$$P = \sum_{n=-\infty}^{\infty} \underbrace{\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j2\pi n f_0 t}\, dt}_{D_n}\, D_n^*$$

$$P = \sum_{n=-\infty}^{\infty} D_n D_n^*$$

$$\therefore \boxed{P = \sum_{n=-\infty}^{\infty} |D_n|^2}$$

-Rayleigh's Theorem:

It is analogous to parseval's theorem. It is proposed for calculating the energy of the non-periodic signals from the signal spectrum.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt$$

$$E = \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi f t} df \right] dt$$

$$E = \int_{-\infty}^{\infty} X^*(f) \left[\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}}_{F.T \text{ of } x(t)} dt \right] df$$

$$E = \int_{-\infty}^{\infty} X^*(f)\, X(f)\, df$$

$$\therefore \boxed{E = \int_{-\infty}^{\infty} |X(f)|^2 df}$$

Example: Find the energy of the following signal, then draw its magnitude and phase spectra.

$$x(t) = 5\, \mathrm{Sinc}(10t - 2)$$

Sol.

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$, but the integration of the sinc function is too hard.

$\therefore$ Apply Rayleigh's Theorem:

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Now, let's find F.T of $x(t)$.

$$X(f) = F.T[5\, \mathrm{Sinc}(10t - 2)]$$

This signal is a time shifted sinc functions, so the duality ① property and the time ② shifting property will applied to find $X(f)$.

$$X(f) = F.T\left[A\tau \, \text{sinc}(\tau(t - t_0))\right]$$

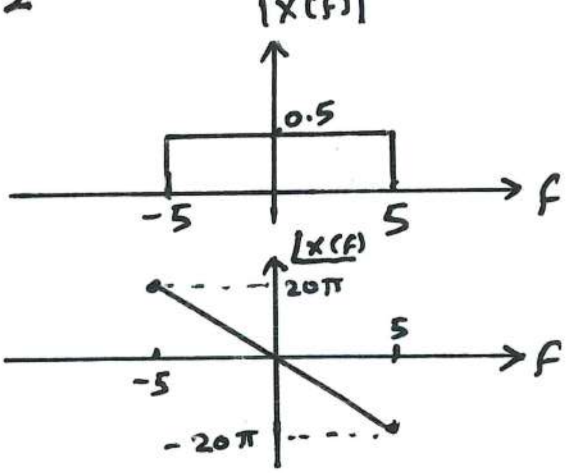
$$\therefore \; X(f) = \frac{1}{10} \; F.T\left[5 \times 10 \, \text{Sinc}\left(10\left(t - \frac{2}{10}\right)\right)\right]$$

Remember that $F.T\left[A\tau \, \text{Sinc}(\tau t)\right] = A\Pi\left(\frac{-f}{\tau}\right)$ {even function}

$$= A\,\Pi\left(\frac{f}{\tau}\right)$$

$$\therefore \; X(f) = \frac{5}{10} \, \Pi\left(\frac{f}{10}\right) e^{-j2\pi f \times 0.2}$$

$$X(f) = 0.5 \, \Pi\left(\frac{f}{10}\right) e^{-j2\pi f(0.2)}$$



$$\therefore \; E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

$$= \int_{-5}^{5} (0.5)^2 \, df$$

$$= 0.25 \; f \Big|_{-5}^{5}$$

$$\therefore \; \boxed{E = 2.5\,\text{J}}$$

Example: Find the spectrum of the following signal:

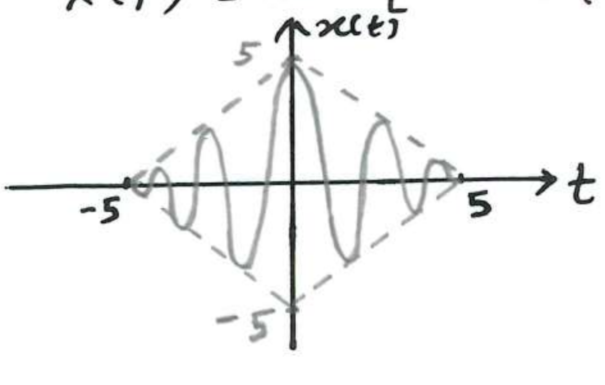
$$x(t) = 5 \, \Lambda\left(\frac{t}{5}\right) \cos(2\pi \times 10^3 t)$$

