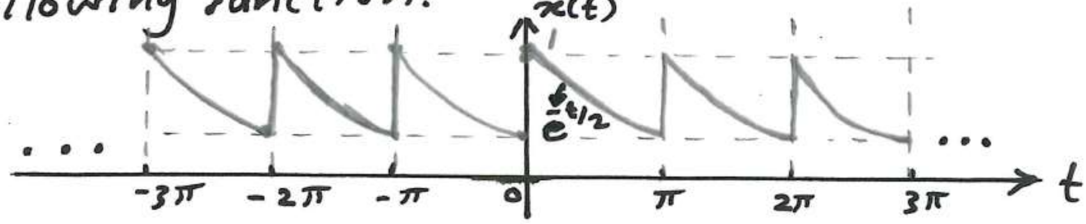
Example: Find the complex (exponential) Fourie series of the following function:



Sol.

$$T_0 = \pi \Rightarrow f_0 = \frac{1}{T_0} = \frac{1}{\pi}$$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi(nf_0)t}$$

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-j2\pi(nf_0)t}\, dt$$

$$D_n = \frac{1}{\pi}\int_0^{\pi} e^{-t/2} \cdot e^{-j2\pi(nf_0)t}\, dt, \quad \text{but } f_0 = \frac{1}{\pi}$$

$$D_n = \frac{1}{\pi}\int_0^{\pi} e^{-t/2} \cdot e^{-j2nt}\, dt = \frac{1}{\pi}\int_0^{\pi} e^{-(\frac{1}{2}+j2n)t}\, dt$$

$$D_n = \frac{-1}{\pi(\frac{1}{2}+j2n)}\left[e^{-(\frac{1}{2}+j2n)t}\right]_0^{\pi}$$

$$D_n = \frac{-2}{\pi(1+j4n)}\left[e^{-(\frac{1}{2}+j2n)\pi} - 1\right]$$

$$D_n = \frac{-2}{\pi(1+j4n)}\left[e^{-\frac{\pi}{2}}\, e^{-j2n\pi} - 1\right]$$

$\because$ n is integer $\Rightarrow e^{-j2\pi n} = 1 = \cos(2\pi n) - j\sin(2\pi n)$

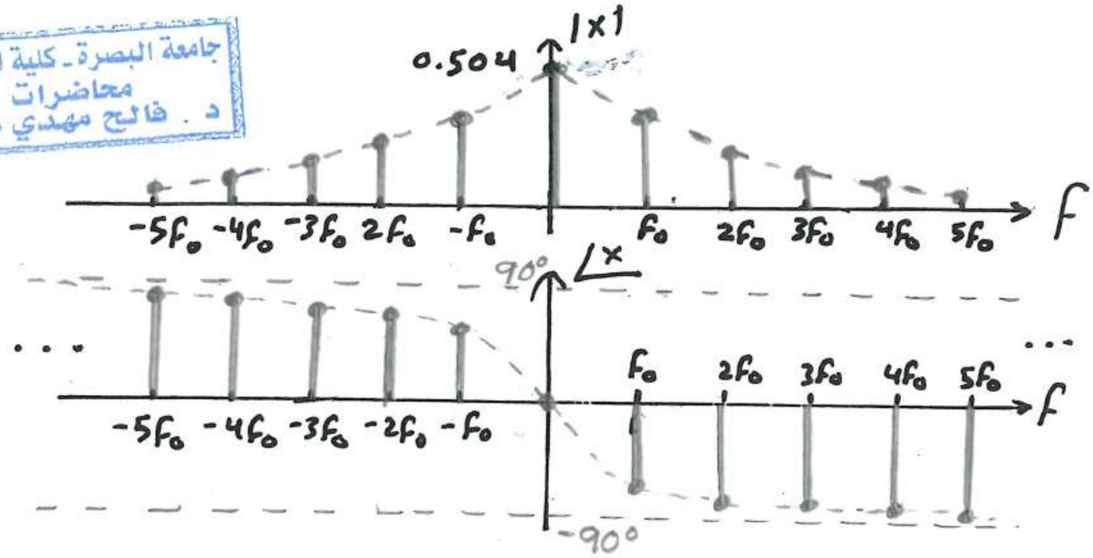
$$\therefore D_n = \frac{-2}{\pi(1+j4n)} \times -0.792$$

$$\therefore D_n = \frac{0.504}{1+j4n}$$

$$|D_n| = \frac{0.504}{\sqrt{1+16n^2}}$$

$$\angle D_n = -\tan^{-1}(4n)$$

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} \frac{0.504}{\sqrt{1+16n^2}}\, e^{-j\tan^{-1}(4n)}\, e^{j2\pi(nf_0)t}$$



