C Energy Spectral Density:

It is the distribution of the energy over the frequency. It represents Fourier transform of the autocorrelation function.

$$G_{xx}(f) = F.T\,[R_{xx}(\tau)]$$

$$\therefore \quad \boxed{R_{xx}(\tau) \xleftrightarrow{F.T} G_{xx}(f)}$$

The above equations constitutes the (Wiener-Khinchine) relations for energy signals.

$$F.T\,[R_{xx}(\tau)] = X(f) \cdot X(-f) = |X(f)|^2$$

$$\therefore \quad \boxed{G_{xx}(f) = |X(f)|^2}$$

Signal energy in time domain analysis:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \equiv \int_{-\infty}^{\infty} x(t)\, x(t)\, dt \equiv R_{xx}(\tau) \text{ when } \tau = 0$$

$$\therefore \quad \boxed{E = R_{xx}(0)}$$

Signal energy in frequency domain analysis:

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df \qquad \text{(Rayleigh theorem)}$$

$$\boxed{E = \int_{-\infty}^{\infty} G_{xx}(f)\, df}$$

- Correlation of Power Signals (Periodic Signals):

A Time-Average Autocorrelation Function $\bar{R}_{xx}(\tau)$:

For Periodic signals with period (T_o):

$$\boxed{\bar{R}_{xx}(\tau) = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t)\, x(t-\tau)\, dt}$$

and Similarly, the Power in time domain. $\boxed{P = \bar{R}_{xx}(0)}$

B Power Spectral Density $\bar{G}_{xx}(f)$:

Wiener-khinchine is applicable for power signals too.

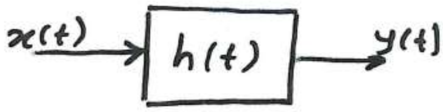
$$\boxed{\bar{R}_{xx}(\tau) \xleftrightarrow{F.T} \bar{G}_{xx}(f)}$$

Then, by using Parseval's theorem:

$$\boxed{P = \int_{-\infty}^{\infty} \bar{G}_{xx}(f)\, df}$$

* The output spectral density of a LTI system is given by:

$$y(t) = h(t) * x(t)$$



$$Y(f) = H(f) \cdot X(f)$$

$$\therefore \boxed{G_{yy}(f) = |H(f)|^2 \cdot G_{xx}(f)}$$

Example: Find the autocorrelation function and the energy spectral density of the following signal. Then find the signal energy.

$$x(t) = e^{-at} u(t) \quad , \quad a>0$$

Sol.

It is easier to find $G_{xx}(f)$ at first in this example.

$$X(f) = F.T[e^{-at} u(t)]$$

$$X(f) = \frac{1}{a + j\omega}$$

$$\therefore G_{xx}(f) = |X(f)|^2 = \frac{1}{|a + j2\pi f|^2}$$

$$\therefore \boxed{G_{xx}(f) = \frac{1}{a^2 + (2\pi f)^2}}$$

$$R_{xx}(\tau) = F.T^{-1}\left[G_{xx}(f) \right]$$

$$R_{xx}(\tau) = \frac{1}{2a} F.T^{-1}\left[\frac{2a}{a^2+(2\pi f)^2}\right]$$

$$\therefore \boxed{R_{xx}(\tau) = \frac{1}{2a} e^{-a|\tau|}}$$

In this example, it is hard to find the energy from $G_{xx}(f)$, so we are going to use the time domain formula :

$$E = R_{xx}(0) = \frac{1}{2a} e^{-a\times 0}$$

$$\therefore \boxed{E = \frac{1}{2a}}$$

<u>Example :</u> Find the average autocorrelation function and the power spectral density of the following signal. Then find its power value.