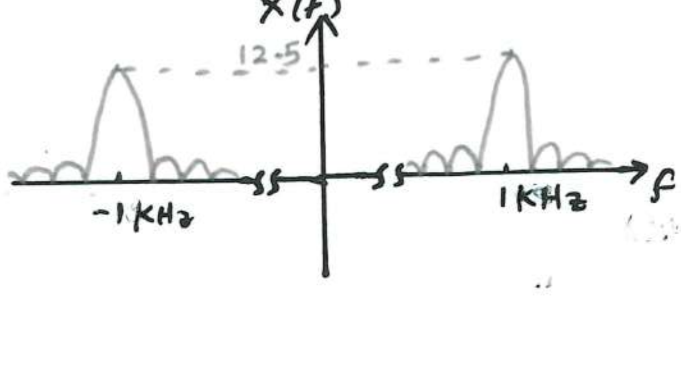
Sol.

$$x(t) = \frac{5}{2} \Lambda\left(\frac{t}{5}\right)\left[e^{j2\pi \times 10^3 t} + e^{-j2\pi \times 10^3 t}\right]$$

$$\therefore \; X(f) = \frac{5 \times 5}{2}\left[\text{Sinc}^2(5(f - 10^3)) + \text{Sinc}^2(5(f + 10^3))\right]$$

$$X(f) = 12.5\left[\text{Sinc}^2(5(f - 10^3)) + \text{Sinc}^2(5(f + 10^3))\right]$$





4. Convolution:

The convolution of two signals $x_1(t)$ and $x_2(t)$ is a new function of time $x(t)$, written as:

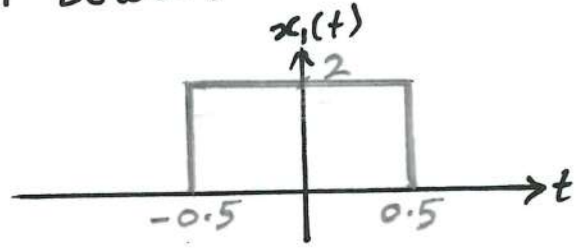
$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

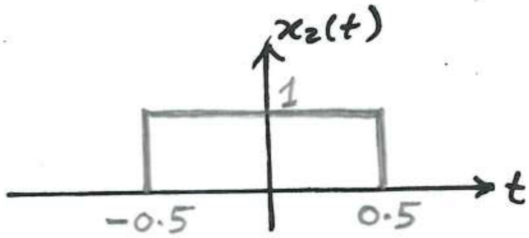
Note that the new function $x(t)$ is still depend on (t), while the variable (τ) is a <u>dummy variable</u> used to perform the integration.

- steps to perform the convolution:

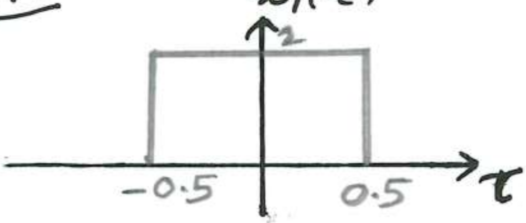
1) Use the dummy variable (τ) instead of (t), $[x_1(\tau), x_2(\tau)]$
2) Do time reversal for one of the signal $x_2(-\tau)$.
3) Do time shifting to the reversed Signal $x_2(t-\tau)$.
4) Multiply $x_1(\tau)$ and $x_2(t-\tau)$, then do integration to find the intersection area between $x_1(\tau)$ and $x_2(t-\tau)$ at that value of (t).

