

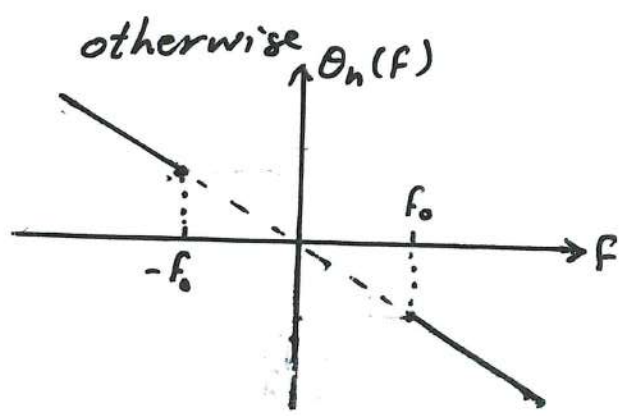
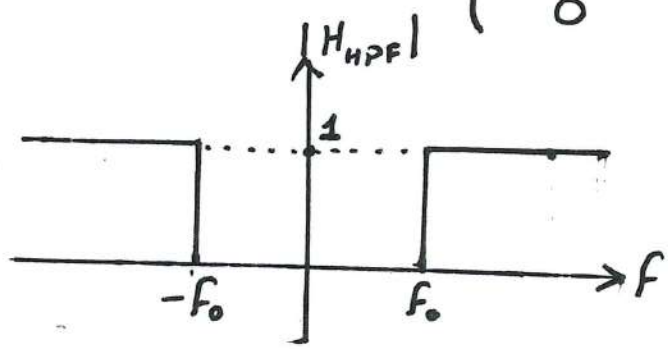
It is clear that ideal LPF requires infinite time domain impulse response as well as it is non causal system ( $h_{LPF}(t) \neq 0$  for  $t < 0$ ).

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- Ideal High-Pass Filter (HPF):

It perfectly rejects the frequency components less than  $f_0$  and pass the components above  $f_0$ .

$$H_{HPF}(f) = \begin{cases} e^{-j2\pi t_d f} & |f| > f_0 \\ 0 & \text{otherwise} \end{cases}$$



$$H_{HPF}(f) = e^{-j2\pi t_d f} - H_{LPF}(f)$$

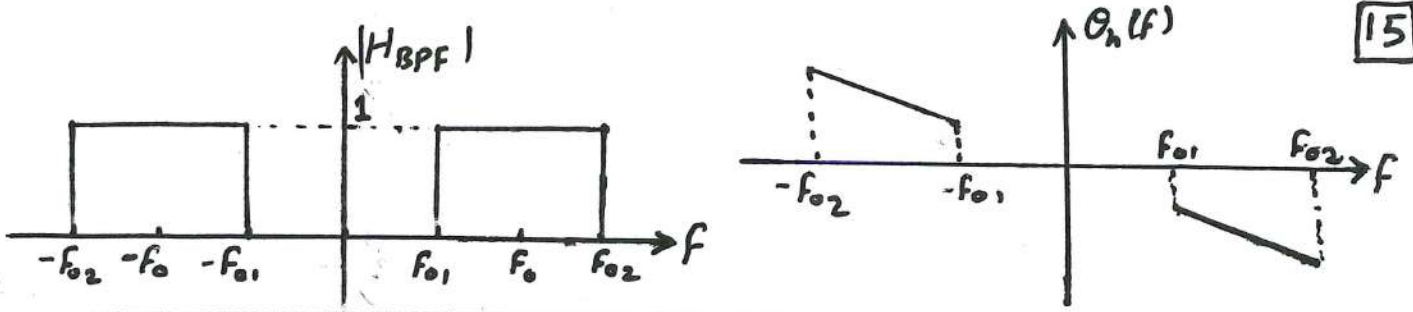
$$H_{HPF}(f) = \left[ 1 - \Pi\left(\frac{f}{2f_0}\right) \right] e^{-j2\pi t_d f}$$

$$h_{HPF}(t) = \delta(t - t_d) - 2f_0 \text{Sinc}(2f_0(t - t_d))$$

- Ideal Band-Pass Filter (BPF):

