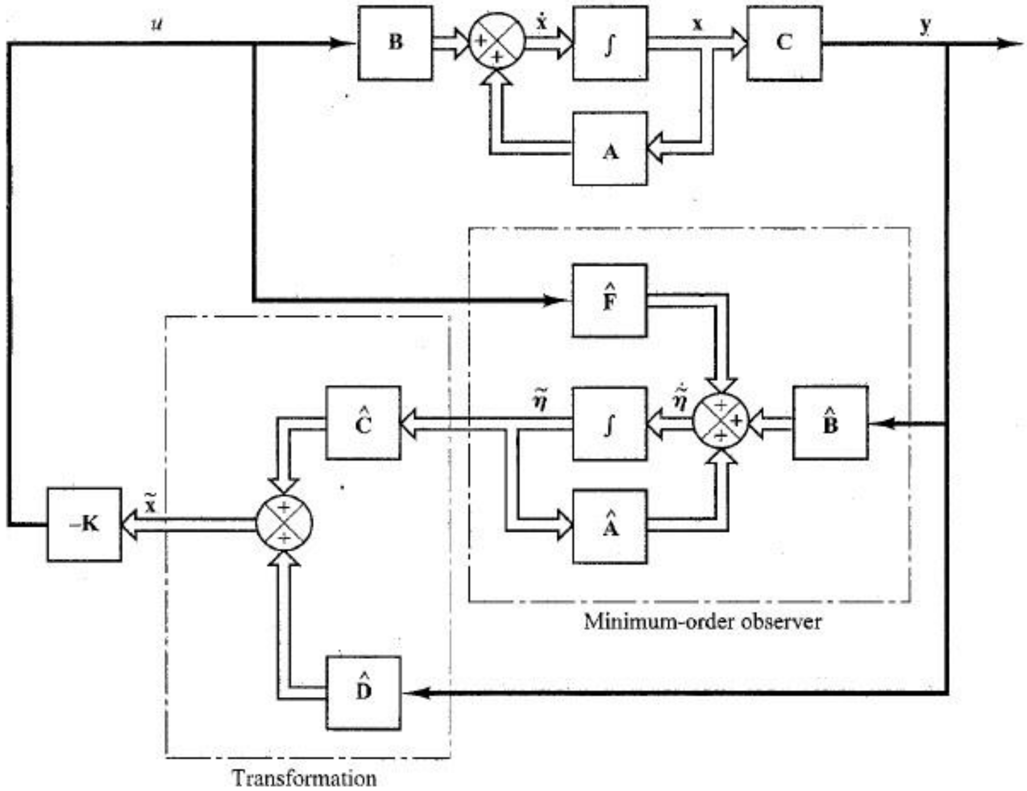


Figure 1.9 block diagram of the observed state feedback control system with the minimum order

From Figure 1.9 the state space equation of the minimum order observer is given by:

$$\frac{d\tilde{\boldsymbol{\eta}}}{dt} = \widehat{\boldsymbol{A}}\tilde{\boldsymbol{\eta}} + \widehat{\boldsymbol{B}}\boldsymbol{y} + \widehat{\boldsymbol{F}}u \qquad (1.30)$$

where: $\widehat{\boldsymbol{A}} = \boldsymbol{A_{bb}} - \boldsymbol{k_e}\boldsymbol{A_{ab}}$;

$\widehat{\boldsymbol{B}} = \widehat{\boldsymbol{A}}\boldsymbol{k_e} + \boldsymbol{A_{ba}} - \boldsymbol{k_e}A_{aa}$

$\widehat{\boldsymbol{F}} = \boldsymbol{B_b} - \boldsymbol{k_e}B_a$

$$\tilde{\boldsymbol{x}} = \widehat{\boldsymbol{C}}\tilde{\boldsymbol{\eta}} + \widehat{\boldsymbol{D}}y \qquad (1.31)$$

where: $\widehat{\boldsymbol{C}} = \begin{bmatrix} 0 \\ \boldsymbol{I_{n-1}} \end{bmatrix}$; and $\widehat{\boldsymbol{D}} = \begin{bmatrix} 1 \\ \boldsymbol{k_e} \end{bmatrix}$

$$u = -\boldsymbol{k}\tilde{\boldsymbol{x}} \qquad (1.32)$$

The characteristic equation for the minimum-order observer is obtained from Equation (1.30) as follows:

$$\left|sI - \tilde{A}\right| = |sI - A_{bb} + k_e A_{ab}| = (s-\mu_1)(s-\mu_2)\dots(s-\mu_{n-1})$$

$$= s^{n-1} + \tilde{\alpha}_1 s^{n-2} + \tilde{\alpha}_2 s^{n-3} + \cdots + \tilde{\alpha}_{n-2}s + \tilde{\alpha}_{n-1} = 0 \qquad (1.33)$$

where $\mu_1, \mu_2, \dots, \mu_{n-1}$ are desired eigenvalues for the minimum-order observer. The observer gain matrix $\boldsymbol{k_e}$ can be determined by first choosing the desired eigenvalues for the minimum-order observer [that is, by placing the roots of the characteristic equation, Equation **(**1.33**),** at the desired locations] and then using the procedure developed for the full-order observer with appropriate modifications.

The observer gain matrix $\boldsymbol{k_e}$ can be obtained using Ackermann's Formula as shown in the following equation:

$$\boldsymbol{k_e} = \boldsymbol{\phi}(\boldsymbol{A_{bb}}) \begin{bmatrix} \boldsymbol{A_{ab}} \\ \boldsymbol{A_{ab}A_{bb}} \\ \boldsymbol{A_{ab}A_{bb}^2} \\ \vdots \\ \boldsymbol{A_{ab}A_{bb}^{n-3}} \\ \boldsymbol{A_{ab}A_{bb}^{n-2}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad (1.34)$$

Where: $\boldsymbol{\phi}(\boldsymbol{A_{bb}}) = \boldsymbol{A_{bb}^{n-1}} + \tilde{\alpha}_1 \boldsymbol{A_{bb}^{n-2}} + \tilde{\alpha}_2 \boldsymbol{A_{bb}^{n-3}} + \cdots + \tilde{\alpha}_{n-2}\boldsymbol{A_{bb}} + \tilde{\alpha}_{n-1}\boldsymbol{I}$

1.4.3.1 Observed-State Feedback Control System with Minimum-Order Observer:

For the case of the observed-state feedback control system with full-order state observer, we have shown that the closed-loop poles of the observed-state feedback control system consist of the poles due to the pole-placement design alone, plus the poles due to the observer design alone. Hence, the pole-placement design and the full-order observer design are independent of each other.

For the observed-state feedback control system with minimum-order observer, the same conclusion applies. The system characteristic equation can be derived as:

$$|s\boldsymbol{I} - \boldsymbol{A} + \boldsymbol{bk}||s\boldsymbol{I} - \boldsymbol{A_{bb}} + \boldsymbol{k_e A_{ab}}| = 0 \qquad (1.35)$$

<u>Example 1.15:</u> Consider the system:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{bx}$$

$$\boldsymbol{y} = \boldsymbol{cx}$$

Where: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}; \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \qquad c = [1 \quad 0 \quad 0]$

Let us assume that we want to place the closed loop poles at:

$$s_{1,2} = -2 \mp j2\sqrt{3}; \qquad s_3 = -6$$

And assume that we choose the desired observer poles to be at:

$$s_{1,2} = -\mu_{1,2} = -10$$

Design minimum-order observer.

Solution:

Using Ackermann's Formula, the state feedback gain matrix can be obtained as follows:

$k = [90 \quad 29 \quad 4]$.

The characteristic equation for the minimum-order observer is:

$$|sI - A_{bb} + k_e A_{ab}| = (s - \mu_1)(s - \mu_2) = (s + 10)(s + 10) = s^2 + 20s + 100 = 0$$

We shall use Ackermann's formula:

$$\boldsymbol{k_e} = \boldsymbol{\phi}(\boldsymbol{A_{bb}}) \begin{bmatrix} \boldsymbol{A_{ab}} \\ \boldsymbol{A_{ab}A_{bb}} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Where; $\boldsymbol{\phi}(\boldsymbol{A_{bb}}) = \boldsymbol{A_{bb}}^2 + 20\boldsymbol{A_{bb}} + 100I$

Since:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}; \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix};$$

$$\therefore A_{aa} = 0;\; A_{ab} = [1 \quad 0]; \quad A_{ba} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}; A_{bb} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix};\; B_a = 0 \;\; and\; B_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore k_e = \left\{ \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}^2 + 20 \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} + 100 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 5 \end{bmatrix}$$

Since;

$$\frac{\boldsymbol{d\tilde{\eta}}}{\boldsymbol{dt}} = \boldsymbol{\hat{A}\tilde{\eta}} + \boldsymbol{\hat{B}y} + \boldsymbol{\hat{F}}u$$

$$\boldsymbol{\hat{A}} = \boldsymbol{A_{bb}} - \boldsymbol{k_e A_{ab}} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix};$$

$$\boldsymbol{\hat{B}} = \boldsymbol{\hat{A}k_e} + \boldsymbol{A_{ba}} - \boldsymbol{k_e} A_{aa} = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} 14 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} * 0 = \begin{bmatrix} -191 \\ -260 \end{bmatrix}$$

$$\boldsymbol{\hat{F}} = \boldsymbol{B_b} - \boldsymbol{k_e} B_a = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 14 \\ 5 \end{bmatrix} * 0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \frac{d\tilde{\eta}_2}{dt} \\ \frac{d\tilde{\eta}_3}{dt} \end{bmatrix} = \begin{bmatrix} -14 & 1 \\ -16 & -6 \end{bmatrix} \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + \begin{bmatrix} -191 \\ -260 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Where: $\begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} - k_e y$, or

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\eta}_2 \\ \tilde{\eta}_3 \end{bmatrix} + k_e x_1$$

If the observed-state feedback is used, then the control signal u becomes:

$$u = -\boldsymbol{k\tilde{x}} = -k \begin{bmatrix} x_1 \\ \tilde{x}_2 \\ \tilde{x}_2 \end{bmatrix}$$

Figure 1.10 is a block diagram showing the configuration of the system with observed-state feedback, where the observer is the minimum-order observer

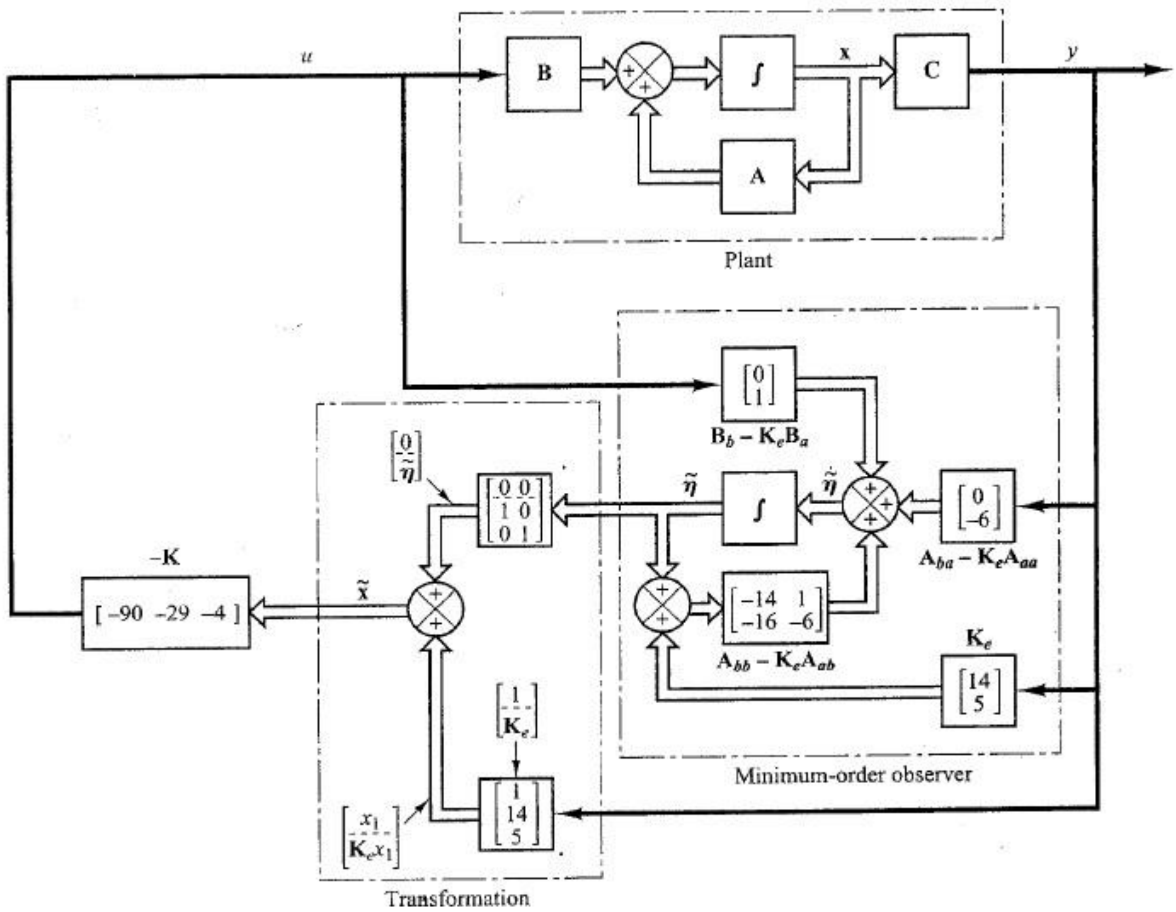


Figure 1.10 block diagram showing the configuration of the system with observed-state feedback

1.4.3.2 Transfer Function of Minimum-Order Observer Based Controller:

From Equations (1.30, 1.31 and 1.32) the transfer function of minimum-order observer is given by:

$$\frac{U(s)}{-Y(s)} = \frac{num}{den} = -\left[\widetilde{\boldsymbol{C}}\left(\boldsymbol{sI} - \widetilde{\boldsymbol{A}}\right)^{-\mathbf{1}}\widetilde{\boldsymbol{B}} + \widetilde{D}\right] \qquad (1.36)$$

Where:

$$\widetilde{\boldsymbol{A}} = \widehat{\boldsymbol{A}} - \widehat{\boldsymbol{F}}\boldsymbol{K_b}$$

$$\widetilde{\boldsymbol{B}} = \widehat{\boldsymbol{B}} - \widehat{\boldsymbol{F}}(K_a + \boldsymbol{K_b k_e})$$

$$\widetilde{\boldsymbol{C}} = -\boldsymbol{K_b}$$

$$\widetilde{D} = -(K_a + \boldsymbol{K_b k_e})$$

We defined the state feedback matrix as:

$$\boldsymbol{k} = \begin{bmatrix} K_a \\ \boldsymbol{K_b} \end{bmatrix}$$

K_a is a scalar; and $\boldsymbol{K_b}$ is $(n-1)\times 1$ matrix.