$$x(t) = A \cos(2\pi f_o t + \theta) , \quad T_o = \frac{1}{f_o}$$

<u>Sol.</u>

$$\bar{R}_{xx}(\tau) = \frac{1}{T_o}\int_{-T_o/2}^{T_o/2} x(t)\, x(t-\tau)\, dt$$

$$\bar{R}_{xx}(\tau) = \frac{1}{T_o}\int_{-T_o/2}^{+T_o/2} \{A \cos(2\pi f_o t+\theta)\}\{A \cos(2\pi f_o(t-\tau)+\theta)\} dt$$

$$\bar{R}_{xx}(\tau) = \frac{A^2}{T_o}\int_{-T_o/2}^{T_o/2} \cos(2\pi f_o t+\theta)\, \cos(2\pi f_o t - 2\pi f_o\tau+\theta)\, dt$$

$$\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\bar{R}_{xx}(\tau) = \frac{A^2}{2T_o}\int_{-T_o/2}^{T_o/2} \left[\cos(+2\pi f_o\tau) + \cos(4\pi f_o t - 2\pi f_o\tau + \theta)\right] dt$$

$$\bar{R}_{xx}(\tau) = \frac{A^2}{2T_o}\left[\int_{-T_o/2}^{T_o/2} \cos(2\pi f_o\tau)dt + \underbrace{\int_{-T_o/2}^{T_o/2} \cos(4\pi f_o t - 2\pi f_o\tau+\theta)\, dt}_{=\text{ zero}}\right]$$

$$\bar{R}_{xx}(\tau) = \frac{A^2}{2T_o} \cos(2\pi f_o\tau) \quad t\Big|_{-T_o/2}^{T_o/2}$$

$$\boxed{\bar{R}_{xx}(\tau) = \frac{A^2}{2}\cos(2\pi f_o \tau)}$$

$$\bar{G}_{xx}(f) = F.T\left[\bar{R}_{xx}(\tau)\right]$$

$$\boxed{\bar{G}_{xx}(f) = \frac{A^2}{4}\left[\delta(f-f_o) + \delta(f+f_o)\right]}$$

Sinusoidal signal power is: $(\text{Amplitude})^2/2$
and it can be verified in this example from $\bar{R}_{xx}(0)$.

$$P = R_{xx}(0) \Rightarrow \boxed{P = \frac{A^2}{2}}$$

Example: A LTI system has an impulse response given by:

$$h(t) = \text{Sinc}(0.5(t-10))$$

The input signal to this system is: $x(t) = e^{-2t}u(t)$.
Find the energy spectral density of the output signal $G_{yy}(f)$

Sol.

$$X(f) = \frac{1}{2 + j(2\pi f)}$$

$$G_{xx}(f) = |X(f)|^2 = \frac{1}{4 + (2\pi f)^2}$$

$$H(f) = F.T\left[h(t)\right] = F.T\left[\text{Sinc}[(t-10)\times 0.5]\right]$$

$$H(f) = F.T\left[0.5\ \text{Sinc}(0.5(t-10))\right] \times \frac{1}{0.5}$$

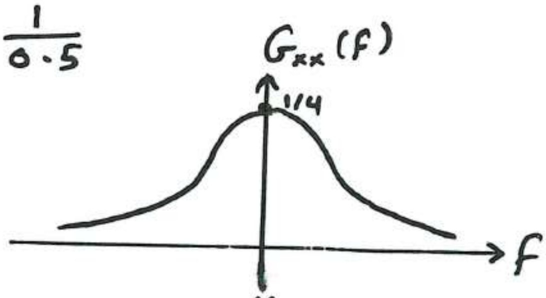
$$H(f) = 2\ \Pi\left(\frac{f}{0.5}\right) e^{-j2\pi(10)f}$$

$$|H(f)| = 2\ \Pi\left(\frac{f}{0.5}\right)$$

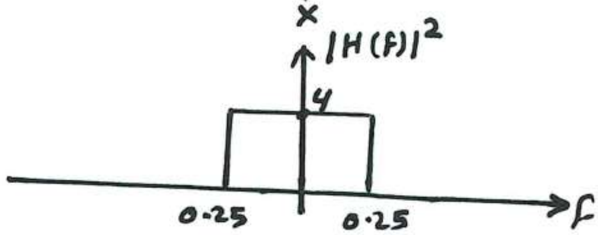
$$|H(f)|^2 = 4\ \Pi\left(\frac{f}{0.5}\right)$$