It passes frequency components between two frequencies ( $f_{01}$  &  $f_{02}$ ) and eliminates all components else where.

$$H_{BPF}(f) = \begin{cases} e^{-j2\pi t_d f} & f_{01} < |f| \leq f_{02} \\ 0 & \text{otherwise.} \end{cases}$$



$$H(f) = \left[ \Pi\left(\frac{f - f_0}{f_{02} - f_{01}}\right) + \Pi\left(\frac{f + f_0}{f_{02} - f_{01}}\right) \right] e^{-j2\pi t_d f}$$

$$h_{BPF}(t) = (f_{02} - f_{01}) \left[ \text{Sinc}((f_{02} - f_{01})(t - t_d)) e^{j2\pi f_0(t - t_d)} + \text{Sinc}((f_{02} - f_{01})(t - t_d)) e^{-j2\pi f_0(t - t_d)} \right]$$

$$h_{BPF}(t) = 2(f_{02} - f_{01}) \text{Sinc}((f_{02} - f_{01})(t - t_d)) \cos(2\pi f_0(t - t_d))$$

- Bandwidth of Ideal Filters:

\* For LPF and BPF, the bandwidth  $\{(BW) \text{ or } (B)\}$  is the positive frequency coverage of the filter.

$$BW_{LPF} = f_0$$

$$BW_{BPF} = f_{02} - f_{01}$$

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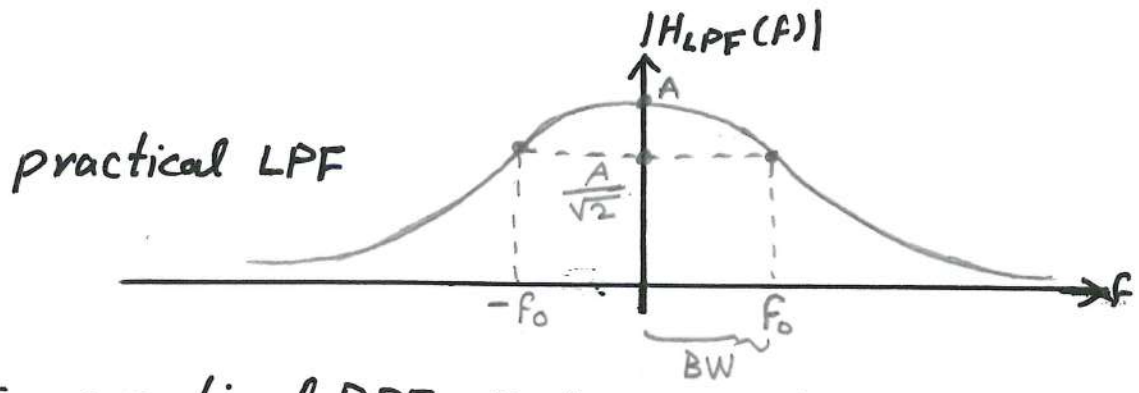
\*  $f_0$  of the BPF is considered the center frequency of the BPF  $\left\{ f_0 = \frac{f_{01} + f_{02}}{2} \right\}$ .

\* The HPF has no definite BW.

