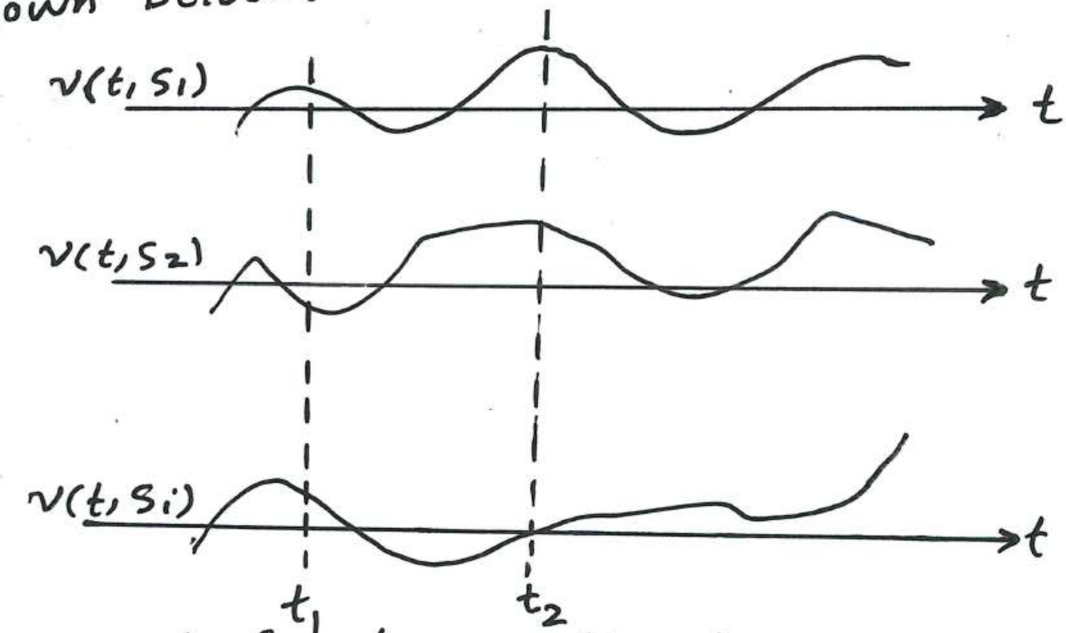
Random Signals and Noise

1. Random Process:

Random signal is a manifestation of random electrical signal that take place over time. It is also called a random process and stochastic process.

* A random process maps experimental outcomes into real functions of time. The collection of time functions is known as ensemble, and each member is called sample function. The ensembles are represented by $v(t,S)$.

* Consider an experiment to generate random signals shown below:



The sample function $v_i(t, S_i)$ having the value of $v(t_1, S_i)$ at time instant t_1. At time t_1 we could expect any value from the ensemble.

$$v(t, S_i) \longrightarrow \text{Random Process}$$

$$v(t_1, S_i) \longrightarrow \text{Random Variable}$$

In other words, the random variable is a spectial case of random process.

2. Ensemble Average and Correlation Functions:

* The mean value (average) of the random process $v(x,t)$ is commonly called the "expectation", and it is denoted by "E".

$$\boxed{\bar{v}(x,t) \triangleq E[v(x,t)] = \int_{-\infty}^{\infty} P_x(x)\, v(x,t)\, dx}$$

where x is random variable.

$P_x(x)$ is the probability density function (PDF) of the variable x.

* The statistical relationship between $v(x,t_1)$ and $v(x,t_2)$ is know as the "autocorrelation function" (R).

$$\boxed{R_v(t_1,t_2) \triangleq E[v(x,t_1)\; v(x,t_2)]}$$

$$\boxed{R_v(t_1,t_2) = \int_{-\infty}^{\infty} P_x(x)\, v(x,t_1)\, v(x,t_2)\, dx}$$

If $v(x,t_1)$ and $v(x,t_2)$ are statistically independent,

$$R_v(t_1,t_2) = E[v(x,t_1)] \cdot E[v(x,t_2)]$$

$$\therefore \quad R_v(t_1,t_2) = \bar{v}(x,t_1) \cdot \bar{v}(x,t_2)$$ for independent processes only.