2. Fourier Transform:

Fourier transform is a generalization of Fourier series for non-periodic signals. It converts the signal to the frequency domain.

$$F.T\,[x(t)] = \boxed{X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

$$\text{OR} \quad = \boxed{X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt}$$

The inverse Fourier transform of $X(f)$ (or $X(\omega)$) is:

$$F.T^{-1}[X(f)] = \boxed{x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df}$$

$$\text{OR} \quad \boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{+j\omega t}\, d\omega}$$

- Properties of Fourier Transform:

Fourier transform pair can be represented by:

$$x(t) \xleftrightarrow{F.T} X(f) \quad \text{or} \quad X(\omega)$$

1 Linearity:

$$a\,x_1(t) + b\,x_2(t) \xleftrightarrow{F.T} a\,X_1(f) + b\,X_2(f)$$

where a and b are constants

2 Time Scaling:

If $\quad x(t) \xleftrightarrow{F.T} X(f)$

then $\quad x(at) \xleftrightarrow{F.T} \frac{1}{|a|} X\left(\frac{f}{a}\right)$

OR $\frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

3 Time Shifting:

If $\quad x(t) \xleftrightarrow{F.T} X(f)$

then, $\quad x(t-t_o) \xleftrightarrow{F.T} X(f)\, e^{-j2\pi f t_o}$

OR $X(\omega)\, e^{-j\omega t_o}$

4 Frequency Shifting:

If $\quad x(t) \xleftrightarrow{F.T} X(f)$

then $\quad x(t)\, e^{+j2\pi f_o t} \xleftrightarrow{F.T} X(f-f_o)$ OR $x(t) e^{j\omega_o t} \longleftrightarrow X(\omega-\omega_o)$

$x(t)\, e^{-j2\pi f_o t} \xleftrightarrow{F.T} X(f+f_o)$

5 Duality:

If $\quad x(t) \xleftrightarrow{F.T} X(f)$ OR $x(t) \xleftrightarrow{F.T} X(\omega)$

$X(t) \xleftrightarrow{F.T} x(-f)$ OR $X(t) \xleftrightarrow{F.T} 2\pi\, x(-\omega)$

e.g: $\quad F.T\left[\Pi\left(\frac{t}{\tau}\right)\right] = \tau\, \mathrm{Sinc}(f\tau)$

$F.T^{-1}\left[\tau\, \mathrm{Sinc}(t\tau)\right] = \Pi\left(\frac{-f}{\tau}\right)$

since the gate function has an even symmetry,

$\Pi\left(\frac{-f}{\tau}\right) = \Pi\left(\frac{f}{\tau}\right)$

6 Differentiation:

If $\quad x(t) \xleftrightarrow{F.T} X(f)$, $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

$\frac{d}{dt}(x(t)) \xleftrightarrow{F.T} j2\pi f\, X(f)$, $\quad \frac{d}{dt}(x(t)) \xleftrightarrow{F.T} j\omega\, X(\omega)$

$\frac{d^n}{dt^n}(x(t)) \xleftrightarrow{F.T} (j2\pi f)^n X(f)$, $\quad \frac{d^n}{dt^n}(x(t)) \xleftrightarrow{F.T} (j\omega)^n X(\omega)$

6 Integration:

$$x(t) \xleftrightarrow{F.T} X(f)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F.T} \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0)\,\delta(f)$$

OR $\quad x(t) \xleftrightarrow{F.T} X(\omega)$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F.T} \frac{1}{j\omega} X(\omega) + \pi X(0)\,\delta(\omega)$$

7 Area Under the Signal and Spectrum:

If $\quad x(t) \xleftrightarrow{F.T} X(f)$

The DC component of the signal is:

$$DC = \int_{-\infty}^{\infty} x(t)\,dt$$

The DC component is $f=0$ component, so

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt .$$

3. Fourier Transform of Some Important Signals:

Note that, Fourier transform of the following signals should be memorized.