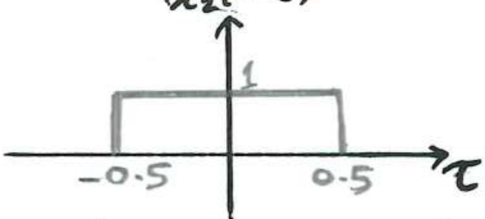
Example: Find the convolution of the two square pulses shown below:

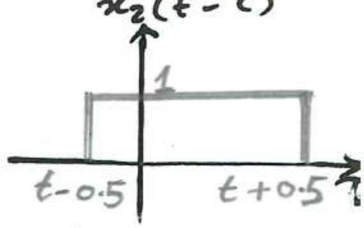




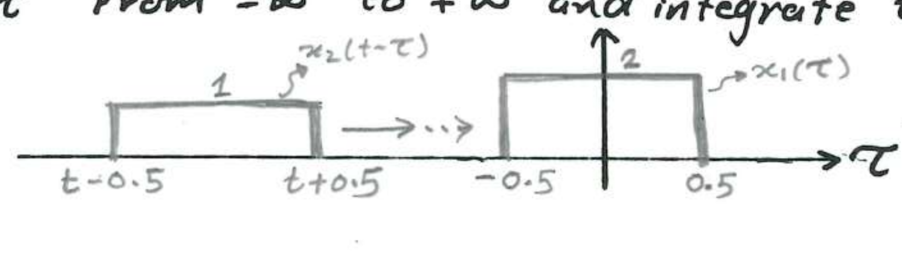
Sol.







Now change t from $-\infty$ to $+\infty$ and integrate the intersection.



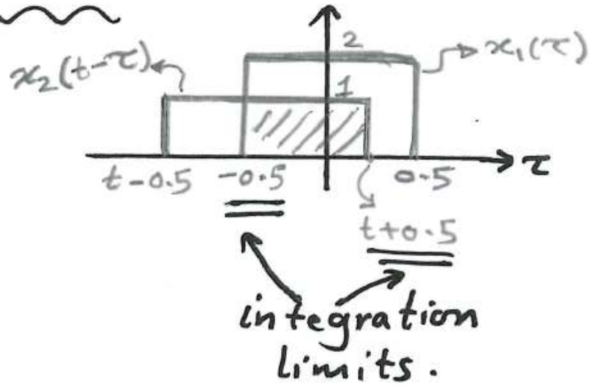
1,, $-\infty < (t+0.5) \leq 0.5 \quad \{-\infty < t \leq -1\}$:

There is no overlap area between $x_1(\tau)$ and $x_2(t-\tau)$, so

$$x(t) = x_1(t) * x_2(t) = 0 \quad \text{for } \infty < t \leq -1.$$

2,, $-0.5 < (t+0.5) \leq 0.5 \quad \{-1 < t \leq 0\}$:

Integrate the overlap area.



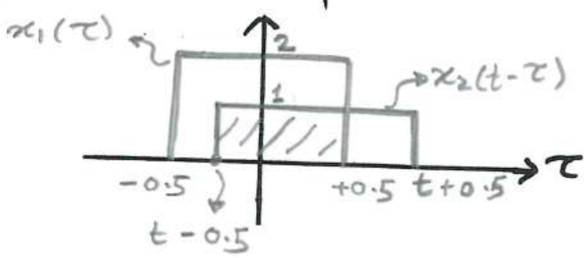
$$x(t) = x_1(t) * x_2(t)$$

$$= \int_{-0.5}^{t+0.5} 2 \times 1 \cdot d\tau = 2\,\tau \Big|_{-0.5}^{t+0.5}$$

$$\therefore \quad x(t) = 2(1+t) \quad \text{for } -1 < t \leq 0.5$$

3,, $(t+0.5) > 0.5$ & $(t-0.5) \leq 0.5 \quad \{0 < t \leq 1\}$:

$$x(t) = \int_{t-0.5}^{0.5} 2 \times 1 \; d\tau$$



$$x(t) = 2\,\tau \Big|_{t-0.5}^{0.5}$$

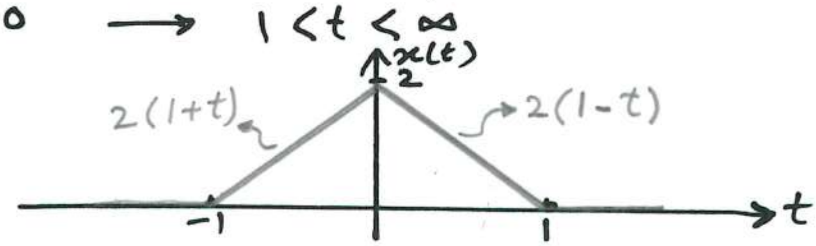
$$\therefore \quad x(t) = 2(1-t) \quad \text{for } 0 < t \leq 1$$