1.4.3.3 Design of Regulator Systems with Observer:

In this section we shall consider a problem of designing regulator systems by using the pole-placement-with-observer approach.

Consider the regulator system shown in Figure 1.11. (The reference input is zero).

The plant transfer function is:

$$G(s) = \frac{10(s+2)}{s(s+4)(s+6)}$$

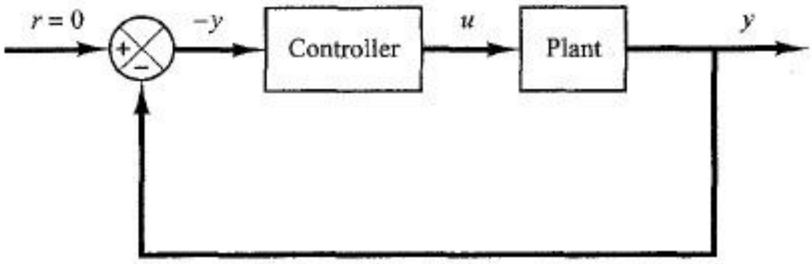


Figure 1.11 Regulator System

Using the pole placement approach, design a controller . Assume that we use the minimum-order observer. (We assume that only the output y is measurable).

We shall use the following design procedure:

1. Derive a state-space model of the plant.
2. Choose the desired closed-loop poles for pole placement. Choose the desired observer poles.
3. Determine the state feedback gain matrix $\boldsymbol{k}$ and the observer gain matrix $\boldsymbol{k_e}$.
4. Using the gain matrices $\boldsymbol{k}$ and $\boldsymbol{k_e}$ obtained in step *3*, derive the transfer function of the observer controller. it must be stable controller.

Design Step 1: the state space equation and the output equation of the given system is obtained as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -24 & -10\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 10\\ -80\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + [0]u$$

Design Step 2: assume that the desired closed loop poles at:

$s_{1,2} = -1 \mp j2, \quad s_3 = -5$

Then, the sate feedback gain matrix is given as (using Ackermann's Formula):

$$k = [1.25 \quad 1.25 \quad 0.19375]$$

<u>Design Step 3</u>: Assume that the desired closed loop poles for observer pole locations as follows: $s_{1,2} = -4.5$

Then, the observer gain matrix is given as (using Ackermann's Formula):

$$k_e = \begin{bmatrix} -1 \\ 6.25 \end{bmatrix}$$

Design Step 3: We shall determine the transfer function of the observer controller from the following equation:

$$G_c(s) = \frac{U(s)}{-Y(s)} = \frac{num}{den} = -\left[\widetilde{\boldsymbol{C}}\left(\boldsymbol{sI} - \widetilde{\boldsymbol{A}}\right)^{-\mathbf{1}}\widetilde{\boldsymbol{B}} + \widetilde{D}\right]$$

Therefore;

$$G_c(s) = \frac{1.2109s^2 + 11.2125s + 25.3125}{s^2 + 6s + 2.1406} = \frac{1.2109(s + 5.3582)(s + 3.0012)}{(s + 5.619)(s + 0.381)}$$

As seen, the poles of the controller are located at lift hand side of s-plane; it means the transfer function of the designed controller is stable.

1.5 Design of Control System with Observers:

In Section 1.4 we discussed the design of regulator systems with observers. (The systems did not have reference or command inputs.) In this section we consider the design of control systems with observers when the systems have reference inputs or command inputs.

In Section 1.4 we discussed regulator systems, whose block diagram is shown in Figure 1.11. This system has no reference input, or *r = 0*. When the system has a reference input, several different block diagram configurations are conceivable, each having an observer controller. Two of these configurations are shown in Figures 1.12 (a) and (b); we shall consider them in this section.

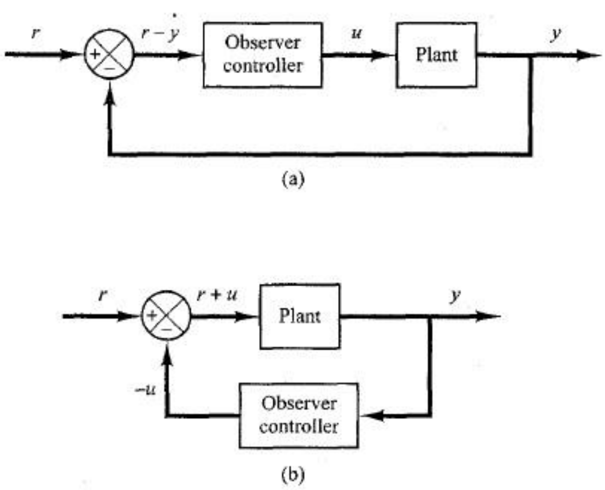


Figure 1.12 a) Control system with observer controller in the feed-forward path.

b) Control system with observer controller in the feed-back path.

Configuration 1: Consider the system shown in Figure 1.13. In this system the reference input is simply added at the summing point. We would like to design the observer controller such that in the unit-step response the maximum overshoot is less than 30% and the settling time is about 5 sec.

In what follows we first design a regulator system. Then, using the observer controller designed, we simply add the reference input ***r*** at the summing point.

Before we design the observer controller, we need to obtain a state-space representation of the plant. Since

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s^2+1)}$$

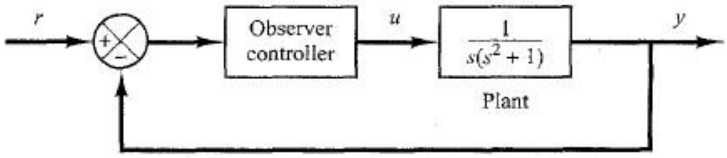


Figure 1.13 Observer Controller in the feed-forward bath

The state equation of the given system is:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u; \qquad y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Assume the desired closed loop poles for pole placement at: $s_{1,2} = -1 \mp j; \quad s_3 = -8$. And the desired observer poles at: $s_{1,2} = -4$.

Using Ackermann's Formula, the state feedback gain matrix k and the observer gain matrix k_e are given as follows:

$$k = [16 \quad 17 \quad 10]; \qquad k_e = \begin{bmatrix}8\\ 15\end{bmatrix}$$

The transfer function of the observer controller is obtained by use the following equation:

$$G_c(s) = \frac{U(s)}{-Y(s)} = \frac{num}{den} = -\left[\widetilde{\boldsymbol{C}}\left(\boldsymbol{sI} - \widetilde{\boldsymbol{A}}\right)^{-\mathbf{1}}\widetilde{\boldsymbol{B}} + \widetilde{D}\right]$$

Therefore;

$$G_c(s) = \frac{302s^2 + 303s + 256}{s^2 + 18s + 113} = \frac{302(s + 0.5017 + j0.5017)(s + 0.5017 - j0.5017)}{(s + 9 + j5.6569)(s + 9 - j5.6569)}$$

The unit-step response curve for this control system is shown in Figure 1.14. The maximum overshoot is about 28% and the settling time is about 4.5 sec. Thus, the designed system satisfies the design requirements.

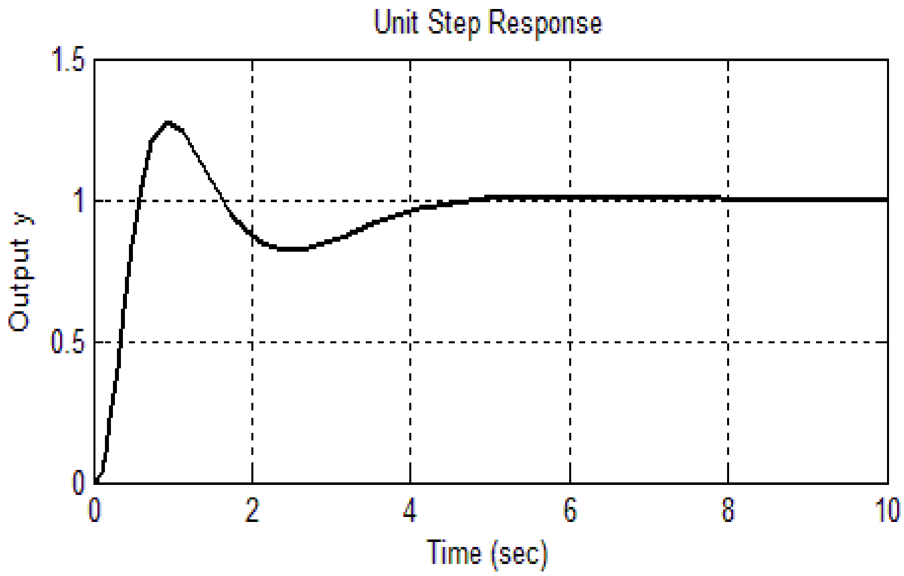


Figure 1.14 Unit step input of the controlled system

Figure 1.15 shows the system without controller:

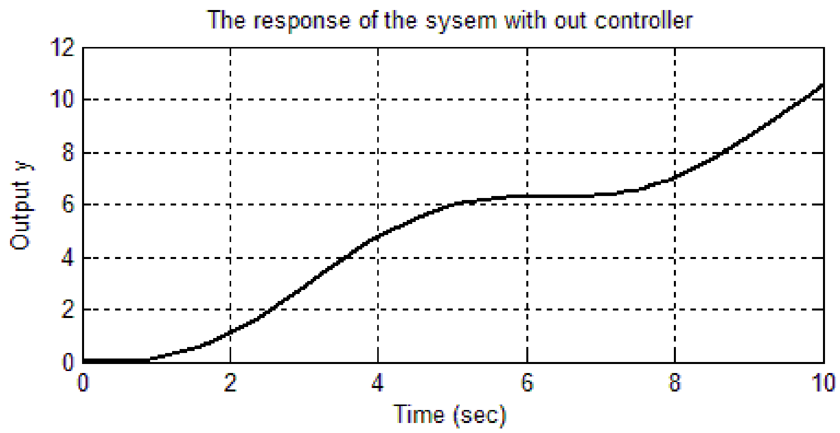


Figure 1.15 the response of the system without controller

The Simulink tools of MATLAB program has been used to plot Figures 1.14 and 1.15 as shown in Figure 1.16 below:

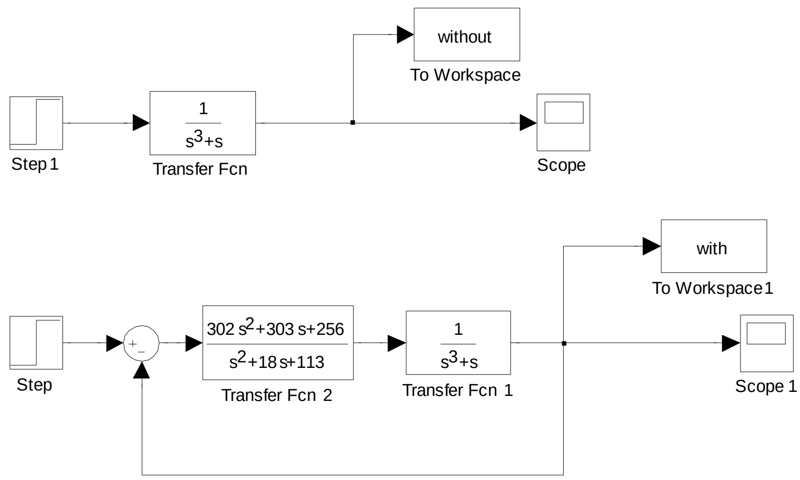


Figure 1.16

Configuration 2: *A* different configuration of the control system is shown in Figure 1.17. The observer controller is placed in the feedback path. The input *r* is introduced into the closed-loop system through the box with gain N. From this block diagram, the closed-loop transfer function is obtained as

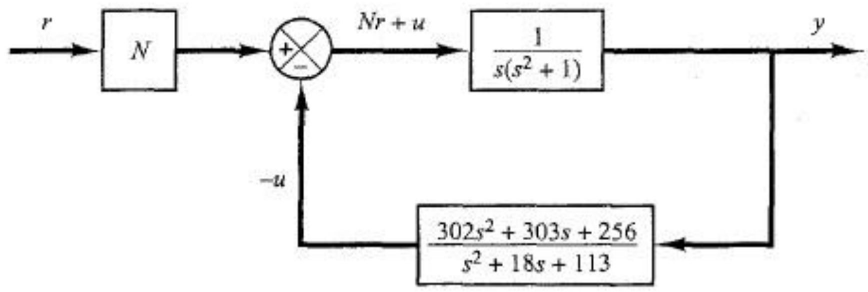


Figure 1.17 Control system with observer controller in the feedback

$$\frac{Y(s)}{R(s)} = \frac{N(s^2 + 18s + 113)}{s(s^2 + 1)(s^2 + 18s + 113) + 302s^2 + 303s + 256}$$

We determine the value of constant N such that for a unit-step input *r*, the output *y* is unity as *t* approaches infinity.