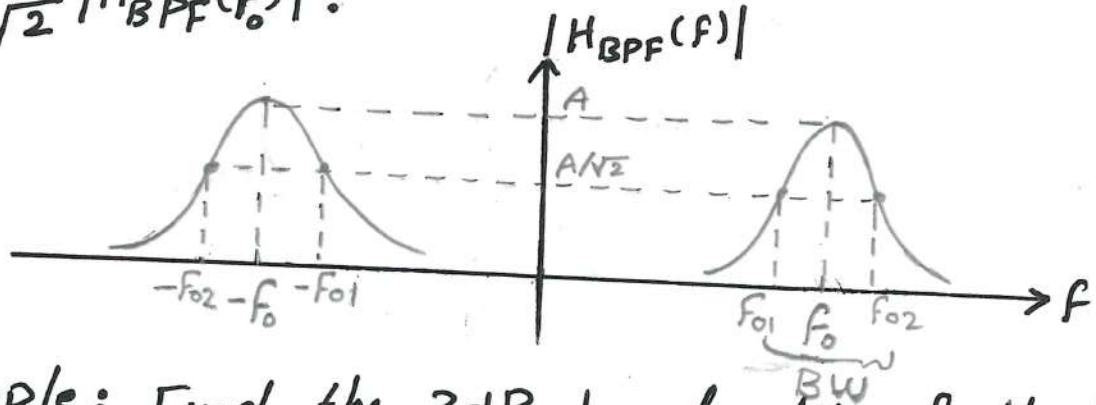
[B] Practical Filters:

The most common definition of filter bandwidth is called the 3dB bandwidth ( $BW_{3dB}$ ).

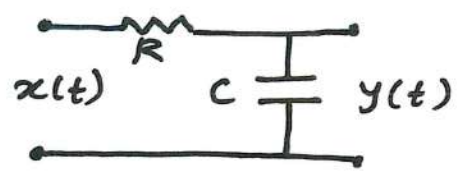
- For LPF:  $BW_{3dB}$  is the positive frequency at which the amplitude spectrum  $|H_{LPF}(f)|$  drops to value equal to  $\frac{1}{\sqrt{2}}|H(0)|$ .



- For practical BPF:  $BW_{3dB}$  is the difference between frequencies at which  $|H_{BPF}(f)|$  drops to a value equal to  $\frac{1}{\sqrt{2}} |H_{BPF}(f_0)|$ .



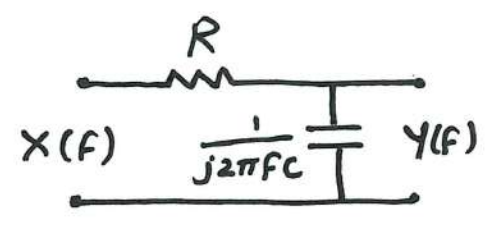
Example: Find the 3dB bandwidth of the following LPF circuit.



Sol.

$$H(f) = \frac{Y(f)}{X(f)} = \frac{1}{\frac{1}{j2\pi f C} + R}$$

$$H(f) = \frac{1}{1 + j2\pi f RC}$$



$$|H(f)| = \frac{1}{\sqrt{1 + (2\pi RC)^2 f^2}}, \quad \theta_h(f) = -\tan^{-1}(2\pi RC f)$$

at the 3dB BW,  $|H(f_0)| = \frac{|H(0)|}{\sqrt{2}}$

where  $H(0) = 1$

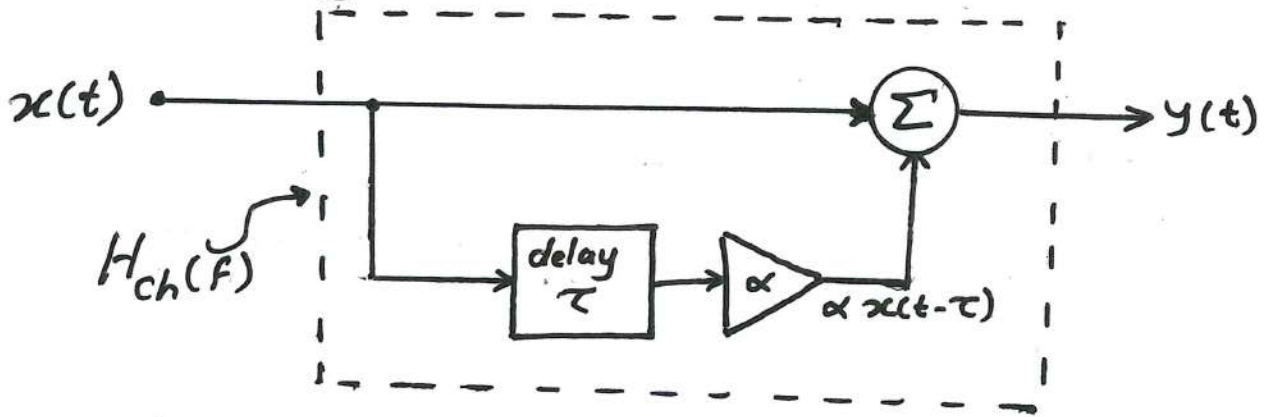
$$\therefore \frac{1}{\sqrt{1 + (2\pi RC)^2 f_0^2}} = \frac{1}{\sqrt{2}}$$