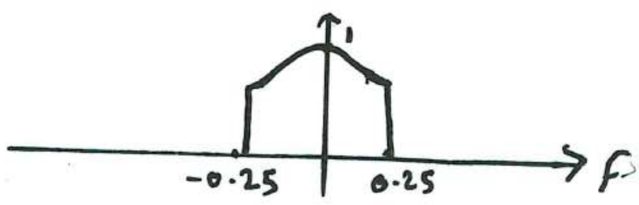
$$G_{yy}(f) = |H(f)|^2\ G_{xx}(f)$$

$$G_{yy}(f) = \frac{4}{4 + (2\pi f)^2}\Pi\left(\frac{f}{0.5}\right)$$



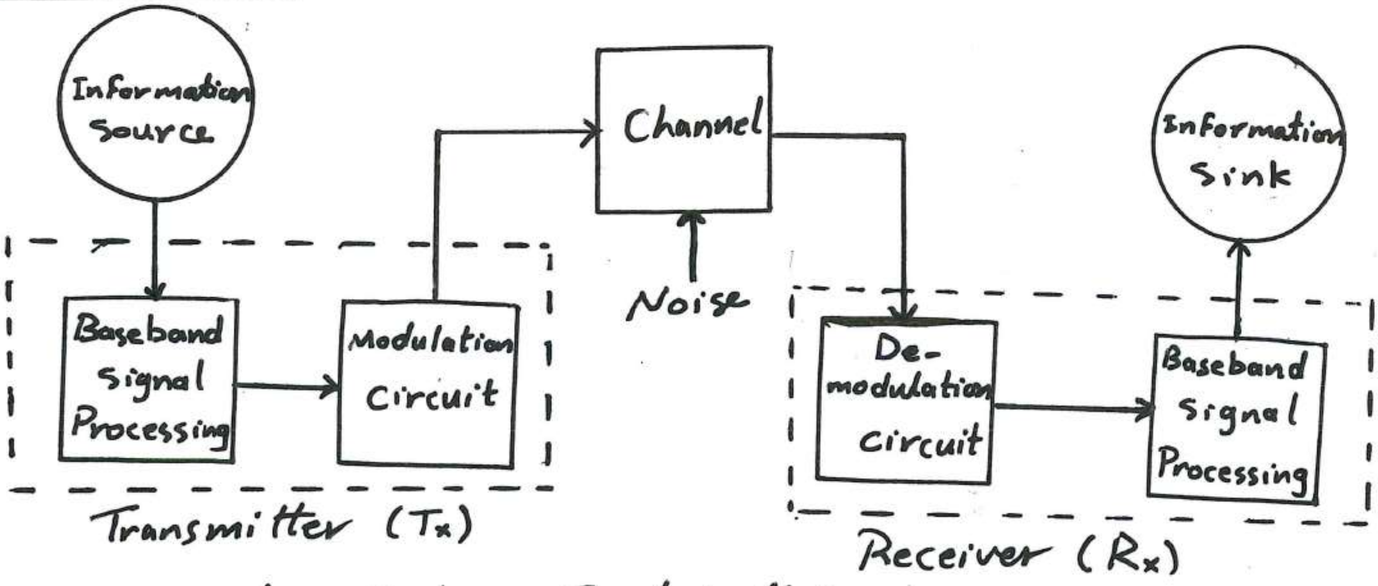
x





Linear Modulation

1. Definitions:



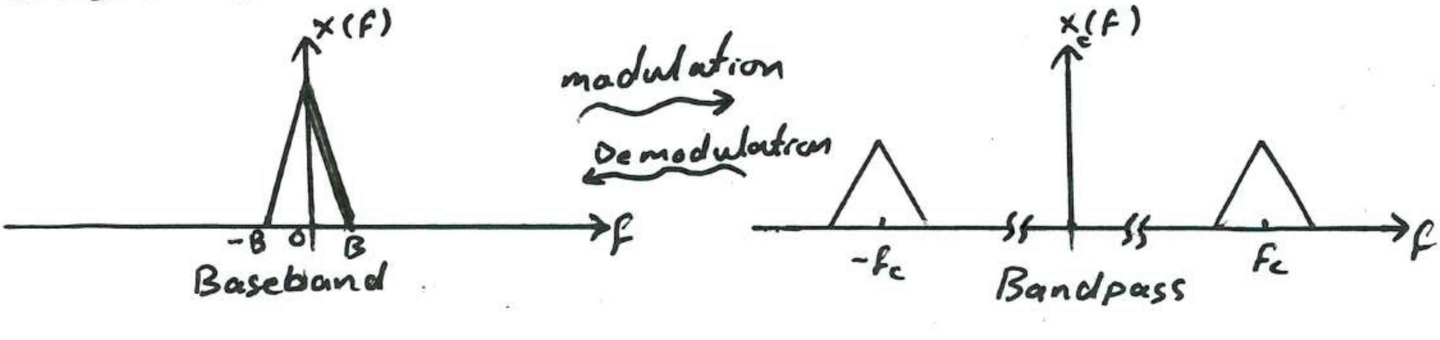
Communication Sytem Goal is " Design a system to transmit information signal with as little deterioration as possible within design constraints of signal power, signal bandwidth, and system cost ".