Note: if

$$G_c(s) = \frac{s^m + a_1 s^{m-1} + a_2 s^{m-2} + \cdots + a_{m-1}s + a_m}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1}s + b_n}$$

Then:

$$N = \frac{a_m}{b_n}$$

Therefore,

$$N = \frac{256}{113} = 2.2655$$

The unit-step response of the system is shown in Figure 1.18. Notice that the maximum overshoot is very small; approximately 4%. The settling time is about 5 sec.

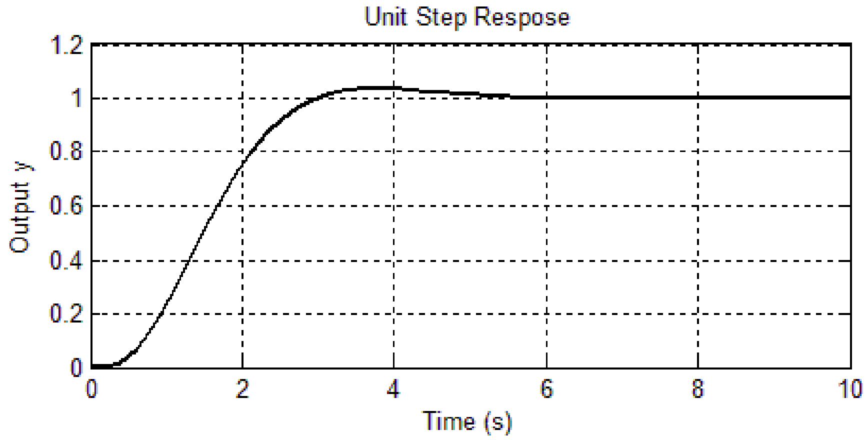


Figure 1.18 Unit step response

The Simulink tools of MATLAB program has been used to plot Figure 1.18 as shown in Figure 1.19 below:

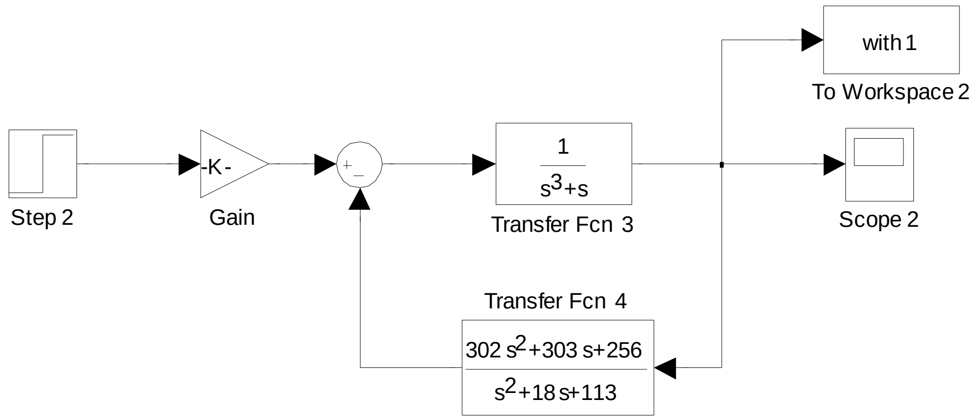


Figure 1.19

1.6 Quadratic Optimal Regulator System:

An advantage of the quadratic optimal control method over the pole-placement method is that the former provides a systematic way of computing the state feedback control gain matrix.

We shall now consider the optimal regulator problem that, given the system equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{bu} \qquad (1.37)$$

determines the matrix $\boldsymbol{k}$ of the optimal control vector:

$$u(t) = -\boldsymbol{kx(t)} \qquad (1.38)$$

so as to minimize the performance index:

$$J = \int_0^\infty (\boldsymbol{x^T Q x} + \boldsymbol{u^T R u})dt \qquad (1.39)$$

where $\boldsymbol{Q}$ is a positive-definite (or positive-semidefinite) Hermitian or real symmetric matrix and $\boldsymbol{R}$ is a positive-definite Hermitian or real symmetric matrix.

As will be seen later, the linear control law given by Equation (1.38) is the optimal control law. Therefore, if the unknown elements of the matrix $\boldsymbol{k}$ are determined so as to minimize the performance index, then $u(t) = -\boldsymbol{kx(t)}$ is optimal for any initial state **x(0).**

The design steps may be stated as follows:

1. Solve the following Equation for the matrix P. [If a positive-definite matrix **P** ($n \times n\, matrix$) exists (certain systems may not have a positive definite matrix P), the system is stable, or matrix ***A*** - ***Bk*** is stable.].

$$\boldsymbol{A^T P + PA - PBR^{-1}B^T P + Q = 0} \qquad (1.40)$$

2. Substitute this matrix **P** into the following Equation. The resulting matrix **K** is the optimal matrix,

$$\boldsymbol{k = R^{-1}B^T P} \qquad (1.41)$$

Example 1.16: Consider the following system,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

determine the optimal feedback gain matrix $\boldsymbol{k}$ such that the following performance index is minimized:

$$J = \int_0^\infty (x^T Q x + u^2)dt$$

Where

$$Q = \begin{bmatrix}1 & 0\\ 0 & \mu\end{bmatrix} \qquad (\mu \geq 0)$$

Solution:

Design Step 1: Solve Equation 1.40 to find the matrix P:

$$\boldsymbol{A^T P + PA - PBR^{-1}B^T P + Q = 0}$$

From the given equation of ***J***, **R**=[1].

$$\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}p_1 & p_2\\ p_4 & p_3\end{bmatrix} + \begin{bmatrix}p_1 & p_2\\ p_4 & p_3\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - \begin{bmatrix}p_1 & p_2\\ p_4 & p_3\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}[1][0 \quad 1]\begin{bmatrix}p_1 & p_2\\ p_4 & p_3\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & \mu\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

This equation can be simplified to:

$$\begin{bmatrix}0 & 0\\ p_1 & p_2\end{bmatrix} + \begin{bmatrix}0 & p_1\\ 0 & p_2\end{bmatrix} - \begin{bmatrix}p_2^2 & p_2 p_3\\ p_2 p_3 & p_3^2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & \mu\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

From which we obtain the following three equations:

$$1 - p_2^2 = 0$$

$$p_1 - p_2 p_3 = 0$$

$$\mu + 2p_2 - p_3^2 = 0$$

Solving these three equations, than:

$$P = \begin{bmatrix}\sqrt{\mu + 2} & 1\\ 1 & \sqrt{\mu + 2}\end{bmatrix}$$

Design Step 2: The sole equation 1.41 to obtain ***k:***

$$\boldsymbol{k} = \boldsymbol{R^{-1}B^TP} = [1][0 \quad 1]\begin{bmatrix} \sqrt{\mu+2} & 1 \\ 1 & \sqrt{\mu+2} \end{bmatrix} = [1 \quad \sqrt{\mu+2}]$$

The optimal control signal is:

$$u = -kx = -x_1 - \sqrt{\mu+2}x_2$$

Since the characteristic equation is

$$|sI - A + bk| = s^2 + \sqrt{\mu+2}\, s + 1 = 0$$

If $\mu = 1$, the two closed loop poles are located at:

$$s_{1,2} = -0.866 \mp j$$

The system is stable.

<u>Example 1.16:</u> Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

determine the optimal feedback gain matrix ***k*** such that the following performance index is minimized:

$$J = \int_0^\infty (x^TQx + u^TRu)dt$$

Where

$$Q = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R = 0.01$$

Then plot the unit step response of the given system.

Solution:

Design Step 1: Solve Equation 1.40 to find the matrix P:

$$\boldsymbol{A^TP + PA - PBR^{-1}B^TP + Q = 0}$$

$$\therefore P = \begin{bmatrix} 55.12 & 14.6711 & 1 \\ 14.6711 & 7.0267 & 0.5312 \\ 1 & 0.5312 & 0.1167 \end{bmatrix}$$

Design Step 2: The sole equation 1.41 to obtain ***k:***

$$\boldsymbol{k = R^{-1}B^TP}$$

$$\therefore k = [100 \quad 53.12 \quad 11.6711]$$

Figure 1.20 shows the closed loop system:

When the reference input r is a unit step function so the unit step response of the given system can be obtained using the following MATLAB program as shown in Figure 1.21.

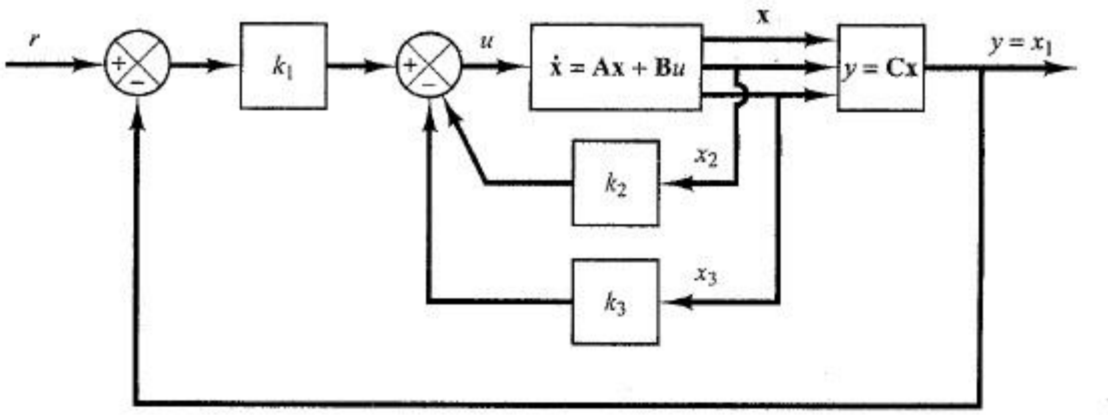


Figure 1.20 the control system

```
A=[0 1 0; 0 0 1;0 -2 -3];
B=[0;0;1];
C=[ 1 0 0];
D=[0];
K=[100 53.12 11.6711];
k1=K(1); k2=K(2); k3=K(3);
AA=A-B*K;
BB=B*k1;
CC=C;
DD=D;
t=0:0.01:8;
y=step(AA,BB,CC,DD,1,t);
plot(t,y)
```

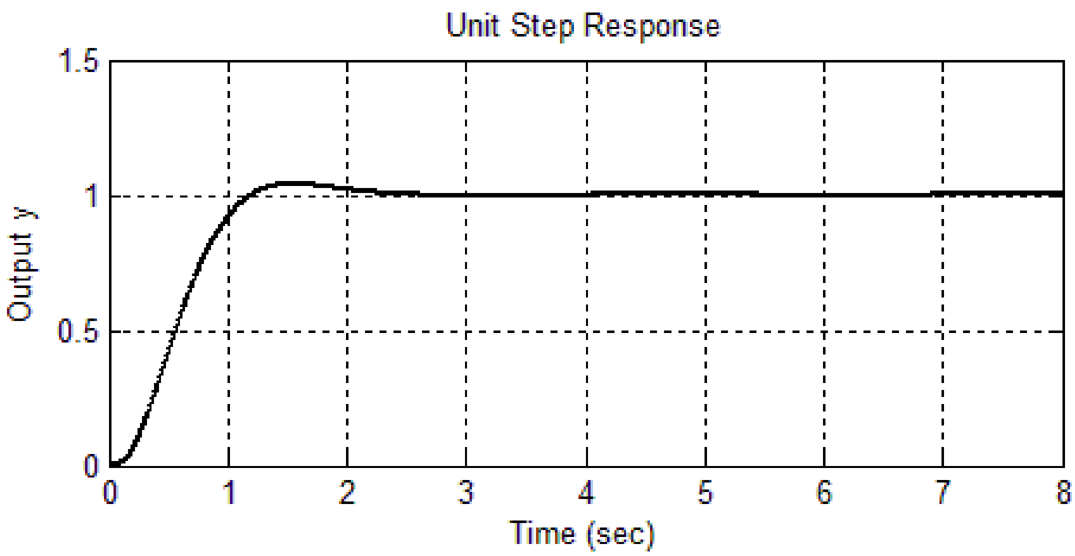


Figure 1.21 Unit step response

Chapter Two
PID Control System Design

2.1 Introduction:

It is interesting to note that more than half of the industrial controllers in use today utilize PID or modified PID control schemes.

Because most PID controllers are adjusted on-site, many different types of tuning rules have been proposed in the literature. Using these tuning rules, delicate and fine tuning of PID controllers can be made on-site. Also, automatic tuning methods have been developed and some of the PID controllers may possess on-line automatic tuning capabilities. Modified forms of PID control, such as I-PD control and two degrees of freedom PID control, are currently in use in industry.

In this chapter we first present the design of a PID controlled system. We next discuss modified PID controls such as PI-D control and I-PD control. Then we introduce two-degrees-of-freedom control systems, which can satisfy conflicting requirements that single-degree-of-freedom control systems cannot.

2.2 Tuning Rules for PID Controllers:

Figure 2.1 shows a PID control of a plant. If a mathematical model of the plant can be derived, then it is possible to apply various design techniques for determining parameters of the controller that will meet the transient and steady-state specifications of the closed-loop system. However, if the plant is so complicated that its mathematical model cannot be easily obtained, then an analytical approach to the design of a PID controller is not possible. Then we must resort to experimental approaches to the tuning of PID controllers.