4,, $(t-0.5) > 0.5 \quad \{1 < t < \infty\}$:

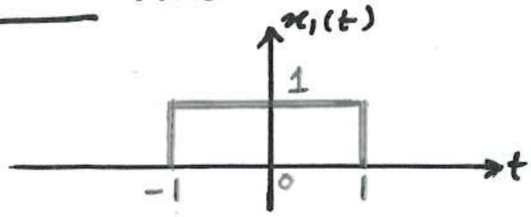
No overlaped area, so

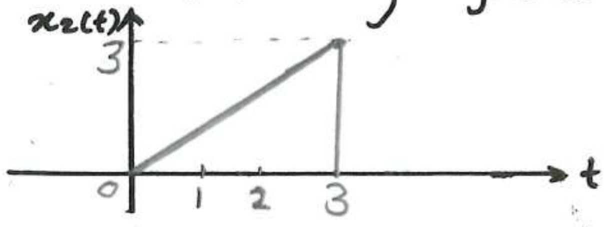
$$x(t) = x_1(t) * x_2(t) = 0 \quad \text{for } 1 < t < \infty$$

$$\therefore \quad x(t) = \begin{cases} 0 & \rightarrow -\infty < t \leq -1 \\ 2(1+t) & \rightarrow -1 < t \leq 0 \\ 2(1-t) & \rightarrow 0 < t \leq 1 \\ 0 & \rightarrow 1 < t < \infty \end{cases}$$

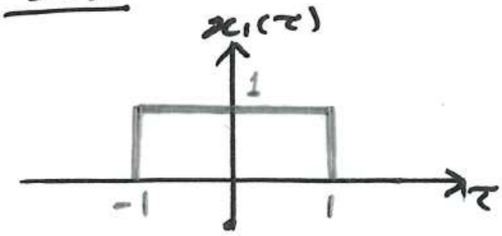


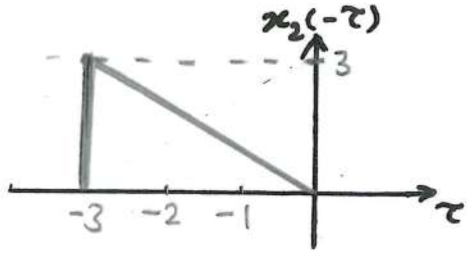
Example: Find the convolution of the following signals:

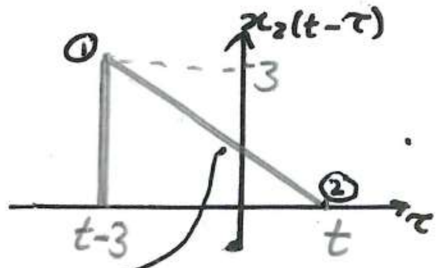




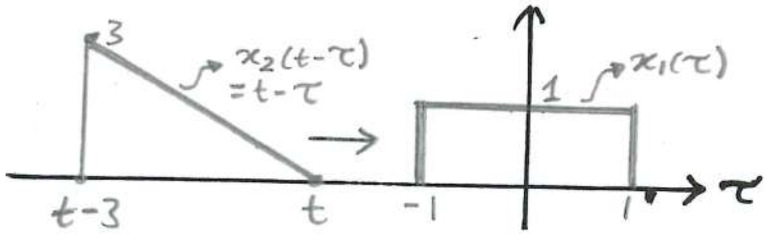
Sol.







then move $x_2(t-\tau)$, $-\infty < t < \infty$



$$\text{Slop} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-3}{t-(t-3)} = -1$$

$$(y - y_1) = \text{slop}\,(x - x_1)$$

$$x_2(t-\tau) - (+3) = -1(\tau - (t-3))$$

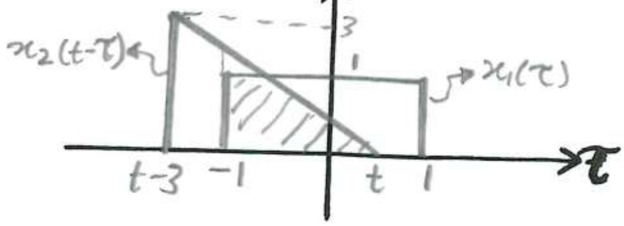
$$\therefore\ x_2(t-\tau) = (t-\tau)$$

1// $\infty < t < -1$:

There is no overlap, so

$$x(t) = x_1(t) * x_2(t) = 0$$

2// $-1 < t < 1$:



$$\therefore\ x(t) = \int_{-1}^{t} 1 \cdot (t-\tau)\, d\tau$$

$$x(t) = \int_{-1}^{t} t\, d\tau - \int_{-1}^{t} \tau\, d\tau$$

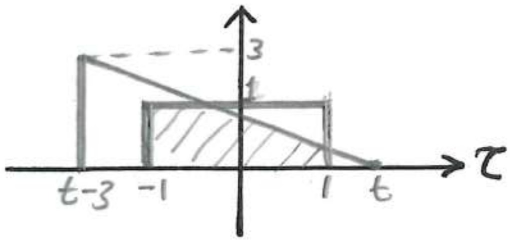
$$x(t) = t\ \tau \Big|_{-1}^{t} - \frac{1}{2}\ \tau^2 \Big|_{-1}^{t} = t(t+1) - \frac{1}{2}(t^2-1)$$

$$x(t) = t(t+1) - \frac{1}{2}(t+1)(t-1) = (t+1)\left[t - \frac{1}{2}(t-1)\right]$$

$$x(t) = (t+1)\left[\frac{1}{2}t + \frac{1}{2}\right] = (t+1)\left[\frac{t+1}{2}\right]$$

$$x(t) = \frac{(t+1)^2}{2} \qquad \text{for } -1 < t < 1$$

3. $t > 1$ & $t-3 \leq -1$ $\{1 < t \leq 2\}$:



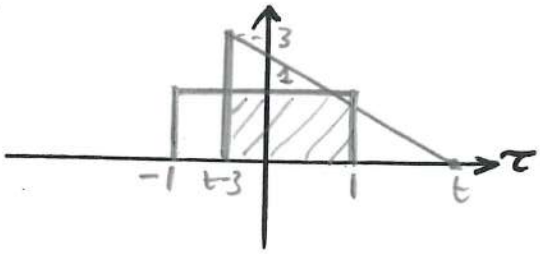
$$x(t) = \int_{-1}^{1} 1 \cdot (t-\tau)\, d\tau$$

$$x(t) = \int_{-1}^{1} t\, d\tau - \int_{-1}^{1} \tau\, d\tau$$

$$x(t) = t\, \tau \Big|_{-1}^{1} - \frac{1}{2}\tau^2 \Big|_{-1}^{1}$$

$$\therefore\ x(t) = 2t \qquad \text{for } 1 < t \leq 2$$

4. $t-3 > -1$ & $t-3 \leq 1$ $\{2 < t \leq 4\}$:



$$x(t) = \int_{t-3}^{1} 1 \cdot (t-\tau)\, d\tau$$

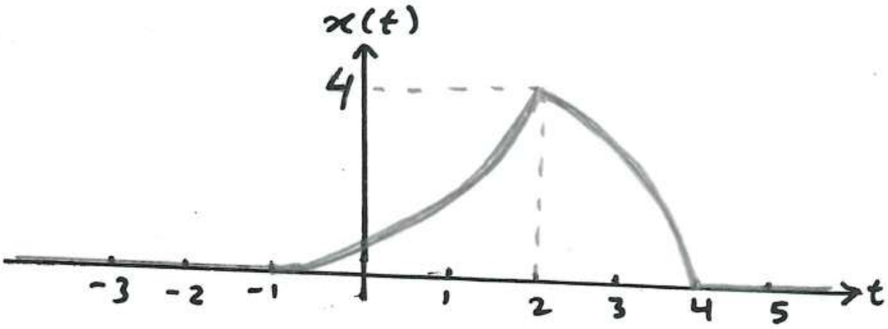
$$x(t) = t \cdot \tau \Big|_{t-3}^{1} - \frac{1}{2}\tau^2 \Big|_{t-3}^{1}$$

$$x(t) = 4 + t - \frac{1}{2}t^2 \qquad \text{for } 2 < t \leq 4$$

5. $t-3 > 1$ $\{4 < t < \infty\}$:

There is no overlap, so

$$x(t) = x_1(t) * x_2(t) = 0 \qquad \text{for } 4 < t < \infty$$



Properties of Convolution:

a) Commutative Law:

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

b) Distributive Law:

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t)$$

c) Associative Law:

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t)$$

d) Time Convolution Theorem:

Convolution in time domain $\longleftrightarrow$ multiplicat in freq. domain

$$F.T\{x_1(t) * x_2(t)\} = X_1(f)\, X_2(f)$$

e) Frequency Convolution Theorem:

Multiplication in time domain $\longleftrightarrow$ convolution in freq. domain

$$F.T\{x_1(t)\, x_2(t)\} = X_1(f) * X_2(f)$$

f) Convolution with $\delta(t)$:

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

The convolution with the impulse shifts the signal to where $\delta(t)$ exists.

<u>Example:</u> Find the spectrum of the following signal.

$$x(t) = 3\,\Pi\left(\frac{t}{4}\right) \cos(2\pi \times 10^4 t)$$

<u>Sol.</u>

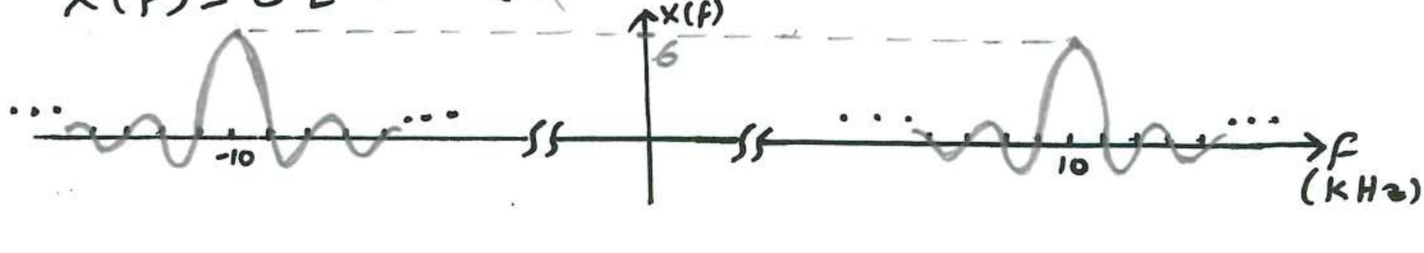
Let's use the convolution properties in solving this problem.

$$x(t) = \underbrace{3\,\Pi\left(\frac{t}{4}\right)}_{x_1(t)} \cdot \underbrace{\cos(2\pi\times 10^4 t)}_{x_2(t)}$$

$$X(f) = 12\,\text{Sinc}(4f) * \frac{1}{2}\left[\delta(f-10^4) + \delta(f+10^4)\right]$$

$$X(f) = 6\left[\text{Sinc}(4f) * \delta(f-10^4) + \text{Sinc}(4f) * \delta(f+10^4)\right]$$

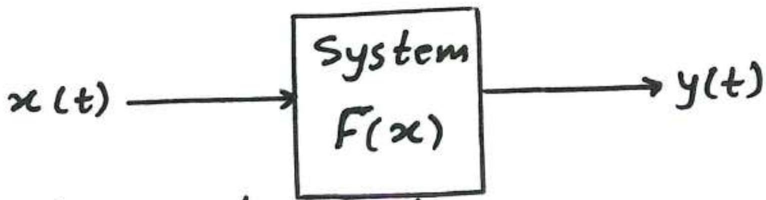
$$\therefore\; X(f) = 6\left[\text{Sinc}(4(f-10^4)) + \text{Sinc}(4(f+10^4))\right]$$



Signal Transmission and Filtering

1. System Representation:

A system is a mathematical model of a physical process that relates the input signal to the output signal (response).



$x(t)$ is the input signal.

$y(t)$ is the output signal (response).

$F(x)$ is the function that produces an output $y(t)$ from an input $x(t)$.