Baseband Signal (Modulating Signal):

Waveform whose spectral magnitude is non-zero for frequencies at or near $f = 0\,Hz$, and negligible spectral magnitude at $f \gg 0\,Hz$.

Band-Pass Signal (Modulated Signal):

Waveform whose spectral magnitude is non-zero for frequencies concentrated in a band around $f = \mp f_c$ where $f_c \gg 0$ and called the carrier frequency.



Modulation:

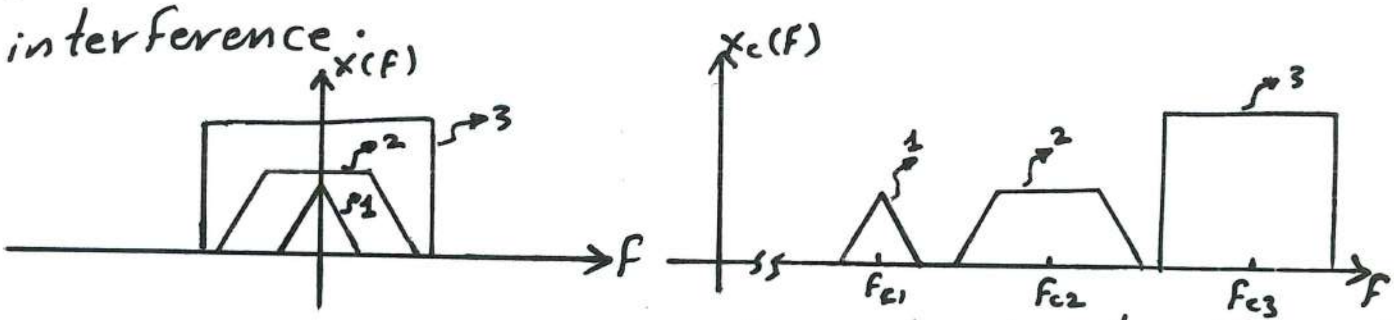
The process of translating a baseband information signal into a bandpass signal with carrier frequency f_c by modifying the carrier signal amplitude, phase, or frequency.

Demodulation:

The process of getting the baseband signal back from the bandpass signal.

Why is the modulation used?

1) Separate the spectrum of the baseband signals to share the same communication channel without any interference.



2) High frequency region provides wider bandwidth and, as a result, higher data rate (Will be discussed in Digital communication class).

3) Higher frequency requires small antenn height at the transmitter side.

Antenna height $\approx 0.1\lambda$, where $\lambda = \frac{c}{f_c}$

For small f_c say 3 kHz, Antenna height $= 0.1 \times \frac{3\times10^8}{3\times10^3}$
$= 10\,km$!!

For large f_c say 3 MHz, Antenna height $= 0.1\ \frac{3\times10^8}{3\times10^6}$
$= 10\,m$

* The carrier signal is a sinusoidal signal:

$$\text{carrier} = A_c \cos(2\pi f_c t + \phi)$$

and the modulated signal (bandpass signal) is given by:

$$x_c(t) = A(t) \cos[2\pi f_c t + \phi(t)]$$

* The information signal will be represented by a message signal $\underline{m(t)}$ in this chapter.

- If $A(t)$ is linearly related to $m(t) \Rightarrow$ Linear Modulation [AM, DSB, SSB, VSB]

- If $\phi(t)$ or its derivative is linearly related to the message signal $m(t) \Rightarrow$ Angle Modulation [PM, FM]

2. Linear Modulation (Amplitude Modulation):

It is often called (Amplitude Modulation). It is defined by setting $\phi(t) = 0$.

$$x_c(t) = A(t) \cos(2\pi f_c t)$$

where $A(t)$ is the instantaneous amplitude of the carrier, which is linearly related to $m(t)$.