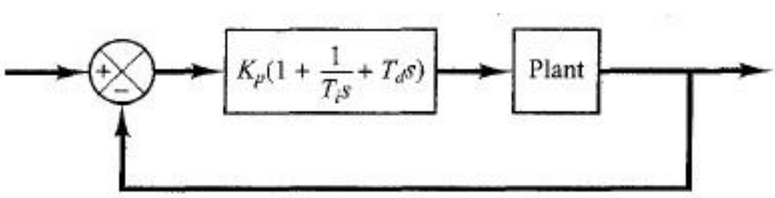


Figure 2.1 plant with PID controller

The process of selecting the controller parameters to meet given performance specifications is known as controller tuning. Ziegler and Nichols suggested rules for tuning PID controllers (meaning to set values K_p, T_i *and* T_d) based on experimental step responses or based on the value of *K*, that results in marginal stability when only proportional control action is used.

2.2.1 Ziegler-Nichols Rules for Tuning PID Controller:

Ziegler and Nichols proposed rules for determining values of the proportional gain K_p, integral time T_i, and derivative time T_d based on the transient response characteristics of a given plant. Such determination of the parameters of PID controllers or tuning of PID controllers can be made by engineers on-site by experiments on the plant.

There are two methods called Ziegler-Nichols tuning rules: the first method and the second method. We shall give a brief presentation of these two methods.

First Method

In the first method, we obtain experimentally the response of the plant to a unit-step input, as shown in Figure 2.2. If the plant involves neither integrator(s) nor dominant complex-conjugate poles, then such a unit-step response curve may look S-shaped, as shown in Figure 2.3.This method applies if the response to a step input exhibits an S-shaped curve. Such step-response curves may be generated experimentally or from a dynamic simulation of the plant.

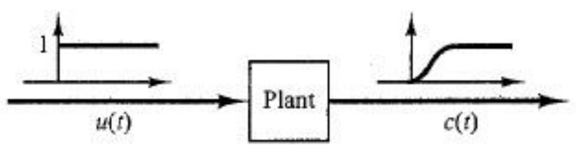


Figure 2.2 Unit step response of the plant

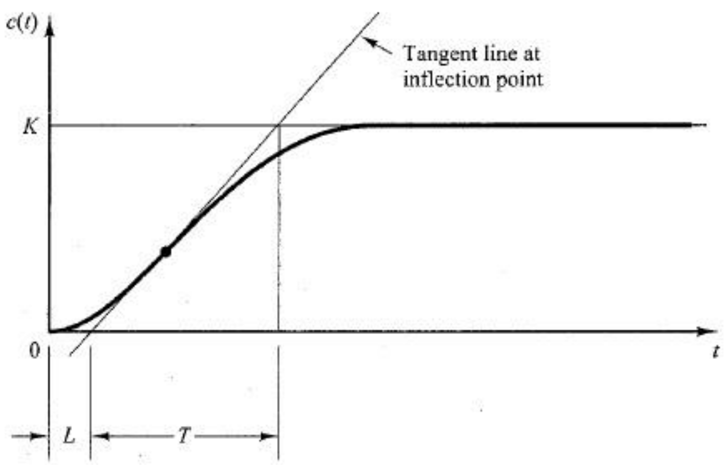


Figure 2.3 S-shape response curve

The S-shaped curve may be characterized by two constants, delay time ***L*** and time constant ***T***. The delay time and time constant are determined by drawing a tangent line at the inflection point of the S-shaped curve and determining the intersections of the tangent line with the time axis and line c (t) = K, as shown in Figure 2.3.

Ziegler and Nichols suggested setting the values of $K_p, T_i \; and \; T_d$ according to the formula shown in Table 2.1.

Type of Controller	K_p	T_i	T_d
P	$\frac{T}{L}$	∞	0
PI	$0.9\frac{T}{L}$	$\frac{L}{0.3}$	0
PID	$1.2\frac{T}{L}$	$2L$	$0.5L$

Table 2.1 Ziegler-Nichols Tuning Rule Based on Step Response of plant (First Method)

Notice that the PID controller tuned by the first method of Ziegler-Nichols rules gives:

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) \qquad (2.1)$$

Or

$$G_c(s) = 1.2\frac{T}{L}\left(1 + \frac{1}{2Ls} + 0.5Ls\right) \qquad (2.2)$$

Or

$$G_c(s) = 0.6T\frac{\left(s + \frac{1}{L}\right)^2}{s} \qquad (2.3)$$

From eq. 2.3, it is clear that the PID controller has a pole at the origin and double zeros at s = -1/L.

Second Method

In the second method, we first set $T_i = \infty$ and $T_d = 0$. Using the proportional control action only (see Figure 2.4), increase K_p from 0 to a critical value K_{cr} at which the output first exhibits sustained oscillations. (If the output does not exhibit sustained oscillations for whatever value K_p may take, then this method does not apply.) Thus, the critical gain K_{cr} and the corresponding period P_{cr} are experimentally determined (see Figure 2.5). Ziegler and Nichols suggested that we set the values of the parameters K_p, T_i and T_d according to the formula shown in Table 2.2.

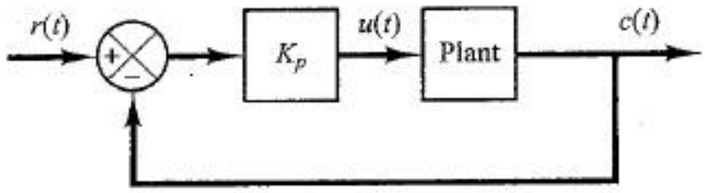


Figure 4 closed loop system with a proportional controller

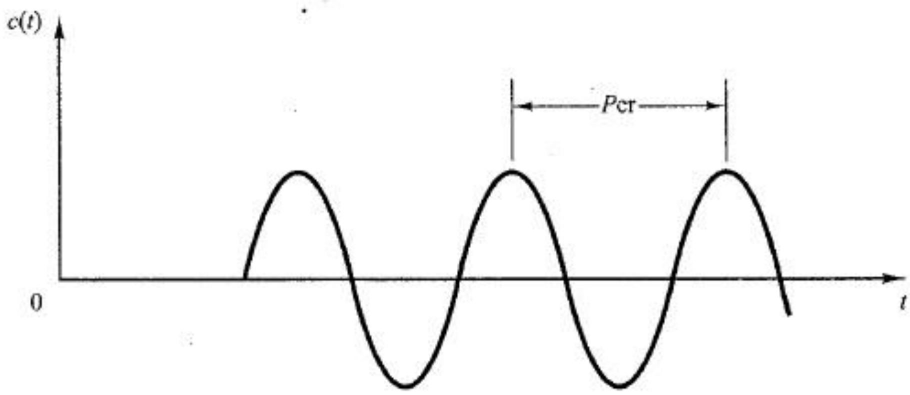


Figure 5 Sustained oscillations with period P_{cr}

Type of Controller	K_p	T_i	T_d
P	$0.5K_{cr}$	∞	0
PI	$0.45K_{cr}$	$\frac{1}{1.2}P_{cr}$	0
PID	$0.6K_{cr}$	$0.5P_{cr}$	$0.125P_{cr}$

Table 2.2 Ziegler-Nichols Tuning Rule Based on Critical gain and Critical Period (Second Method)

Notice that the PID controller tuned by the second method of Ziegler-Nichols rules gives:

$$G_c(s) = 0.6K_{cr}\left(1 + \frac{1}{0.5P_{cr}s} + 0.125P_{cr}s\right) \qquad (2.4)$$

Or

$$G_c(s) = 0.075K_{cr}P_{cr}\frac{\left(s+\frac{4}{P_{cr}}\right)^2}{s} \qquad (2.5)$$

Thus, the PID controller has a pole at the origin and double zeros at $s = -4/P_{cr}$.

Note that if the system has a known mathematical model (such as the transfer function), then we can use the root-locus method to find the critical gain K_{cr} and the frequency of the sustained oscillations ω_{cr}, where $\omega_{cr} = 2\pi/P_{cr}$. These values can be found from the crossing points of the root-locus branches with the $j\omega$axis. (Obviously, if the root-locus branches do not cross the $j\omega$ axis, this method does not apply).

Example 2.1: Consider the following system:

$$G_p(s) = \frac{1}{s(s+1)(s+5)}$$

Design a PID controller for the present system using a Ziegler-Nichols tuning rule for the determination of the values of parameters $K_p, T_i \ and \ T_d$. Then obtain a unit-step response curve and check to see if the designed system exhibits approximately 25% maximum overshoot. If the maximum overshoot is excessive (40% or more), make a fine tuning and reduce the amount of the maximum overshoot to approximately 25% or less.

Solution:

Since the plant has an integrator, we use the second method of Ziegler-Nichols tuning rules. By setting $T_i = \infty$ and $T_d = 0$, we obtain the closed-loop transfer function as follows:

$$\frac{C(s)}{R(s)} = \frac{K_p}{s(s+1)(s+5)+K_p}$$

The value of *K*, that makes the system marginally stable so that sustained oscillation occurs can be obtained by use of Routh's stability criterion. Since the characteristic equation for the closed-loop system is:

$$s^3 + 6s^2 + 5s + K_p = 0$$

The Routh array becomes as follows:

$$\begin{array}{lcc} s^3 & 1 & 5 \\ s^2 & 6 & K_p \\ s^1 & \frac{30-K_p}{6} & \\ s^0 & K_p & \end{array}$$

Examining the coefficients of the first column of the Routh table, we find that sustained oscillation will occur if $K_p = 30$. Thus, the critical gain $K_{cr} = 30$.

At $K_p = 30$, the auxiliary equation is:

$6s^2 + 30 = 0 --\rightarrow s_{1,2} = \mp j\sqrt{5}$

Therefore; from which we find the frequency of the sustained oscillation $\omega_{cr} = \sqrt{5}$. Hence, the period of sustained oscillation is:

$$P_{cr} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5}} = 2.8099$$

Referring to Table 2.*2,* we determine K_p, T_i *and* T_d as follows:

$$K_p = 0.6K_{cr} = 18$$

$$T_i = 0.5\, P_{cr} = 1.405$$

$$T_d = 0.125P_{cr} = 0.35124$$

The transfer function of the PID controller is thus:

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

$$\therefore G_c(s) = 18\left(1 + \frac{1}{1.405s} + 0.35124s\right)$$

$$\therefore G_c(s) = 6.3223\frac{(s + 1.4235)^2}{s}$$

A block diagram of the control system with the designed PID controller is shown in Figure *2.6.*

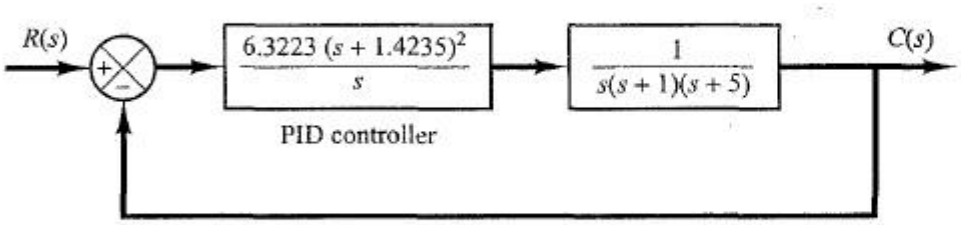


Figure 2.6 the system with designed PID controller

Next, let us examine the unit-step response of the system. The closed-loop transfer function *C(s) /R(s)* is given by:

$$\frac{C(s)}{R(s)} = \frac{6.3223s^2 + 18s + 12.811}{s^4 + 6s^3 + 11.3223s^2 + 18s + 12.811}$$

The unit-step response of this system can be obtained easily with MATLAB. See MATLAB Program below. The resulting unit-step response curve is shown in Figure 2.7. The maximum overshoot in the unit-step response is approximately 62%. The amount of maximum overshoot is excessive. It can be reduced by fine tuning the controller parameters. Such fine tuning can be made on the computer. We find that by keeping ,K_p = 18 and by moving the double zero of the PID controller to s = -0.65, that is, using the PID controller:

$$\therefore G_c(s) = 18\left(1 + \frac{1}{3.077s} + 0.7692s\right) = 13..846\frac{(s+0.65)^2}{s}$$

the maximum overshoot in the unit-step response can be reduced to approximately 18% (see Figure 2.8.

```
num=[0 0 6.3223 18 12.811];
den=[1 6 11.3223 18 12.811];
step(num,den)
```