* The correlation function between two different random process is called the "cross correlation function".

$$\boxed{R_{vw}(t_1,t_2) \triangleq E[v(x,t_1) \cdot w(x,t_2)]}$$

Note that the correlation is a measure of the similarity between two signals.

* The independent random processes are also known as uncorrelated process.

$$R_{vw}(t_1,t_2) = E[v(x,t_1)] \cdot E[w(x,t_2)]$$
$$= \overline{v}(x,t_1) \cdot \overline{w}(x,t_2)$$

For independent (uncorrelated) random process

* The processes are said to be orthogonal if:

$$\boxed{R_{vw}(t_1,t_2) = 0}$$

for orthogonal processes only.

* The Covariance:

It is a measure of how much two random processes change together, and it is denoted by (C_{vw}).

$$\boxed{C_{vw}(t_1,t_2) = R_{vw}(t_1,t_2) - \overline{v}(x,t_1)\ \overline{w}(x,t_2)}$$

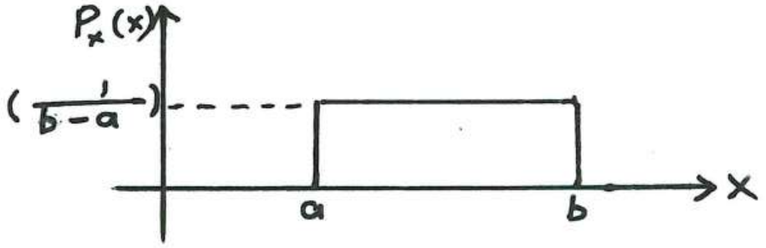
It can be deduced that

$$\boxed{C_{vw}(t_1,t_2) = 0}$$

For uncorrelated (independent) random processes.

3. Uniformly Distributed Random Process:

If the random variable of the random process can take any value between two limits (a and b) with equal probability, the process is called uniformly distributed.
The PDF of the uniformly distributed random variable (x) is given by:

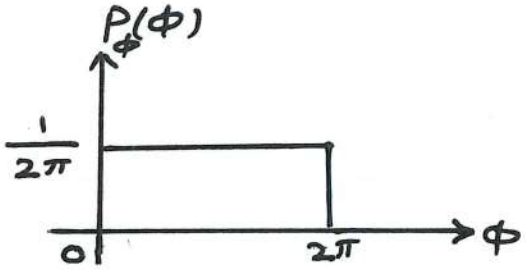


Example: Among a large collection of osilloscop's signals, one signal is randomly selected.

$$v_i(t,\phi) = A\cos(w_o t + \phi_i)$$

where ϕ is a random variable with uniform PDF from 0 to 2π radian. Find: a) The mean of $v(t,\phi)$.
b) The auto correlation of $v(t,\phi)$.

Sol.



$a = 0$, $b = 2\pi$

a) $\bar{v}(t,\phi) = E[v(t,\phi)]$

$$\bar{v}(t,\phi) = \int_{-\infty}^{\infty} P_\phi(\phi)\, v(t,\phi)\, d\phi$$

$$\bar{v}(t,\phi) = \int_0^{2\pi} \frac{1}{2\pi} \cdot A\cos(w_o t + \phi)\, d\phi$$

$$\bar{v}(t,\phi) = \frac{A}{2\pi}\int_0^{2\pi} \cos(w_o t + \phi)\, d\phi \Rightarrow \boxed{\bar{v}(t,\phi) = 0}$$

b) $R_v(t_1,t_2) = E[v(t_1,\phi)\, v(t_2,\phi)]$

$$R_v(t_1,t_2) = \int_{-\infty}^{\infty} P_\phi(\phi)\, v(t_1,\phi)\, v(t_2,\phi)\, d\phi$$