$$\therefore 1 + (2\pi RC)^2 f_0^2 = 2 \Rightarrow \boxed{f_0 = \frac{1}{2\pi RC} = BW}$$

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## 6. Multipath Transmission:

It is the phenomenon occurs when a transmitted signal arrives at the receiver through two or more different delays.



$y(t) = x(t) + \alpha x(t - \tau)$ , where  $\alpha$  is an attenuation factor.

$$Y(f) = X(f) + \alpha X(f) e^{-j2\pi f\tau}$$

$$Y(f) = X(f) [1 + \alpha e^{-j2\pi f\tau}]$$

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$$H_{ch}(f) = \frac{Y(f)}{X(f)} = [1 + \alpha e^{-j2\pi f\tau}]$$

$$H_{ch}(f) = 1 + \alpha \cos(2\pi f\tau) + j\alpha \sin(2\pi f\tau)$$

$$\therefore |H_{ch}(f)| = [(1 + \alpha \cos(2\pi f\tau))^2 + (\alpha \sin(2\pi f\tau))^2]^{1/2}$$

$$|H_{ch}(f)| = [1 + 2\alpha \cos(2\pi f\tau) + \alpha^2 \cos^2(2\pi f\tau) + \alpha^2 \sin^2(2\pi f\tau)]^{1/2}$$

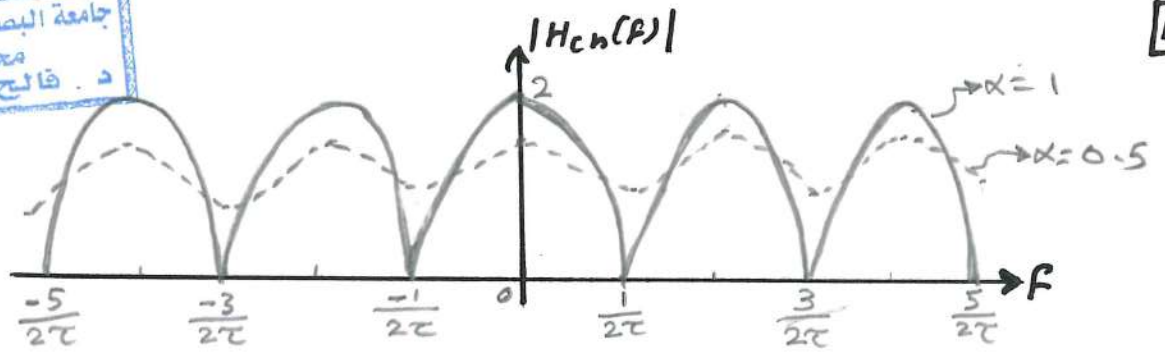
$$|H_{ch}(f)| = [1 + 2\alpha \cos(2\pi f\tau) + \alpha^2]^{1/2}$$