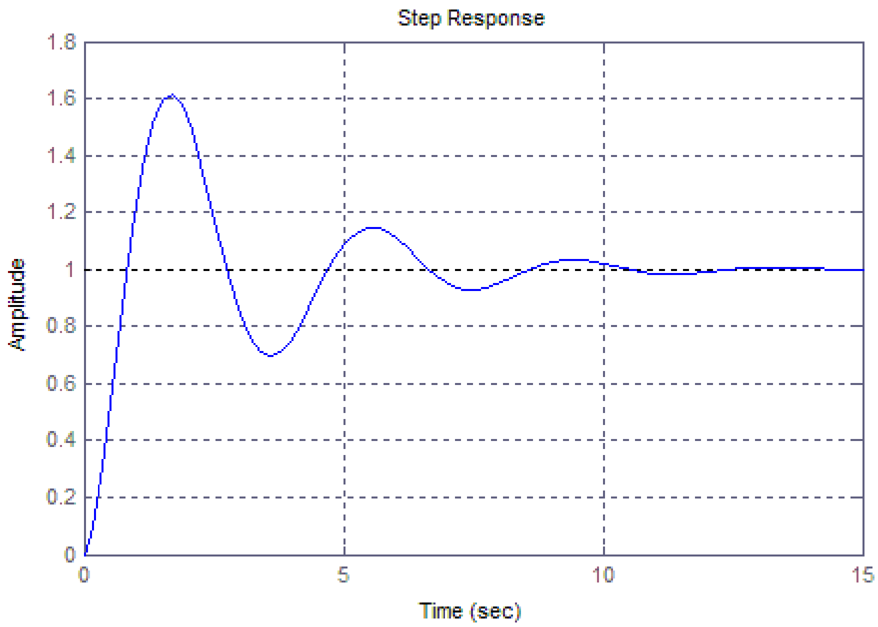


Figure 2.7 Unit step response

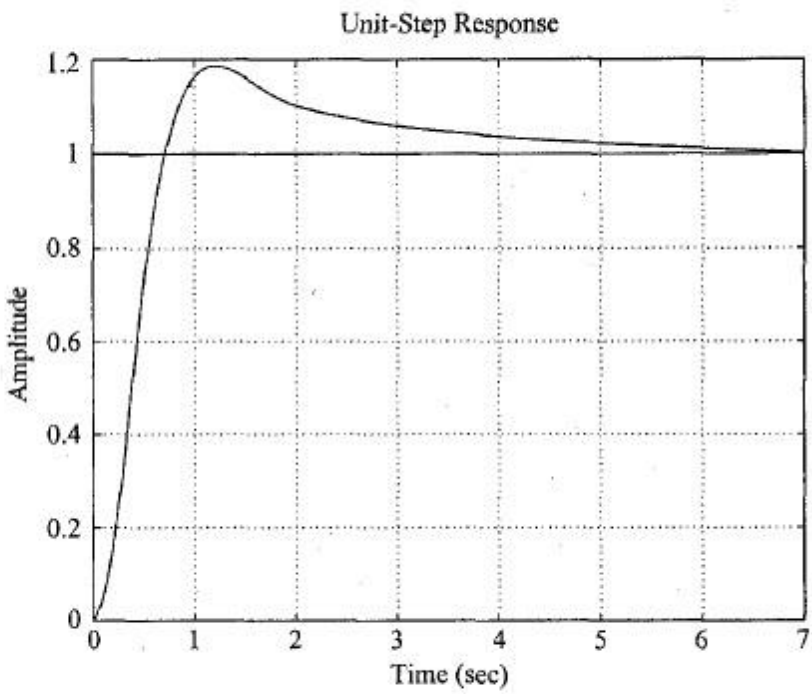


Figure 2.8 Unit step response

If the proportional gain K, is increased to 39.42, without changing the location of the double zero (s = -0.65), that is, using the PID controller:

$$\therefore G_c(s) = 39.42\left(1 + \frac{1}{3.077s} + 0.7692s\right) = 30.322\frac{(s+0.65)^2}{s}$$

then the speed of response is increased, but the maximum overshoot is also increased to approximately 28%, as shown in Figure 2.9. Since the maximum overshoot in this case is fairly close to 25%.

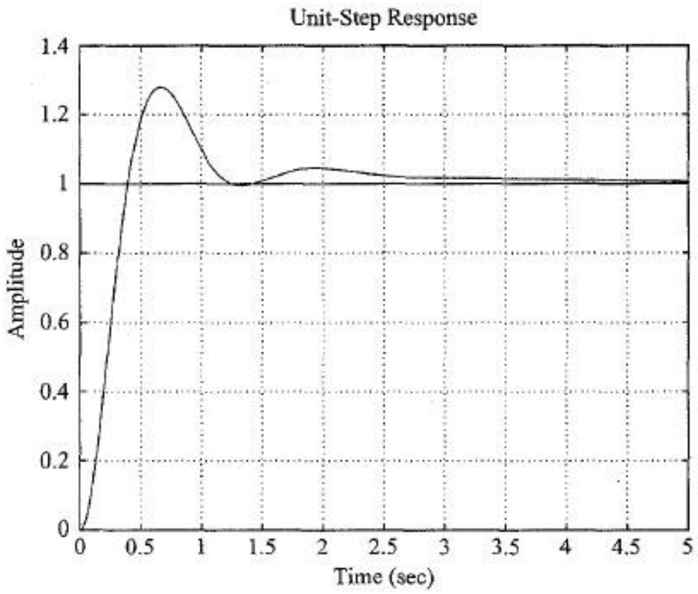


Figure 2.8 Unit step response

Then the values of the PID control parameters are: $K_p = 39.42; \quad T_i = 3.077; \quad T_d = 0.7692$
It is interesting to observe that these values respectively are approximately twice the values suggested by the second method of the Ziegler-Nichols tuning rule. The important thing to note here is that the Ziegler-Nichols tuning rule has provided a starting point for fine tuning.

Example 2.2: Consider the following system:

$$G_p(s) = \frac{10}{(s+1)(s+5)}$$

Design a PID controller for the present system using a Ziegler-Nichols tuning rule for the determination of the values of parameters K_p, T_i and T_d. Then obtain a unit-step response curve and check to see if the designed system exhibits approximately 25% maximum overshoot. If the maximum overshoot is excessive (40% or more), make a fine tuning and reduce the amount of the maximum overshoot to approximately 25% or less.

Solution:

The given system does not have integrator part or complex poles, so that the first method of Ziegler-Nichols can be used.

By using the following code in MATLAB the S-shaped response curve can be obtained as shown in Figure 2.9.

```
num=[10];
den=[1 6 5];
step(num,den)
```

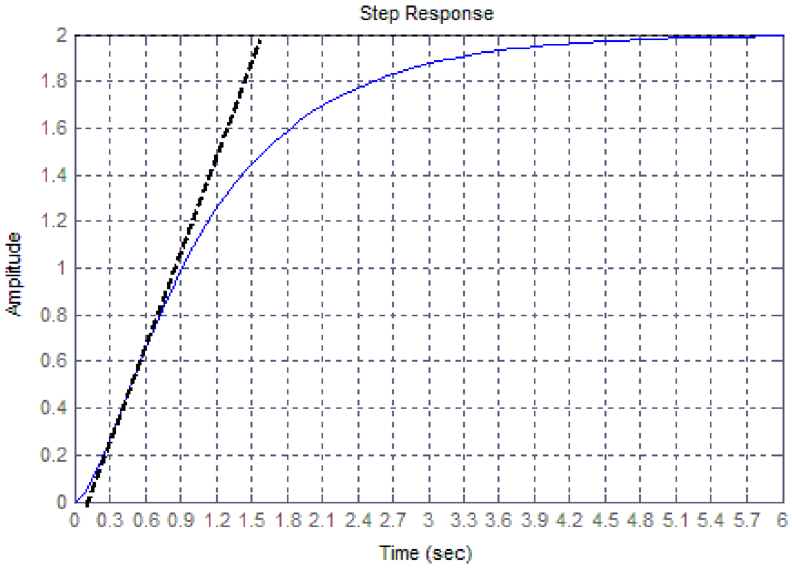


Figure 2.9 S-Shaped Response of the given system

From the figure 2.9, the values of the shape parameter are:

$$L = 0.1 \quad and \quad T = 1.5$$

From table 2.1 the parameters of PID controller can be computed as:

$$K_p = 1.2 * \frac{T}{L} = 1.2 * \frac{1.5}{0.1} = 18$$

$$T_i = 2 * L = 2 * 0.1 = 0.2$$

$$T_d = 0.5 * L = 0.5 * 0.1 = 0.05$$

The transfer function of the PID controller is given as:

$$G_c(s) = K_p(1 + \frac{1}{T_i s} + T_d s)$$

$$G_c(s) = 18\left(1 + \frac{1}{0.2s} + 0.05s\right) = 18 + \frac{3.6}{s} + 0.9s$$

We can find the unit step response of the closed loop system by using the Simulink in MATLAP as shown in Figures 2.10 and 2.11.

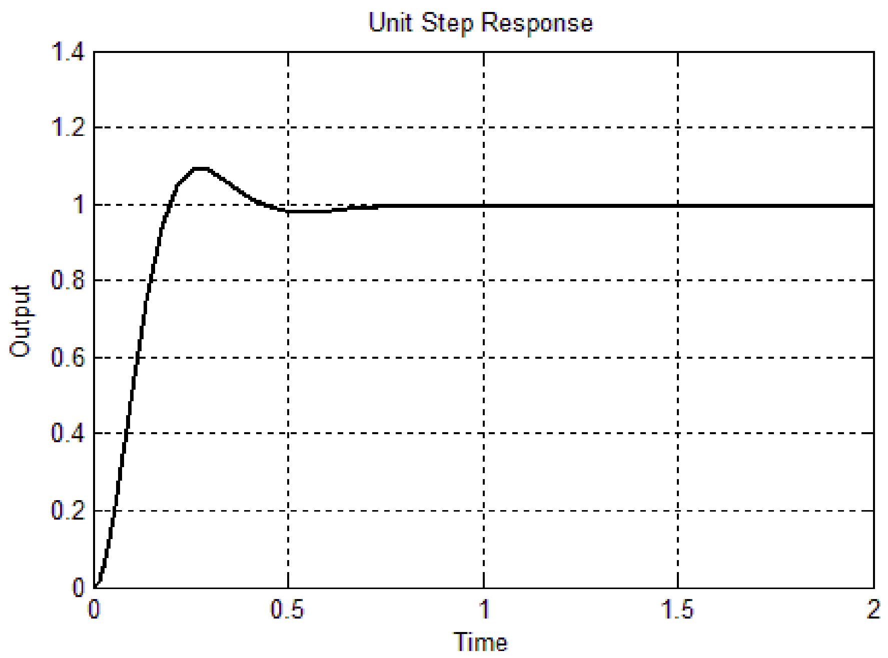


Figure 2.10 Unit Step Response of the given System

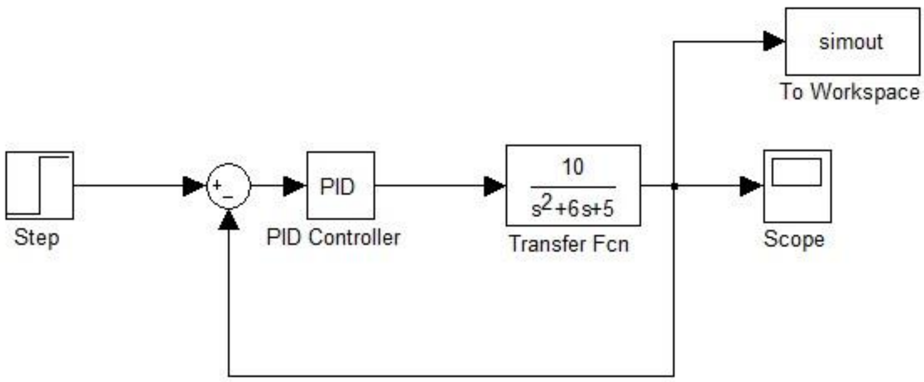


Figure 2.11 Simulink of the closed loop system

From figure 2.10 we can see that the maximum overshoot is 10%. The PID parameters do not need fine tuning.

Now we have to test if the second can be used or not.

Assume that $T_i = \infty\ and\ T_d = 0$. The characteristic equation can be written as:

$$1 + K_p G(s) = 0$$

$$1 + \frac{K_p * 10}{s^2 + 6s + 5} = 0$$

$$s^2 + 6s + (5 + 10K_p) = 0$$

Using Routh Method:

$$\begin{array}{lll} s^2 & 1 & 5 + 10K_p \\ s^1 & 6 & 0 \\ s^0 & 5 + 10K_p & 0 \end{array}$$

$$\therefore K_p = -0.5$$

Since, $K_p = -0.5$ (negative) the second method cannot be used.

Example 2.3: Consider the following system:

$$G_p(s) = \frac{(s+2)(s+3)}{s(s+1)(s+5)}$$

Prove that neither first method nor the second method of Zeigler-Nichols Formula can be used to design the PID controller for the given system?

Solution:

Because of the presence of an integrator, the first method does not apply. Also, if the second method is attempted, the closed loop system with proportional controller will not exhibit sustained oscillation wherever value the gain K_p may take. This can be seen from the following analysis. Since the characteristic equation is:

$$s(s+1)(s+5) + K_p(s+2)(s+3) = 0$$

or

$$s^3 + (6 + K_p)s^2 + (5 + 5K_p)s + 6K_p = 0$$

The Routh array becomes:

$$\begin{array}{lll} s^3 & 1 & 5 + 5K_p \\ s^2 & 6 + K_p & 6K_p \\ s^1 & \dfrac{30 + 29K_p + 5K_p^2}{6 + K_p} & 0 \\ s^0 & 6K_p & \end{array}$$

The coefficients in the first column are positive for all values of positive K_p. Thus, in the present case the closed loop system will not exhibit sustained oscillations and therefore, the critical gain value K_{cr} does not exist. Hence, the second method does not apply.