$$R_v(t_1,t_2) = \frac{A^2}{2\pi}\int_0^{2\pi} \cos(w_o t_1 + \phi)\cos(w_o t_2 + \phi)\, d\phi$$

$$R_v(t_1,t_2) = \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}\left[\cos(w_o(t_1 - t_2)) + \cos(w_o(t_1 + t_2) + 2\phi)\right] d\phi$$

$$R_v(t_1,t_2) = \frac{A^2}{4\pi}\left[\int_0^{2\pi} \cos(w_o(t_1 - t_2))\, d\phi + \underbrace{\int_0^{2\pi} \cos(w_o(t_1 + t_2) + 2\phi)\, d\phi}_{=0}\right]$$

$$R_v(t_1,t_2) = \frac{A^2}{4\pi}\cos(w_o(t_1 - t_2)) \cdot \phi \Big|_0^{2\pi}$$

$\cos(w_o(t_1 - t_2)) = \cos(w_o(t_2 - t_1))$ because the cosine is even function.

$$\therefore \quad \boxed{R_v(t_1,t_2) = \frac{A^2}{2}\cos(w_o(t_2 - t_1))}$$

- We are going to recall this example later on in this chapter.

Example: Consider the process $v(t)$ and $w(t)$ defined by: $v(t,x) = t + x$ $\quad w(t,y) = tY$

where x and Y are random variables.

a) Find mean and the autocorrelation of each signal.

b) Find the cross correlation between v and w.

c) Verify that the coveriance of the two processes is equal to zero if the two variables (x,y) are independent.

Sol.

a) $\bar{v}(t,x) = E[t+x]$ $\quad$ but t is not random variable.

$$\bar{v}(t,x) = E[t] + E[x]$$

$$\therefore \boxed{\bar{v}(t,x) = t + \bar{x}}$$

$$R_v(t_1,t_2) = E[(t_1+x)(t_2+x)]$$

$$R_v(t_1,t_2) = E[t_1t_2 + (t_1+t_2)x + x^2]$$

$$R_v(t_1,t_2) = E[t_1t_2] + E[(t_1+t_2)x] + E[x^2]$$

$$R_v(t_1,t_2) = E[t_1t_2] + (t_1+t_2)E[x] + E[x^2]$$

$$\therefore \boxed{R_v(t_1,t_2) = t_1t_2 + (t_1+t_2)\bar{x} + \overline{x^2}}$$

$$\bar{w}(t,y) = E[tY]$$

$$\bar{w}(t,y) = t\,E[Y] \Rightarrow \boxed{\bar{w}(t,y) = t\,\bar{Y}}$$

$$R_w(t_1,t_2) = E[t_1Y \cdot t_2Y]$$

$$R_w(t_1,t_2) = E[t_1t_2Y^2]$$

$$R_w(t_1,t_2) = t_1t_2\,E[Y^2] \Rightarrow \boxed{R_w(t_1,t_2) = t_1t_2\,\overline{Y^2}}$$

b) $R_{vw}(t_1,t_2) = E[(t_1+x)\cdot(t_2Y)]$

$$R_{vw}(t_1,t_2) = E[t_1t_2Y + t_2xY]$$

$$\therefore \boxed{R_{vw}(t_1,t_2) = t_1t_2\bar{Y} + t_2\overline{xY}}$$

c) $C_{vw}(t_1, t_2) = R_{vw}(t_1, t_2) - \bar{v}(t_1, x)\, \bar{w}(t_2, y)$

$$C_{vw}(t_1, t_2) = t_1 t_2 \bar{y} + t_2 \overline{xy} - [(t_1 + \bar{x}) \cdot (t_2 \bar{y})]$$

$$C_{vw}(t_1, t_2) = t_1 t_2 \bar{y} + t_2 \overline{xy} - [t_1 t_2 \bar{y} + t_2 \bar{x}\, \bar{y}]$$