For  $\alpha = 1$ ,  $|H_{ch}(f)| = [2 + 2\cos(2\pi f\tau)]^{1/2}$

$$|H_{ch}(f)| = [2(1 + \cos(2\pi f\tau))]^{1/2} = [2 \times 2 \cos^2(\pi f\tau)]^{1/2}$$

$$|H_{ch}(f)| = 2 |\cos(\pi f\tau)|$$

For  $\alpha = 0.5$ ,  $|H_{ch}(f)| = (1.25 + \cos 2\pi f\tau)^{1/2}$

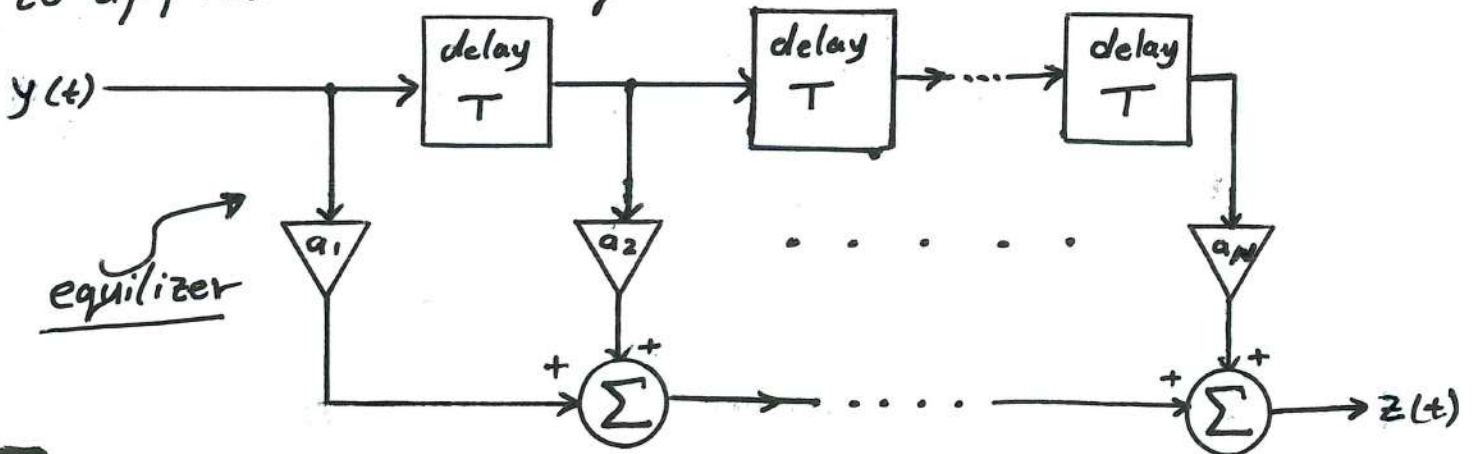


### Equalization:

\* It is a technique used for compensating the channel distortion caused by the multipath transmission.

$$\therefore H_{eq}(F) = \frac{1}{H_{ch}(F)}$$

\* Practically, a tapped delay-line is commonly used to approximate the equalization filter.



$$z(t) = a_1 y(t) + a_2 y(t-T) + a_3 y(t-2T) + \dots + a_N y(t-(N-1)T)$$

$$z(t) = \sum_{n=1}^N a_n y(t - (n-1)T)$$

$$\therefore Z(F) = \sum_{n=1}^N a_n Y(F) e^{-j2\pi F(n-1)T}$$

$$\therefore H_{eq}(F) = \frac{Z(F)}{Y(F)} = \sum_{n=1}^N a_n e^{-j2\pi F(n-1)T}$$

OR 
$$H_{eq}(F) = a_1 + a_2 e^{-j2\pi F(T)} + a_3 e^{-j2\pi F(2T)} + \dots + a_N e^{-j2\pi F(N-1)T}$$

The equalizer modifies  $a_1, a_2, \dots, a_N$  until reaching  $H_{eq}(F) = \frac{1}{H_{ch}(F)}$ , so the overall T.F is flat.

Example: Find the equalization coefficients require to compensate the distortion of two-path channel.

Sol.