A Normal Amplitude Modulation (AM):

The general equation of this type of modulation is:

$$x_c(t) = m(t) \cos(2\pi f_c t) + A_c \cos(2\pi f_c t)$$

OR

$$\boxed{x_c(t) = [A_c + m(t)] \cos(2\pi f_c t)}$$

* Some times the AM modulation is called Double Side Band Large Carrier (DSB-LC) because:
- both sides of the signal spectrum is transmitted.
- a large carrier value is transmitted with the signal.

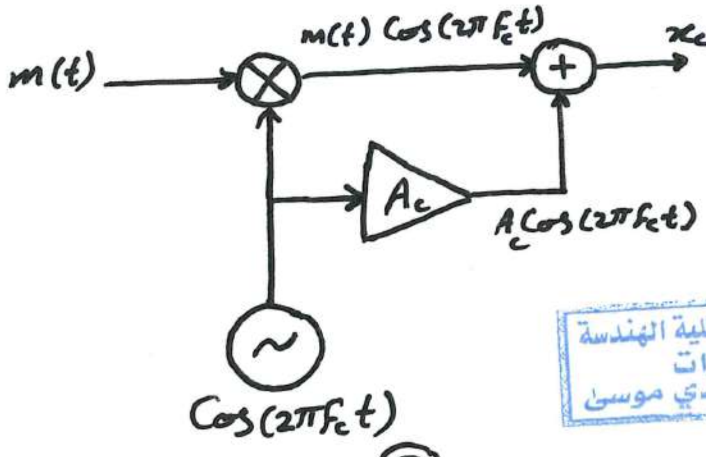
- AM Generation:

Let's recall the AM general equations:

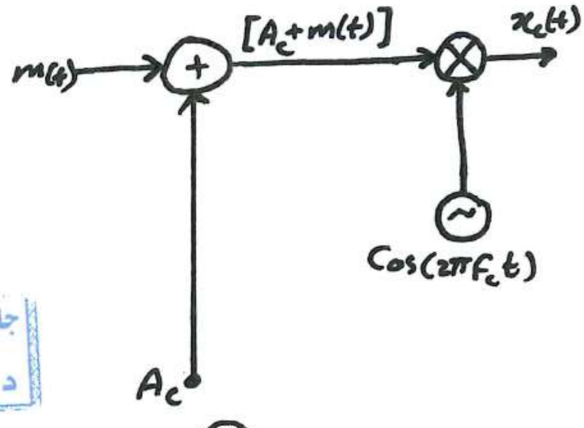
$$x_c(t) = m(t)\cos(2\pi f_c t) + A_c\cos(2\pi f_c t) \quad \text{— ⓐ}$$

OR $$x_c(t) = [A_c + m(t)]\cos(2\pi f_c t) \quad \text{— ⓑ}$$

The generation of eq. ⓐ and ⓑ are shown below:



ⓐ



ⓑ

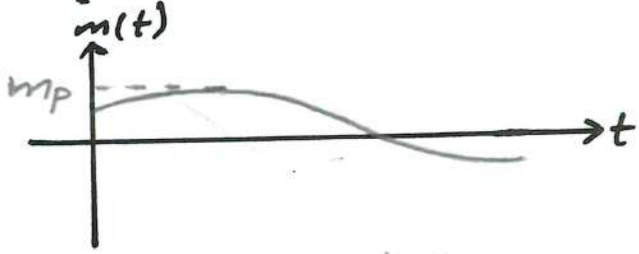
- Frequency Spectrum of AM:

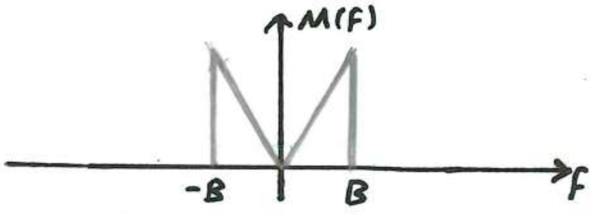
By taking Fourier transform for the modulated signal:

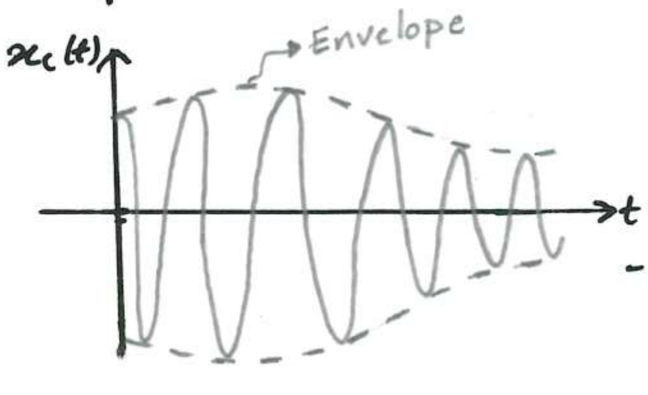
$$X_c(f) = F.T\,[m(t)\cos(2\pi f_c t)] + F.T\,[A_c\cos(2\pi f_c t)]$$

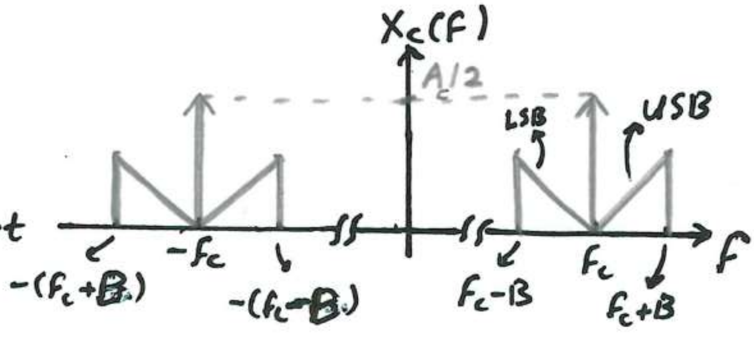
$$X_c(f) = \left[M(f) * \tfrac{1}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right)\right] + \left[\tfrac{A_c}{2}\left(\delta(f-f_c) + \delta(f+f_c)\right)\right]$$

$$X_c(f) = \left[\tfrac{1}{2}M(f-f_c) + \tfrac{1}{2}M(f+f_c)\right] + \tfrac{A_c}{2}\delta(f-f_c) + \tfrac{A_c}{2}\delta(f+f_c)$$









* It can be noticed that $x_c(t)$ has two envelopes:

$[A+m(t)]$ and $-[A+m(t)]$

and the sine wave of the carrier signal oscillates in between.

- Modulation Index (μ):

If $m(t) = m_p\, m_n(t)$

where m_p is the peak value of $m(t)$.

$m_n(t)$ is the normalized value of $m(t)$ $\left\{ m_n(t) = \frac{m(t)}{m_p} \right\}$

then $x_c(t) = [A_c + m_p\, m_n(t)] \cos(2\pi f_c t)$.

$$x_c(t) = A_c \left[1 + \frac{m_p}{A_c}\, m_n(t)\right] \cos(2\pi f_c t)$$

$$\boxed{x_c(t) = A_c [1 + \mu\, m_n(t)] \cos(2\pi f_c t)}$$

where μ is the modulation index.

$$\boxed{\mu = \frac{m_p}{A_c}}$$

Case 1: $\mu < 1$ $\{A_c > m_p\}$

$\therefore A_c[1 + \mu\, m_n(t)] > 0$

So the envelope can easily be detected.

Case 2: $\mu = 1$ $\{A_c = m_p\}$

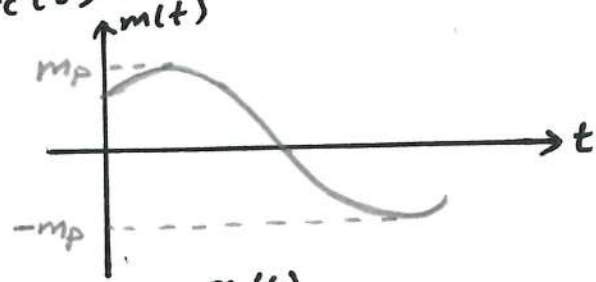
$A_c[1 + \mu\, m_n(t)] \geq 0$

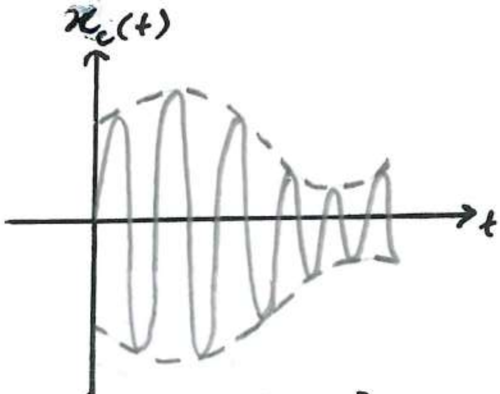
The envelope starts from zero in this case.