<u>Example 2.3:</u> Consider the electronic PID controller shown in Figure 2.12. Determine the values of R_1, R_2, R_3, R_4, C_l and of the controller such that the transfer function $G_c(s)$, where:

$$G_c(s) = 39.42\left(1 + \frac{1}{3.077s} + 0.7692s\right) = 30.3215\left(\frac{(s + 0.65)^2}{s}\right)$$

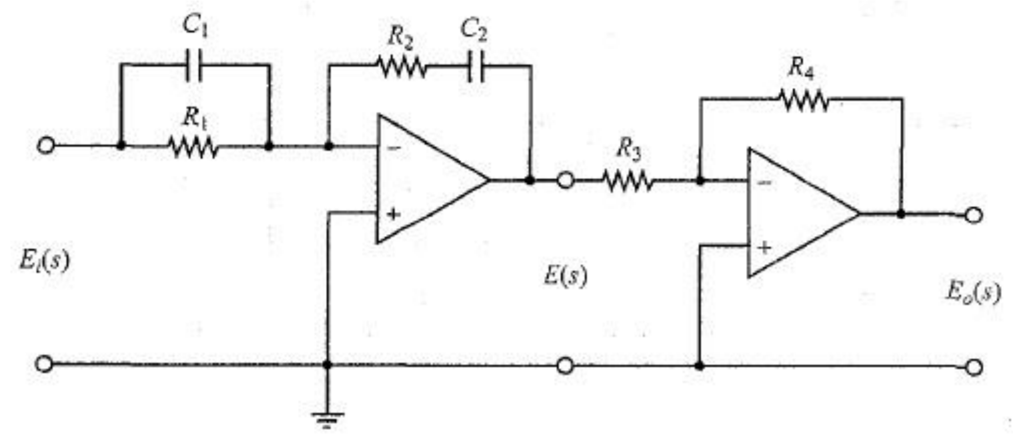


Figure 2.12 Electronic circuit of PID Controller

Solution:

$$\frac{E(s)}{E_i(s)} = -\frac{Z_2}{Z_1}$$

$$Z_2 = R_2 + \frac{1}{C_2 s} = \frac{1}{C_2 s}(1 + R_2 C_2 s)$$

$$Z_1 = \frac{R_1 * \frac{1}{C_1 s}}{R_1 + \frac{1}{C_1 s}} = \frac{R_1}{(R_1 C_1 s + 1)}$$

$$\therefore \frac{E(s)}{E_i(s)} = \frac{(1 + R_2 C_2 s)(R_1 C_1 s + 1)}{C_2 R_1 s} = -C_1 R_2 * \frac{(s + \frac{1}{C_2 R_2})(s + \frac{1}{C_1 R_1})}{s}$$

$$\frac{E_0(s)}{E(s)} = -\frac{R_4}{R_3}$$

Since:

$$\frac{E_0(s)}{E_i(s)} = \frac{E_0(s)}{E(s)} * \frac{E(s)}{E_i(s)}$$

$$\therefore \frac{E_0(s)}{E_i(s)} = \frac{C_1R_4R_2}{R_3} * \frac{(s + \frac{1}{C_2R_2})(s + \frac{1}{C_1R_1})}{s}$$

The $G_c(s)$ can be rewritten as:

$$G_c(s) = 30.3215\left(\frac{(s + 0.65)^2}{s}\right) \qquad (I)$$

If we assume that $\frac{1}{C_1R_1} = \frac{1}{C_2R_2}$, then

$$G_c(s) = \frac{E_0(s)}{E_i(s)} = \frac{C_1R_4R_2}{R_3} * \frac{\left(s + \frac{1}{C_1R_1}\right)^2}{s} \qquad (II)$$

Compare eq. (I) and eq. (II), we obtain that:

$$\frac{1}{C_1R_1} = \frac{1}{C_2R_2} = 0.65$$

$$\frac{C_1R_4R_2}{R_3} = 30.3215$$

If we assume that $R_1 = 100k\Omega$, than $C_1 = 1.53 \times 10^{-5}F$.

If we assume that $R_2 = 200k\Omega$, than $C_2 = 7.69 \times 10^{-6}F$.

And

$$\frac{R_4}{R_3} = 9.91$$

If we assume that $R_3 = 100k\Omega$, than $R_4 = 990k\Omega$

2.3 Modifications of PID Control Schemes:

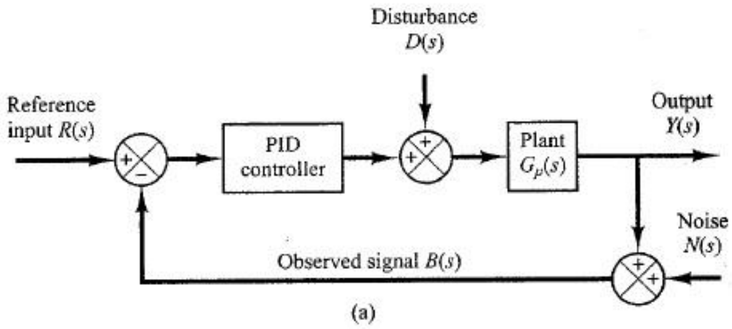
Consider the basic PID control system shown in Figure 2.13(a), where the system is subjected to disturbances and noises. Figure 2.13(b) is a modified block diagram of the same system. In the basic PID control system such as the one shown in Figure 2.13(b), if the reference input is a step function, then, because of the presence of the derivative term in the control action, the manipulated variable *u(t)* will involve an impulse function (delta function). In an actual PID controller, instead of the pure derivative term $T_d s$ we employ:

$$\frac{T_d s}{1 + \gamma T_d s}$$

where the value of γ is somewhere around 0.1. Therefore, when the reference input is a step function, the manipulated variable *u(t)* will not involve an impulse function, but will involve a sharp pulse function. Such a phenomenon is called *set-point kick.*

2.3.1 PI-D Controller:

To avoid the set-point kick phenomenon, we may wish to operate the derivative action only in the feedback path so that differentiation occurs only on the feedback signal and not on the reference signal. The control scheme arranged in this way is called the PI-D control. Figure 2.14 shows a PI-D-controlled system.



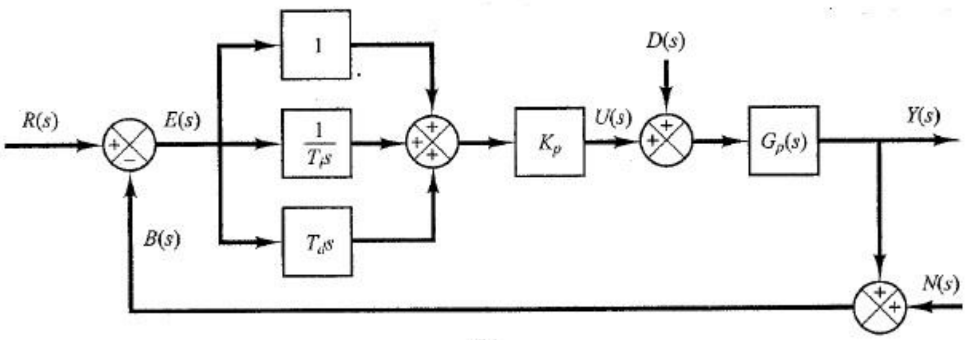


Figure 13 a) PID controller system

b) Equivalent block diagram.

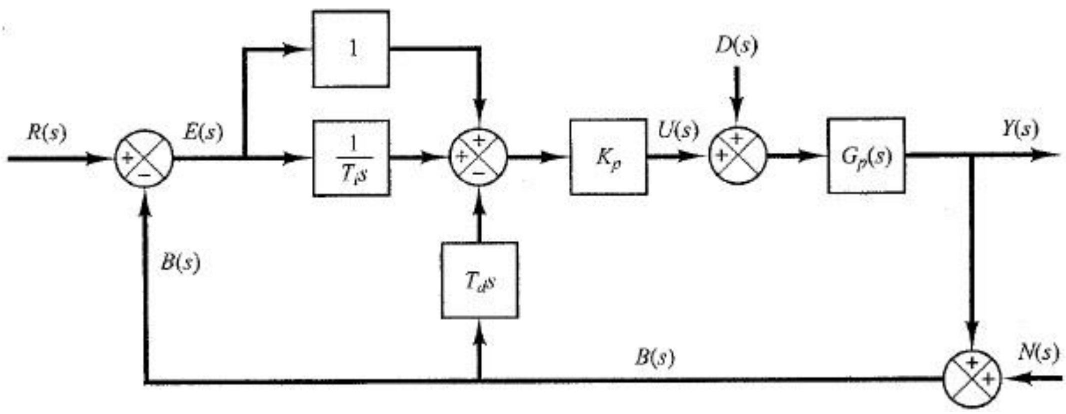


Figure 2.14 PI-D controller System

From Figure 2.14, it can be seen that the manipulated signal *U(s)* is given by:

$$U(s) = K_p\left(1 + \frac{1}{T_i s}\right)R(s) - K_p\left(1 + \frac{1}{T_i s} + T_d s\right)B(s) \qquad (2.6)$$

Notice that in the absence of the disturbances and noises, the closed-loop transfer function of the basic PID control system [shown in Figure 2.13(b)] and the PI-D control system (shown in Figure 2.14) are given, respectively, by:

$$\frac{Y(s)}{R(s)} = \left(1 + \frac{1}{T_i s} + T_d s\right)\frac{K_p G_p(s)}{1 + \left(1 + \frac{1}{T_i s} + T_d s\right)K_p G_p(s)} \qquad (2.7)$$

$$\frac{Y(s)}{R(s)} = \left(1 + \frac{1}{T_i s}\right)\frac{K_p G_p(s)}{1 + \left(1 + \frac{1}{T_i s} + T_d s\right)K_p G_p(s)} \qquad (2.8)$$

It is important to point out that in the absence of the reference input and noises, the closed-loop transfer function between the disturbance *D(s)* and the output *Y (s)* in either case is the same and is given by;

$$\frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + \left(1 + \frac{1}{T_i s} + T_d s\right)K_p G_p(s)} \qquad (2.9)$$

2.3.2 I-PD Control:

Consider again the case where the reference input is a step function. Both PID control and PI-D control involve a step function in the manipulated signal. Such a step change in the manipulated signal may not be desirable in many occasions. Therefore, it may be advantageous to move the proportional action and derivative action to the feedback path so that these actions affect the feedback signal only. Figure 2.15 shows such a control scheme. It is called the I-PD control. The manipulated signal is given by:

$$U(s) = \left(\frac{K_p}{T_i s}\right)R(s) - K_p\left(1 + \frac{1}{T_i s} + T_d s\right)B(s) \qquad (2.10)$$

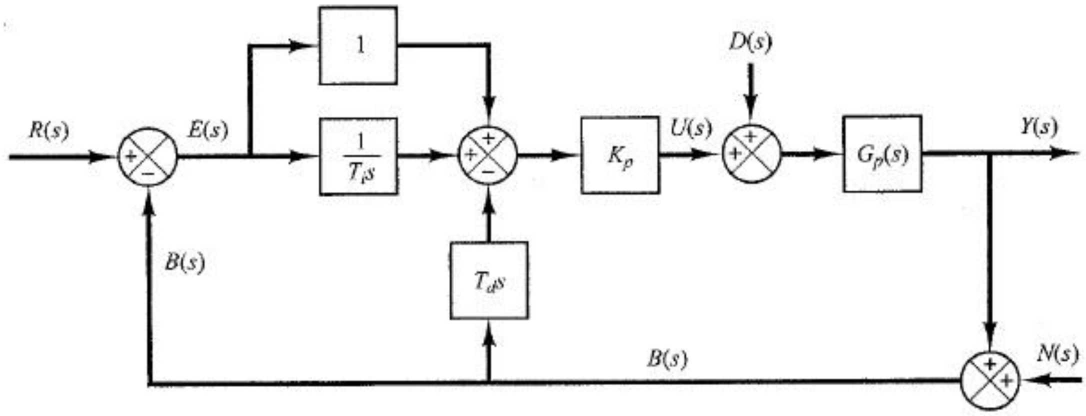


Figure 2.15 I-PD Control System

Notice that the reference input *R(s)* appears only in the integral control part. Thus, in I-PD control, it is imperative to have the integral control action for proper operation of the control system.

The closed-loop transfer function $\frac{Y(s)}{R(s)}$ in the absence of the disturbance input and noise input is given by:

$$\frac{Y(s)}{R(s)} = \left(\frac{1}{T_i s}\right) \frac{K_p G_p(s)}{1 + \left(1 + \frac{1}{T_i s} + T_d s\right) K_p G_p(s)} \quad (2.11)$$

It is noted that in the absence of the reference input and noise signals, the closed-loop transfer function between the disturbance input and the output is given by

$$\frac{Y(s)}{D(s)} = \frac{G_p(s)}{1 + \left(1 + \frac{1}{T_i s} + T_d s\right) K_p G_p(s)} \quad (2.12)$$

Chapter Three
“Control Systems Design by Root-Locus Method”

3.1 Introduction:

The primary objective of this chapter is to present procedures for the design and compensation of single-input-single-output linear time-invariant control systems. Compensation is the modification of the system dynamics to satisfy the given specifications. The approach to the control system design and compensation used in this chapter is the root-locus approach.

Control systems are designed to perform specific tasks. The requirements imposed on the control system are usually spelled out as performance specifications. The specifications may be given in terms of transient response requirements (such as the maximum overshoot and settling time in step response) and of steady-state requirements (such as steady-state error in following ramp input).

The design by the root-locus method is based on reshaping the root locus of the system by adding poles and zeros to the system's open loop transfer function and forcing the root loci to pass through desired closed-loop poles in the s plane. The characteristic of the root-locus design is its being based on the assumption that the closed-loop system has a pair of dominant closed-loop poles.

Setting the gain is the first step in adjusting the system for satisfactory performance. In many practical cases, however, the adjustment of the gain alone may not provide sufficient alteration of the system behaviour to meet the given specifications. As is frequently the case, increasing the gain value will improve the steady-state behaviour but will result in poor stability or even instability. It is then necessary to redesign the system (by modifying the structure or by incorporating additional devices or components) to alter the overall behaviour so that the system will behave as desired. Such a redesign or addition of a suitable device is called compensation. A device inserted into the system for the purpose of satisfying the specifications is called a compensator. The compensator compensates for deficit performance of the original system.