For x and y are independent random variables

$$\overline{xy} = \bar{x}\, \bar{y}$$

$$C_{vw}(t_1, t_2) = t_1 t_2 \bar{y} + t_2 \bar{x}\, \bar{y} - [t_1 t_2 \bar{y} + t_2 \bar{x}\, \bar{y}]$$

$$\therefore \quad \boxed{C_{vw}(t_1, t_2) = 0}$$

4. Ergodic and Stationary Random Process:

If the time average of the random process $v(t,x)$ is:

$$\langle v(t,x) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} v(t,x)\, dt$$

and the statistical average (expectation) of this signal is given by:

$$E[v(t,x)] = \int_{-\infty}^{\infty} P_x(x)\; v(t,x)\, dx$$

The random process $v(t,x)$ is said to be "ergodic" if the time average of $v(t,x)$ is equal to its statistical average.

$$\boxed{\langle v(t,x) \rangle = E[v(t,x)]} \quad \text{For ergodic process}$$

* Let m_v is the mean of $v(t,x)$ (dc component).

σ_v is the standard deviation (rms value) of $v(t,x)$

σ_v^2 is the variance of $v(t,x)$ (ac power).

* The total average power of the random process $v(t,x)$ is:

$$\boxed{P_T = \overline{v^2(t,x)}}$$

so the variance = total power − dc power

where $\boxed{\text{dc power} = m_v^2 = \left(\bar{v}(t,x)\right)^2}$

$\therefore \boxed{\sigma_v^2 = P_T - m_v^2}$ and $\sigma_v = \sqrt{\sigma_v^2}$

The "wide-sense stationary" (WSS) process is a random process with ① mean $E[v(t,x)]$ is independent of time and ② autocorrelation $R_v(t_1,t_2)$ depends only on the time difference (t_2-t_1), i.e $R_v(t_2-t_1) = R_v(\tau)$

* "strictly Stationary" process is a stationary process for $-\infty \leq t \leq \infty$, so it is not exist in practice.

$$\therefore \boxed{\begin{aligned} &E[v(t,x)] = m_v = \text{constant} \\ &R_v(t_1,t_2) = R_v(t_2-t_1) = R_v(\tau) \end{aligned}}$$

For WSS process where $\tau = t_2 - t_1$

* All ergodic process are WSS, but not all WSS process are ergotic.

*For wide-sense stationary (WSS) process:

$$E[v(t,x)] = m_v$$

$$R_v(\tau) = E[v(t)\, v(t-\tau)] = E[v(t+\tau)\, v(t)]$$

*$R_v(\tau)$ is an even function, so $\boxed{R_v(-\tau) = R_v(\tau)}$

and the total power is:

$$\boxed{P_T = R_v(0) = \sigma_v^2 + m_v^2} = \overline{v^2}$$

and for energy signals:

$$\boxed{E = R_v(0)}$$

Note that the autocorrelation function has maximum value at $\tau = 0$.

$$\therefore \boxed{|R_v(\tau)| \leq R_v(0)}$$

* The random process of the example given in page 4 is:

WSS because $E[v(t,x)] = m_v = 0$

and $R_v(t_1, t_2) = \frac{A^2}{2} \cos(\omega_o (t_2 - t_1)) = \frac{A^2}{2} \cos(\omega_o \tau)$

* The process is ergodic too because:

$$\langle v(t,x) \rangle = E[v(t,x)] = 0$$

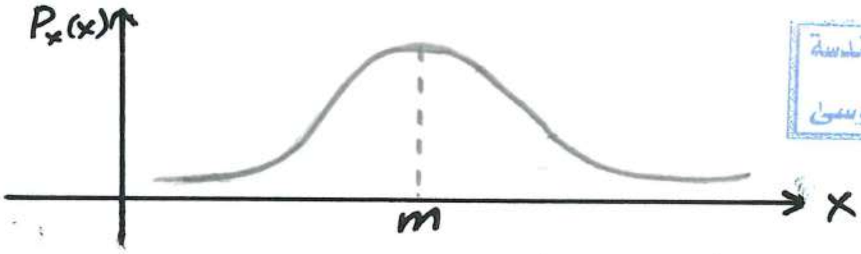
* However, the signals of example in page 5 is non-stationary because their mean values are time dependent.