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$$H_{ch}(f) = 1 + \alpha e^{-j2\pi f \tau}$$

$$H_{eq}(f) = \frac{1}{H_{ch}(f)} \Rightarrow H_{eq}(f) = \frac{1}{1 + \alpha e^{-j2\pi f \tau}}$$

At first the equalizer time delay is set as:  $T = \tau$

$$\therefore H_{eq}(f) = \frac{1}{1 + \alpha e^{-j2\pi f T}}$$

but  $\left\{ \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \dots \right\}$  for  $|x| < 1$

$$\therefore H_{eq}(f) = 1 - \alpha e^{-j2\pi f T} + \alpha^2 e^{-j2\pi f (2T)} - \alpha^3 e^{-j2\pi f (3T)} + \dots$$

$$H_{eq}(f) = a_1 + a_2 e^{-j2\pi f T} + a_3 e^{-j2\pi f (2T)} + a_4 e^{-j2\pi f (3T)} + \dots$$

$$\therefore a_1 = 1, a_2 = -\alpha, a_3 = \alpha^2, \dots, a_n = (-\alpha)^{n-1}$$

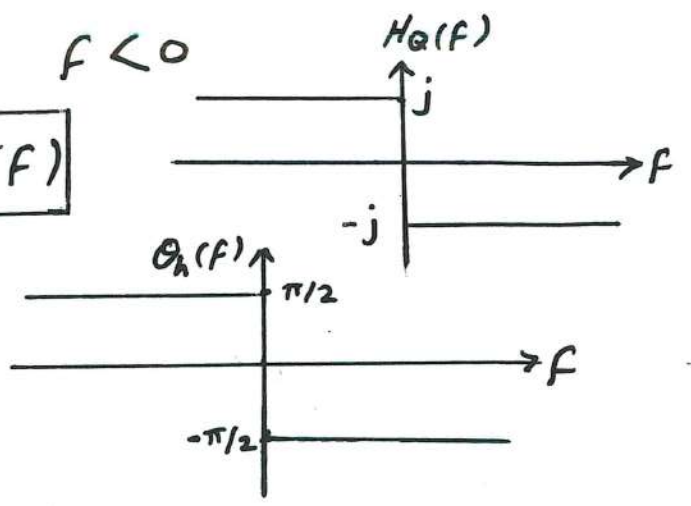
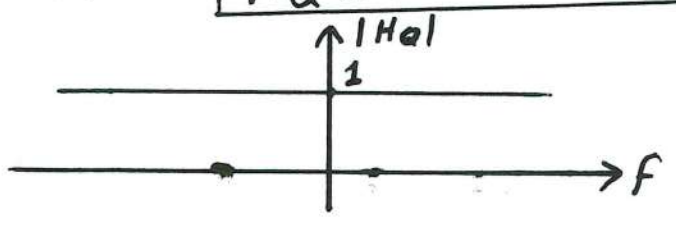
7. Quadrature Filters and Hilbert Transform:

\* Quadrature Filters:

It is an all pass filter which represents a 90° phase shifter whose frequency response is:

$$H_Q(f) = \begin{cases} -j = e^{-j\pi/2} & f > 0 \\ +j = e^{j\pi/2} & f < 0 \end{cases}$$

OR  $H_Q(f) = -j \text{Sgn}(f)$



but  $F.T^{-1}\{Sgn(F)\} = \frac{1}{j\pi(-t)}$  {duality property of F.T}

$$\therefore h_Q(t) = F.T^{-1}\left[H_Q(F)\right]$$

$$h_Q(t) = F.T^{-1}[-j Sgn(F)]$$

$$\therefore \boxed{h_Q(t) = \frac{1}{\pi t}}$$

\* Hilbert Transform  $\hat{x}(t)$ :

The Hilbert transform of a function is obtained by shifting all its frequency components by  $(-90^\circ)$  as shown below:



$$\hat{X}(F) = -j Sgn(F) \cdot X(F)$$

$$\therefore \hat{x}(t) = x(t) * \frac{1}{\pi t}$$

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Properties of Hilbert Transform:

1)  $x(t)$  and  $\hat{x}(t)$  has the same amplitude spectrum  
 $|\hat{X}(F)| = |X(F)|$

2) The energy of  $x(t)$  is equal to the energy of  $\hat{x}(t)$ . The same thing is right for the power of power signal.

$$E_{\hat{x}(t)} = E_{x(t)} \quad \text{or} \quad P_{\hat{x}(t)} = P_{x(t)}$$

3) The signal  $x(t)$  and its Hilbert transform  $\hat{x}(t)$  are orthogonal.

$$\int_{-\infty}^{\infty} x(t) \hat{x}(t) dt = 0 \quad \text{For energy signals.}$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \hat{x}(t) dt = 0 \quad \text{For power signals.}$$

4) Hilber transform of  $\hat{x}(t)$  is equal to  $-x(t)$ , i.e  
 $(\hat{\hat{x}}(t)) = -x(t)$

Proof:

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}$$

$$\hat{X}(f) = -j \operatorname{sgn}(f) \cdot X(f)$$

$$\therefore (\hat{\hat{X}}(f)) = -j \operatorname{sgn}(f) \cdot \hat{X}(f)$$

$$\therefore (\hat{\hat{X}}(f)) = -j \operatorname{sgn}(f) \cdot -j \operatorname{sgn}(f) X(f)$$

$$\operatorname{sgn}(f) \cdot \operatorname{sgn}(f) = 1 \quad , \quad -j \times -j = -1$$

$$\therefore (\hat{\hat{X}}(f)) = -X(f)$$

$$\therefore (\hat{\hat{x}}(t)) = -x(t)$$

Example: Find Hilbert transform of the following signals:

- a)  $x(t) = \cos(2\pi f_0 t)$
- b)  $x(t) = \cos(2\pi f_0 t + \theta)$
- c)  $x(t) = \frac{\sin 2\pi t}{t} \cos(200\pi t)$



Sol.

$$a) x(t) = \cos(2\pi f_0 t)$$

$$X(f) = \frac{1}{2} [\delta(f-f_0) + \delta(f+f_0)]$$

$$\hat{X}(f) = -j \operatorname{sgn}(f) X(f)$$

$$\hat{X}(f) = \frac{-j}{2} \operatorname{sgn}(f) [\delta(f-f_0) + \delta(f+f_0)] , \operatorname{sgn}(f) = \begin{cases} 1 & f > 0 \\ -1 & f < 0 \end{cases}$$

$$\hat{X}(f) = \frac{1}{j2} [\delta(f-f_0) - \delta(f+f_0)]$$

$$\therefore \hat{x}(t) = \sin(2\pi f_0 t)$$

b)  $x(t) = \cos(2\pi f_0 t + \theta)$   
 $x(t) = \cos(2\pi f_0 (t + \frac{\theta}{2\pi f_0}))$  → time shifting property of F.T



$$X(f) = \frac{1}{2} [\delta(f-f_0) + \delta(f+f_0)] e^{+j2\pi f \frac{\theta}{2\pi f_0}}$$

$$X(f) = \frac{1}{2} [\delta(f-f_0) + \delta(f+f_0)] e^{+j \frac{f}{f_0} \theta}$$

$$\hat{X}(f) = -j \text{sgn}(f) * X(f)$$

$$\hat{X}(f) = \frac{1}{j2} [\delta(f-f_0) - \delta(f+f_0)] e^{+j \frac{f}{f_0} \theta}$$

$$\hat{X}(f) = \frac{1}{j2} [\delta(f-f_0) - \delta(f+f_0)] e^{+j2\pi f \frac{\theta}{2\pi f_0}}$$

$$\therefore \hat{x}(t) = \text{Sin} \left( 2\pi f_0 \left( t + \frac{\theta}{2\pi f_0} \right) \right)$$