Figures 7.l (a) and (b) show compensation schemes commonly used for feedback control systems. Figure 7.l (a) shows the configuration where the compensator Gc(s) is placed in series with the plant. This scheme is called series compensation.

An alternative to series compensation is to feed back the signal(s) from some element (s) and place a compensator in the resulting inner feedback path, as shown in Figure 7.l (b). Such compensation is called parallel compensation or feedback compensation. In this chapter we discuss series compensation in detail.

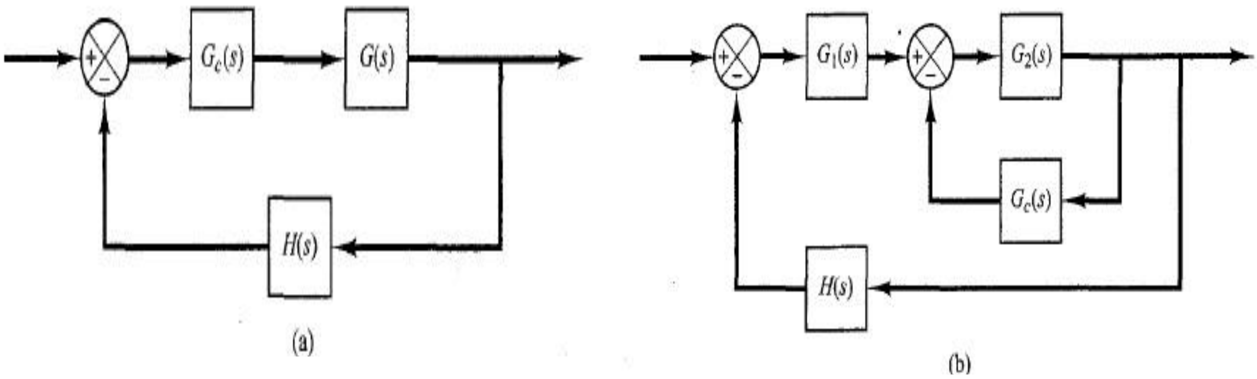


Figure 3.1 (a) Series Compensation.

(b) Parallel Compensation.

3.2 Root-Locus Approach to Control System Design:

The root-locus method is a graphical method for determining the locations of all closed-loop poles from knowledge of the locations of the open-loop poles and zeros as some parameter (usually the gain) is varied from zero to infinity. The method yields a clear indication of the effects of parameter adjustment.

In practice, the root-locus plot of a system may indicate that the desired performance cannot be achieved just by the adjustment of gain. In fact, in some cases, the system may not be stable for all values of gain. Then it is necessary to reshape the root loci to meet the performance specifications.

In designing a control system, if other than a gain adjustment is required, we must modify the original root loci by inserting a suitable compensator. Once the effects on the root locus of the addition of poles and/or zeros are fully understood, we can readily determine the locations of the pole(s) and zero(s) of the compensator that will reshape the root locus as desired. In essence, in the design by the root-locus method, the root loci of the system are reshaped through the use of a compensator so that a pair of dominant closed-loop poles can be placed at the desired location. (Often, the damping ratio and undamped natural frequency of a pair of dominant closed-loop poles are specified).

3.2.1 Effects of the Addition of Poles

The addition of a pole to the open-loop transfer function has the effect of pulling the root locus to the right, tending to lower the system's relative stability and to slow down the settling of the response. (Remember that the addition of integral control adds a pole at the origin, thus making the system less stable). Figure 3.2 shows examples of root loci illustrating the effects of the addition of a pole to a single-pole system and the addition of two poles to a single-pole system.

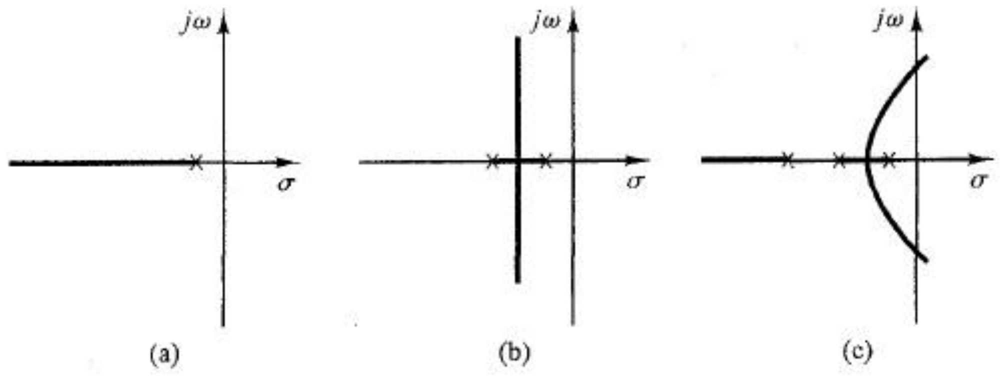


Figure 3.2 (a) Root-locus plot of a single-pole system;
(b) root-locus plot of a two-pole system;
(c) root-locus plot of a three-pole system.

3.2.2 Effects of the Addition of Zeros

The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to speed up the settling of the response. (Physically, the addition of a zero in the feed forward transfer function means the addition of derivative control to the system. The effect of such control is to introduce a degree of anticipation into the system and speed up the transient response.) Figure 3.3 (a) shows the root loci for a system that is stable for small gain but unstable for large gain. Figures 3.3 (b), (c), and (d) show root-locus plots for the system when a zero is added to the open-loop transfer function. Notice that when a zero is added to the system of Figure 3.3 (a), it becomes stable for all values of gain.

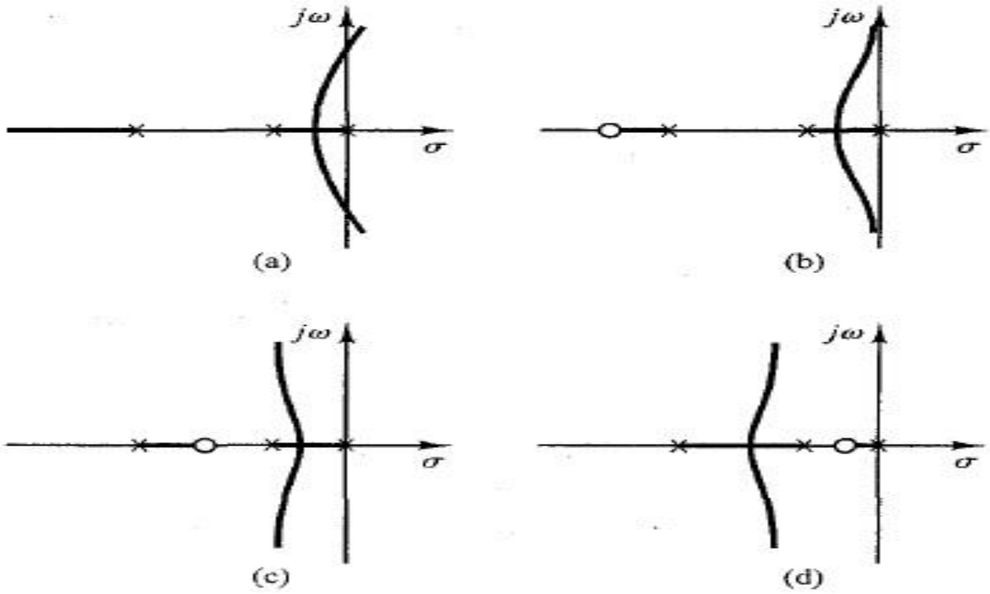


Figure 3.3 (a) Root-locus plot of a three-pole system; (b), (c), and (d) root-locus plots showing effects of addition of a zero to the three-pole system.

3.3 Lead Compensation

There are many ways to realize continuous-time (or analogue) lead compensators, such as electronic networks using operational amplifiers, electrical RC networks, and mechanical spring-dashpot systems.

Figure 3-4 shows an electronic circuit using operational amplifiers. The transfer function for this circuit was obtained as follows:

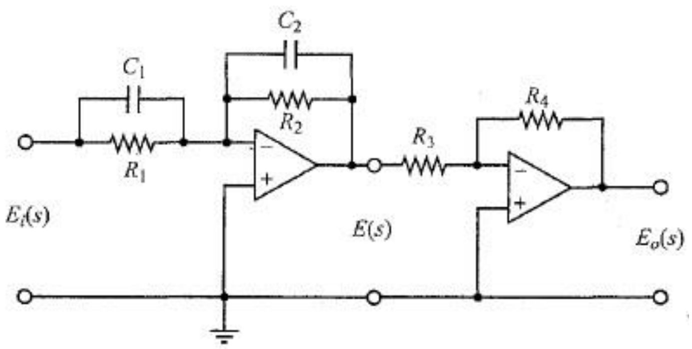


Figure 3.4 Electrical circuit of Lead Compensation

$$\frac{E_0(s)}{E_i(s)} = \frac{R_2R_4(R_1C_1s+1)}{R_1R_3(R_2C_2s+1)} = \frac{R_4C_1}{R_3C_2}\frac{s+\frac{1}{R_1C_1}}{s+\frac{1}{R_2C_2}} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \quad (3.1)$$

Where:

$$T = R_1C_1, \quad \alpha T = R_2C_2, \quad K_c = \frac{R_4C_1}{R_3C_2}$$

From Equation (7-I), we see that this network is a lead network if $R_1C_1 > R_2C_2$, or $\alpha < 1$. It is a lag network if $R_1C_1 < R_2C_2$. The pole-zero configurations of this network when $R_1C_1 > R_2C_2$ and $R_1C_1 < R_2C_2$ are shown in Figure 3-5(a) and (b), respectively.

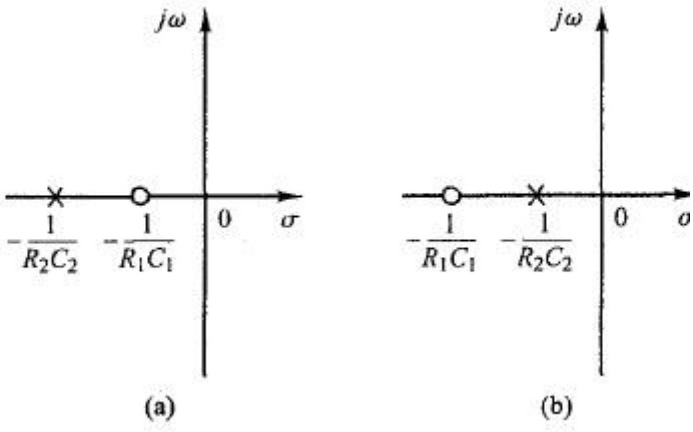


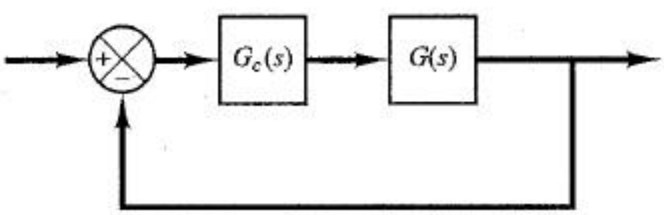
Figure 3.5: Zero pole configuration of a) Lead Compensator b) Lag Compensator

3.3.1 Lead Compensation Techniques Based on the Root-Locus Approach.

The root-locus approach to design is very powerful when the specifications are given in terms of time-domain quantities, such as the damping ratio and undamped natural frequency of the desired dominant closed-loop poles, maximum overshoot, rise time, and settling time.

Consider a design problem in which the original system either is unstable for all values of gain or is stable but has undesirable transient-response characteristics. In such a case, the reshaping of the root locus is necessary in the broad neighbourhood of the jw axis and the origin in order that the dominant closed-loop poles be at desired locations in the complex plane. This problem may be solved by inserting an appropriate lead compensator in cascade with the feed forward transfer function.

The procedures for designing a lead compensator for the system shown in Figure 3.6 by the root-locus method may be stated as follows:



1. From the performance specifications, determine the desired location for the dominant closed-loop poles.

2. By drawing the root-locus plot of the uncompensated system (original system), ascertain whether or not the gain adjustment alone can yield the desired closed loop poles. If not, calculate the angle deficiency 4.This angle must be contributed by the lead compensator if the new root locus is to pass through the desired locations for the dominant closed-loop poles.

3. Assume the lead compensator $G_c(s)$ to be:

$$G_c(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \qquad (0 < \alpha < 1)$$

where α and T are determined from the angle deficiency. Kc is determined from the requirement of the open-loop gain.

4. If static error constants are not specified, determine the location of the pole and zero of the lead compensator so that the lead compensator will contribute the necessary angle$\boldsymbol{\phi}$**.** If no other requirements are imposed on the system, try to make the value of α as large as possible. A larger value of α generally results in a larger value of Kv which is desirable. (If a particular static error constant is specified, it is generally simpler to use the frequency-response approach.)

5. Determine the open-loop gain of the compensated system from the magnitude condition.

Once a compensator has been designed, check to see whether all performance specifications have been met. If the compensated system does not meet the performance specifications, then repeat the design procedure by adjusting the compensator pole and zero until all such specifications are met.

Example (3.1): Consider the system shown in Figure 3-7(a).The feed forward transfer function is:

$$G(s) = \frac{4}{s(s+2)}$$

The root-locus plot for this system is shown in Figure 3-7(b). The closed-loop transfer function becomes:

$$G(s) = \frac{4}{s^2 + 2s + 4} = \frac{4}{(s + 1 + j\sqrt{3})(s + 1 - j\sqrt{3})}$$

The closed loop poles are located at: $s = -1 \mp j\sqrt{3}$

The damping ratio of the closed-loop poles is 0.5.The undamped natural frequency of the closed loop poles is 2 rad/sec. The static velocity error constant is 2 sec^{-1}.