5. Gaussian Process (Normal Process):

The random process is called Gaussian when the PDF of its random variable (x) is gaussian (normal distribution).

$$\boxed{P_x(x) = e^{-(x-m)^2/\sigma^2}}$$

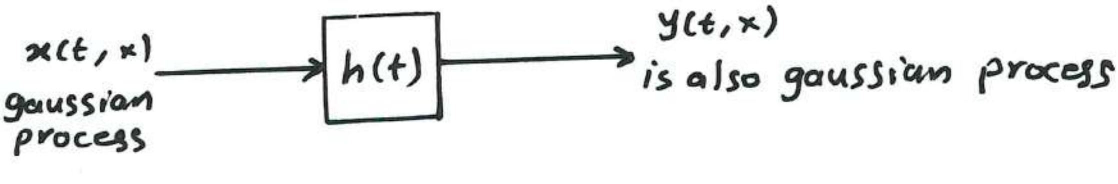
where m is the mean and σ^2 is the variance.



Gaussian process plays an important role in studying communication systems since the gaussian model may be applied to so many random electrical actions.

- The gaussian process is completely described by $E[v(t)]$ and $R_v(t_1, t_2)$.

- Any linear operation on gaussian distribution produces another gaussian distribution.



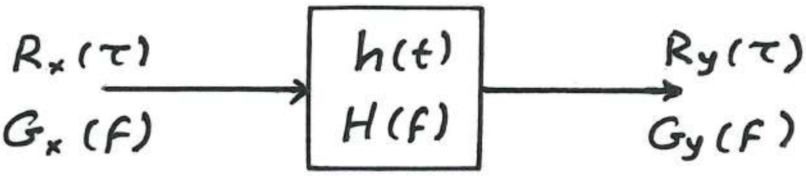
6. Power Spectral Density:

IF $v(t,x)$ is WSS process, then $G_v(f)$ is its power spectral density. It means the distribution of the average power over the frequency domain.

According to Wiener-Kinchine theorem:

$$\boxed{G_v(f) = F.T\left[R_v(\tau)\right]}$$

$$\boxed{P_T = R_v(0) = \int_{-\infty}^{\infty} G_v(f)\, df}$$



$$\boxed{G_y(f) = |H(f)|^2\, G_x(f)}$$

$$m_y = \int_{-\infty}^{\infty} m_x\, h(t)\, dt = m_x \int_{-\infty}^{\infty} h(t)\, dt$$

$$m_y = m_x \underbrace{\int_{-\infty}^{\infty} h(t)\, e^{-j2\pi(0)t}\, dt}_{F.T \text{ with } f=0}$$

$$\therefore \quad \boxed{m_y = H(0)\, m_x}$$

<u>Note:</u> In noise calculations, the direct voltage signals are random, so they can not be described by a mathematical model (equation). However, the statistics solve this problem by calculating the auto correlation function of the signal which can be described by a <u>formula</u>.

Noise in Analog Communications

1. Noise Classification:

In communication systems, noise is originated both in the equipment and in the channel. The noise is undesired variations that distorted the desired signal. The noise effect cannot be canceled completely, but it can be reduced by various means, such as:

- reducing the signal BW by BPF.s
- increasing the transmitter's power.
- using low-noise amplifiers for weak signals.

- Noise
  - External noise
    - man-made noise
    - natural noise
      - atmospheric noise
      - Space noise
  - Internal noise
    - Thermal noise
    - Shot noise
    - Partition noise
    - Flicker noise
    - Transit-Time noise

1.1 External Noise:

External noise is a property of the channel. It can be catogorized into man-made noise and natural noise depending on the source of the noise itself.