$$\hat{x}(t) = \text{Sin} (2\pi f_0 t + \theta)$$

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c)  $x(t) = \frac{\text{Sin}(2\pi t)}{t} \text{Cos}(200\pi t) * \frac{2\pi}{2\pi}$

$$x(t) = 2\pi \frac{\text{Sin}(2\pi t)}{2\pi t} \text{Cos}(2\pi(100)t) * \frac{2}{2}$$

$$x(t) = \pi * 2 \text{Sinc}(2t) * \text{Cos}(2\pi(100)t)$$

$$X(f) = \pi \Pi\left(\frac{f}{2}\right) * \frac{1}{2} [\delta(f-100) + \delta(f+100)]$$

$$X(f) = \frac{\pi}{2} \left[ \Pi\left(\frac{f-100}{2}\right) + \Pi\left(\frac{f+100}{2}\right) \right]$$

$$\hat{X}(f) = -j \text{Sgn}(f) * \frac{\pi}{2} \left[ \Pi\left(\frac{f-100}{2}\right) + \Pi\left(\frac{f+100}{2}\right) \right]$$

$$\hat{X}(f) = \frac{\pi}{j2} \left[ \Pi\left(\frac{f+100}{2}\right) - \Pi\left(\frac{f-100}{2}\right) \right]$$

$$\hat{x}(t) = \frac{\pi}{j2} \left[ 2\text{Sinc}(2t) e^{j2\pi(100)t} - 2\text{Sinc}(2t) e^{-j2\pi(100)t} \right]$$

$$\hat{x}(t) = \frac{\pi}{j2} * 2 \text{Sinc}(2t) \left[ e^{j2\pi(100)t} - e^{-j2\pi(100)t} \right]$$

$$\hat{x}(t) = 2\pi \text{Sinc}(2t) \text{Sin}(2\pi(100)t)$$

$$\hat{x}(t) = 2\pi \frac{\text{Sin}(2\pi t)}{2\pi t} \text{Sin}(200\pi t)$$

$$\therefore \hat{x}(t) = \frac{\text{Sin}(2\pi t)}{t} \text{Sin}(200\pi t)$$

$\therefore$  If  $x(t) = F(t) \text{Cos}(\omega_0 t) \Rightarrow \hat{x}(t) = F(t) \text{Sin}(\omega_0 t)$

### 8. Correlation Functions:

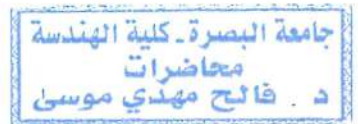
#### - Correlation of Energy Signals:

##### [A] Autocorrelation Function $R_{xx}(\tau)$ :

This function measures the similarity between a signal  $x(t)$  and its delayed version  $x(t-\tau)$ . For real signals:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) x(t-\tau) dt$$

It is clear that:



$$R_{xx}(\tau) = x(\tau) * x(-\tau)$$

$$F.T[R_{xx}(\tau)] = F.T[x(\tau) * x(-\tau)]$$

$$F.T[R_{xx}(\tau)] = F.T[x(\tau)] \cdot F.T[x(-\tau)]$$

$$F.T[R_{xx}(\tau)] = X(f) \cdot X(-f)$$

$$F.T[R_{xx}(\tau)] = |X(f)|^2$$

##### [B] Cross correlation Function $R_{xy}(\tau)$ :

It measures the similarity between one signal and a delayed version of another signal  $y(t-\tau)$ . For real signals:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y(t-\tau) dt$$

$$\therefore R_{xy}(\tau) = x(\tau) * y(-\tau)$$

$$F.T[R_{xy}(\tau)] = F.T[x(\tau)] \cdot F.T[y(-\tau)]$$

$$\therefore F.T[R_{xy}(\tau)] = X(f) \cdot Y(-f)$$

Note that:

$$R_{xx}(\tau) = R_{xx}(-\tau) \rightarrow \text{even function}$$

$$R_{xy}(\tau) = R_{yx}(\tau)$$