2. Linear Time-Invariant (LTI) Systems:

A Linear Systems:

Systems that satify the following two conditions:

1) Additivity:

$$y(t) = F[x_1(t) + x_2(t)] = F[x_1(t)] + F[x_2(t)]$$

$$y(t) = y_1(t) + y_2(t)$$

2) Homogeneity:

$$F[a\,x(t)] = a\,F[x(t)] = a\,y(t)$$

In other words;

$$y(t) = F[x(t)]$$

if $x(t) = a x_1(t) + b x_2(t)$

$$y(t) = F[a x_1(t) + b x_2(t)]$$

The Linear system should satisfy that

$$\boxed{y(t) = a\,F[x_1(t)] + b\,F[x_2(t)]}$$ super-position

B Time-Invariant System:

These systems satisfy the following condition:

if $\quad y(t) = F[x(t)]$

then $\quad F[x(t-t_0)] = y(t-t_0)$

* delay in input gives delay in output.
* The system that does not satisfy the above condition is called a Time-varying system.

* As a result, the (LTI) systems are Linear (1) and Time (2)-Invariant systems.

Example: Check whether the following system Linear and time invariant or not.

$$y(t) = a\,x(t) + b$$

where a and b are constant.

sol.

1) Linearity checking:

*Let $\quad y_1(t) = F[x_1(t)] = a\,x_1(t) + b$

$y_2(t) = F[x_2(t)] = a\,x_2(t) + b$

$\therefore \quad F[x_1(t)] + F[x_2(t)] = a[x_1(t) + x_2(t)] + 2b$

*Now, Let's assume $\quad x(t) = c_1 x_1(t) + c_2 x_2(t)$

$\therefore \quad F[c_1 x_1(t) + c_2 x_2(t)] = a[c_1 x_1(t) + c_2 x_2(t)] + b$

$\therefore \quad F[x_1(t) + x_2(t)] \neq c_1 F[x_1(t)] + c_2 F[x_2(t)]$

so the system is nonlinear.

2) Time-Variant checking:

* $y(t-t_0) = a\,x(t-t_0) + b$
* Let $x(t)$ is delayed by $t_0 \Rightarrow x(t-t_0)$

$\therefore \quad F[x(t-t_0)] = a\,x(t-t_0) + b = y(t-t_0)$

so the system is Time-Invariant.

Example: For the following systems, determine whether the system is LTI or not:

a) $y(t) = t\, x(t)$

b) $y(t) = x(t)\cos(\omega_0 t)$, c) $y(t) = 3\, x(t-4)$

Sol.

a) $y(t) = t\, x(t)$

* Linearity Checking:

- Let $y_1(t) = t\, x_1(t) = F[x_1(t)]$, $y_2(t) = t\, x_2(t) = F[x_2(t)]$

$$F[x_1(t)] + F[x_2(t)] = t\,(x_1(t) + x_2(t)) = t\,x_1(t) + t\,x_2(t)$$

- Let $x(t) = a_1 x_1(t) + a_2 x_2(t)$

$$F[a_1 x_1(t) + a_2 x_2(t)] = a_1\, t\, x_1(t) + a_2\, t\, x_2(t)$$

$$= a_1\, F[x_1] + a_2\, F[x_2]$$

$\therefore$ the system is Linear

* Time-Variant checking:

$$y(t-t_0) = (t-t_0)\, x(t-t_0)$$

Now Let's delay $x(t)$ by $t_0 \rightarrow x(t-t_0)$

$$F[x(t-t_0)] = t\, x(t-t_0) \neq y(t-t_0)$$

$\therefore$ the system is time-varying.

b) $y(t) = x(t)\cos(\omega_0 t)$

* Linearity Checking:

- Let $y_1(t) = x_1(t)\cos(\omega_0 t) = F[x_1]$

$$y_2(t) = x_2(t)\cos(\omega_0 t) = F[x_2]$$

$$F[x_1] + F[x_2] = x_1(t)\cos(\omega_0 t) + x_2(t)\cos(\omega_0 t)$$

- Let $x(t) = a_1 x_1(t) + a_2 x_2(t)$

$$\therefore F[a_1 x_1 + a_2 x_2] = a_1 x_1(t)\cos(\omega_0 t) + a_2 x_2(t)\cos(\omega_0 t)$$

$$= a_1\, F[x_1] + a_2\, F[x_2]$$

$\therefore$ The system is Linear.