It is desired to modify the closed-loop poles so that an undamped natural frequency $\omega_n = 4$ rad/sec is obtained, without changing the value of the damping ratio, $\zeta = 0.5$.

The damping ratio of 0.5 requires that the complex-conjugate poles lie on the lines drawn through the origin making angles of $\pm 60^0$ with the negative real axis.

Since the damping ratio determines the angular location of the complex-conjugate closed loop poles, while the distance of the pole from the origin is determined by the undamped natural frequency ω_n ,the desired locations of the closed-loop poles of this example problem are:

$$s = -2 \mp j2\sqrt{3}$$

A general procedure for determining the lead compensator is as follows: First, find the sum of the angles at the desired location of one of the dominant closed-loop poles with the open-loop poles and zeros of the original system, and determine the necessary angle ϕ to be added so that the total sum of the angles is equal to ∓180°(2k + 1). The lead compensator must contribute this angle ϕ**.** (If the angle ϕ is quite large, then two or more lead networks may be needed rather than a single one.)

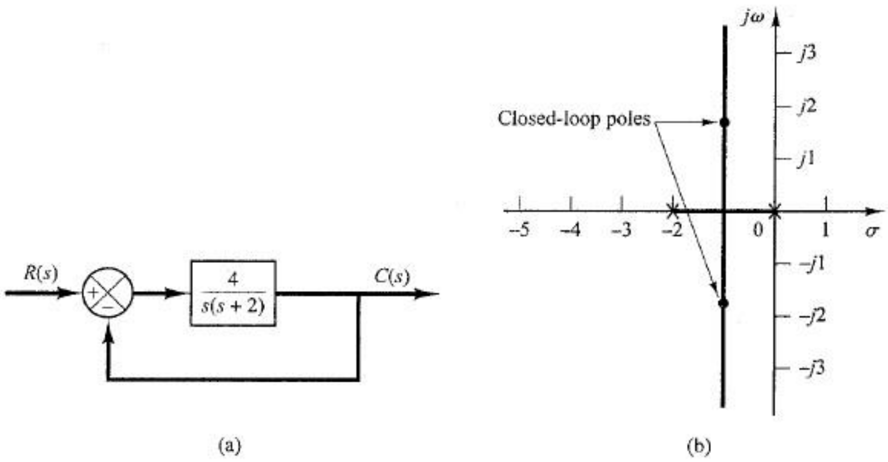


Figure 3.7

If the original system has the open-loop transfer function ***G(s)***, then the compensated system will have the open-loop transfer function:

$$G_c(s)G(s) = \left(K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \right) G(s)$$

The next step is to determine the locations of the zero and pole of the lead compensator. There are many possibilities for the choice of such locations. In what follows, we shall introduce a procedure to obtain the largest possible value for α. First, draw a horizontal line passing through point P, the desired location for one of the dominant closed-loop poles. This is shown as line PA in Figure 3-8. Draw also a line connecting point P and the origin. Bisect the angle between the lines PA and PO, as shown in Figure 3-8. Draw two lines PC and PD that make angles $\pm\phi/2$ with the bisector PB. The intersections of PC and PD with the negative real axis give the necessary locations for the pole and zero of the lead network. The compensator thus designed will make point P a point on the root locus of the compensated system. The open-loop gain is determined by use of the magnitude condition.

In the present system, the angle of $\boldsymbol{G(s)}$ at the desired closed-loop pole is:

$$\arg\left(\frac{4}{s(s+2)}\right)_{\text{at } s=-2+j2\sqrt{3}} = -210^0$$

Thus, if we need to force the root locus to go through the desired closed-loop pole, the lead compensator must contribute $\phi = 30^0$ at this point. By following the foregoing design procedure, we determine the zero and pole of the lead compensator, as shown in Figure 3-9, to be:

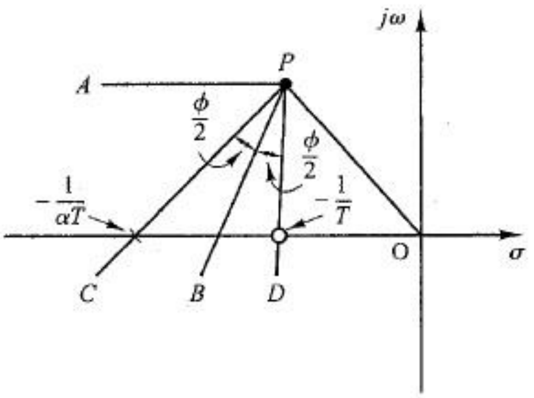


Figure (3.8)

$$Zero\ at\ s = -2.9, \quad pole\ at\ s = -5.4$$

Or

$$T = \frac{1}{2.9} = 0.345, \quad \alpha T = \frac{1}{5.4} = 0.185$$

Thus α = 0.537. The open-loop transfer function of the compensated system becomes:

$$G_s(s)G(s) = K_c \frac{s+2.9}{s+5.4}\frac{4}{(s+2)} = \frac{K(s+2.9)}{s(s+2)(s+5.4)}$$

Where $K = 4K_c$. The root-locus plot for the compensated system is shown in Figure 3.9.The gain K is evaluated from the magnitude condition as follows: Referring to the root-locus plot for the compensated system shown in Figure 3.9, the gain ***K*** is evaluated from the magnitude condition as:

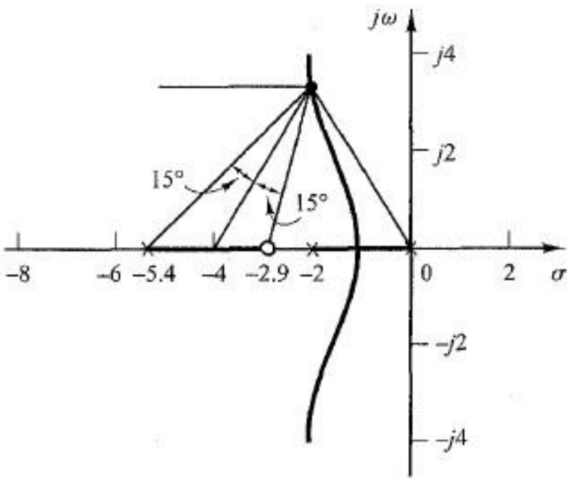


Figure 3.9 Root-locus plot of the compensated system.

$$\left|\frac{K(s+2.9}{s(s+2)(s+5.4)}\right|_{s=-2+j2\sqrt{3}} = 1$$

Or:

K=18.7

It follows that:

$$G(s) = \frac{18.7(s+2.9}{s(s+2)(s+5.4)}$$

The constant K_c of the lead compensator is:

$$K_c = \frac{18.7}{4} = 4.68$$

Hence $K_c\alpha = 2.51$. The lead compensator, therefore, has the transfer function:

$$G_s(s) = 2.51\frac{0.345s+1}{0.185s+1} = 4.68\frac{s+2.9}{s+5.4}$$

If the electronic circuit using operational amplifiers as shown in Figure 3.4 is used as the lead compensator just designed, then the parameter values of the lead compensator are determined from:

$$\frac{E_0(s)}{E_i(s)} = \frac{R_2R_4(R_1C_1s+1)}{R_1R_3(R_2C_2s+1)} = 2.51\frac{0.345s+1}{0.185s+1}$$

As shown in Figure 7.10, where we have arbitrarily chosen $C_1 = C_2 = 10\mu F\ and\ R_3 = 10k\Omega$.

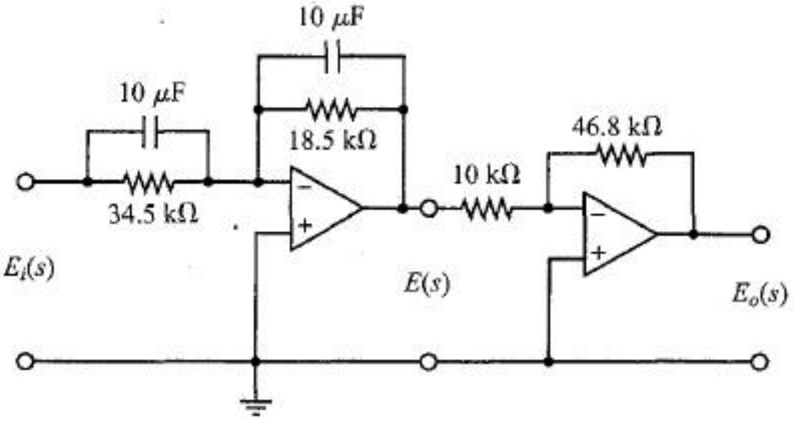


Figure 3.10 Lead compensator circuit

The static velocity error constant K_v is obtained from the expression:

$$K_v = \lim_{s--0} sG_c(s)G(s) = \lim_{s--0} s\frac{18.7(s+2.9}{s(s+2)(s+5.4)} = 5.02sec^{-1}$$

Note that the third closed-loop pole of the designed system is found by dividing the characteristic equation by the known factors as follows:

$$s(s+2)(s+5.4)+18.7(s+2.9) = (s+2+j2\sqrt{3})(s+2-j2\sqrt{3})(s+3.4)$$

In what follows we shall examine the unit-step responses of the compensated and uncompensated systems with MATLAB.

The closed-loop transfer function of the compensated system is:

$$\frac{C(s)}{R(s)} = \frac{18.7(s+2.9)}{s(s+2)(s+5.4)+18.7(s+2.9)} = \frac{18.7(s+2.9)}{s^3+7.4s^2+29.5s+54.23}$$

Hence,

Numc=[0 0 18.7 54.23]

Denc=[1 7.4 29.5 54.23]

For the uncompensated system, the closed-loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{4}{s^2+2s+4}$$

Numc=[0 0 4]

Denc=[1 2 4]

MATLAB Program produces the unit-step response curves for the two systems. The resulting plot is shown in Figure 3.11. Notice that the compensated system exhibits slightly larger maximum overshoot. The settling time of the compensated system is one-half that of the original system, as expected.

```
% - - - - - - - - - - - Unit-step response ---------
% ***** Unit-step responses of compensated and uncompensated
% systems *****
numc = [0 0 18.7 54.231];
denc = [1 7.4 29.5 54.231];
num = [0 0 41];
den = [ 1 2 41];
t = 0:0.05:5;
[cl ,X1 t]= step(numc,denc,t);
[c2,X2,t] = step(num,den,t);
plot(t,cl ,t,cl ,'o',t,c2,t,c2,'x')
grid
title('Unit-Step Responses of Compensated and Uncompensated Systems')
xlabel('t Sec')
ylabel('Outputs cl and c2')
text(0.7,1.32,'Compensated system')
text(1.3,0.68,'Uncompensated system')
```

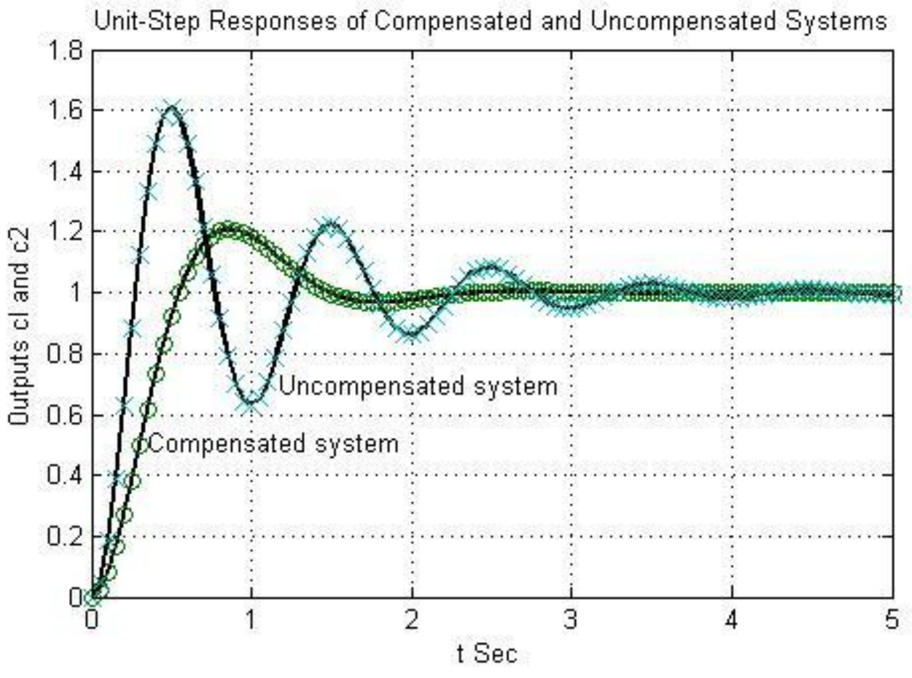


Figure 3.11 Unit-Step Responses of Compensated and Uncompensated Systems

3.4 Lag Compensation

The configuration of the electronic lag compensator using operational amplifiers is the same as that for the lead compensator shown in Figure 3.4. If we choose $R_2C_2 > R_1C_1$ in the circuit shown in Figure 3.4, it becomes a lag compensator. Referring to Figure 3.4, the transfer function of the lag compensator is given by:

$$\frac{E_o(s)}{E_i(s)} = \widehat{K}_c \frac{s + \frac{1}{T}}{s + \frac{1}{\beta T}}$$

Where:

$$T = R_1 C_1, \quad \beta T = R_2 C_2, \quad \beta = \frac{R_2 C_2}{R_1 C_1} > 1, \quad \widehat{K}_c = \frac{R_4 C_1}{R_3 C_2}$$