Case 3: $\mu > 1$ $\{A_c < m_p\}$ Overmodulation case

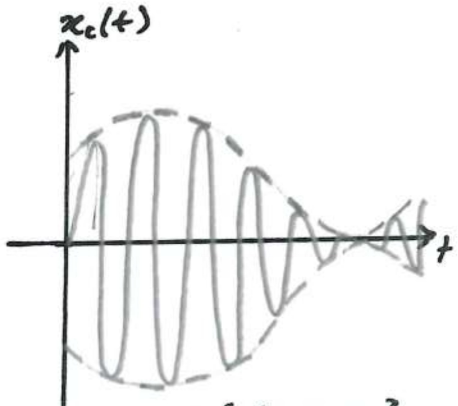
The envelope $A_c[1 + \mu\, m_n(t)]$ falls below zero.

Note: In overmodulation case ($\mu > 1$), a phase reversion occurs due to the presence of the negative values of the envelope. As a result, the signal information resides in the ① amplitude and the ② phase of the band-pass signal $x_c(t)$.

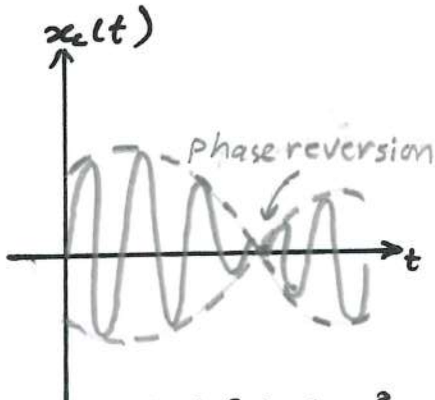




$\mu < 1$ $\{A_c > m_p\}$



$\mu = 1$ $\{A_c = m_p\}$



$\mu > 1$ $\{A_c < m_p\}$

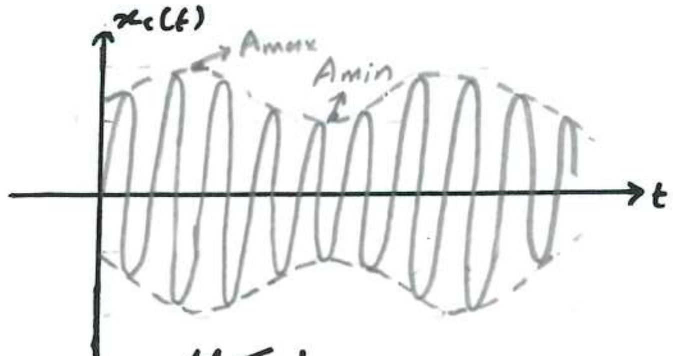
- Single-Tone AM Modulation:

Here the message signal is pure sinusoid, and given by:

$$m(t) = a_m \cos(2\pi f_m t)$$

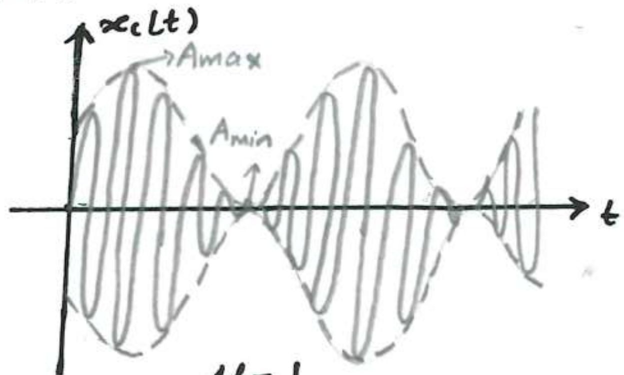
$$\therefore \mu = \frac{a_m}{A_c} \quad \text{or} \quad \boxed{a_m = \mu A_c}$$

$$\therefore x_c(t) = A_c [1 + \mu \cos(2\pi f_m t)] \cos(2\pi f_c t)$$



$\mu < 1$

$A_{max} > 0$ and $A_{min} > 0$



$\mu = 1$

$A_{max} > 0$ and $A_{min} = 0$