1 Impulse Function $\delta(t)$:

$$x(t) = \delta(t)$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\,dt = e^{-j2\pi f(0)}$$

{ see the properties of the delta function

$\therefore \quad X(f) = 1 \quad , \quad X(\omega) = 1$

$$\boxed{\delta(t) \xleftrightarrow{F.T} 1}$$

Using the duality property of Fourier transform:

$$x(t) = 1$$

$X(f) = \delta(-f)$ $\quad$ but $\delta(f)$ is even function

$$\therefore \quad X(f) = \delta(f)$$

$$\therefore \quad \boxed{1 \xleftrightarrow{F.T} \delta(f)}$$

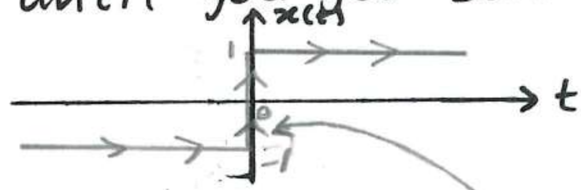
OR $\quad 1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)$

* Note that, the shorter the time domain duration, the wider the frequency domain coverage is and vice versa.

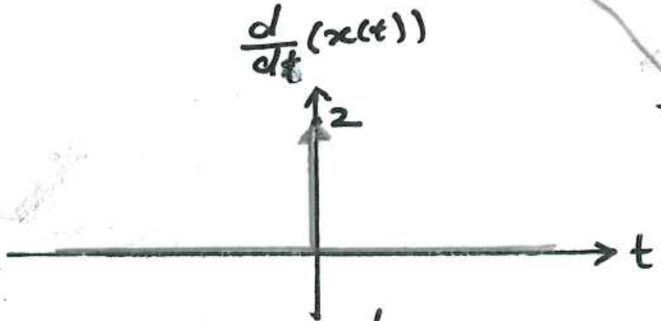
2 Signum Function Sgn(t):

Let's use the differentiation property to derive F.T of this function. In this kind of solution, diffrentiate the ploted signal until you get $\delta(t)$.

* The signal is starting from $-\infty$ and ending at ∞



* the sign of $\delta(t)$ depend on the direction of the arrow.



* The derivative of the discontinuity is $\delta(t)$ with an amplitude equal to the difference between the two levels

$$\therefore \frac{d}{dt} x(t) = 2\,\delta(t) \qquad \text{F.T for both sides}$$

$$F.T\left\{\frac{d}{dt} x(t)\right\} = F.T\,\{2\,\delta(t)\}$$

$$j2\pi f \; X(f) = 2 \times 1 \;\Rightarrow\; X(f) = \frac{1}{j\pi f}$$

$$\therefore \boxed{Sgn(t) \xleftrightarrow{F.T} \frac{1}{j\pi f}} \quad \text{OR} \quad Sgn(t) \xleftrightarrow{F.T} \frac{2}{j\omega}$$

3 Unit step Function (u(t)):

Since, $u(t) = \frac{1}{2}\left[sgn(t) + 1\right]$

$$F.T\left[u(t)\right] = \frac{1}{2}\left(F.T\left[Sgn(t)\right] + F.T\left[1\right]\right)$$

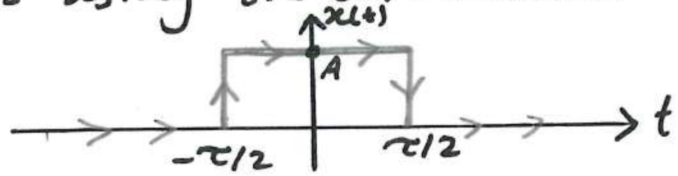
$$F.T\left[u(t)\right] = \frac{1}{2}\left(\frac{1}{j\pi f} + \delta(f)\right)$$

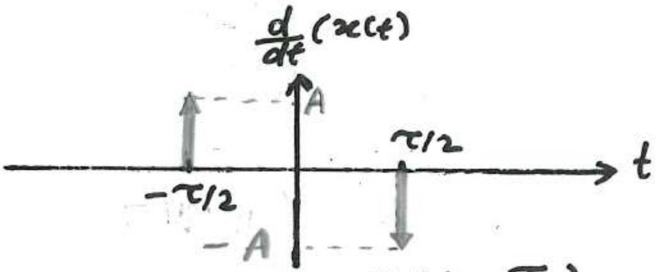
$$\therefore \boxed{u(t) \xleftrightarrow{F.T} \left(\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)\right)} \quad \text{OR} \quad u(t) \xleftrightarrow{F.T} \left(\frac{1}{j\omega} + \pi\,\delta(\omega)\right)$$

4 Rectangular (Gate) Function $\Pi(\frac{t}{\tau})$:

Let's derive it using the differentiation property.

$x(t) = A\,\Pi(\frac{t}{\tau})$





$$\frac{d}{dt}[x(t)] = A\delta(t+\frac{\tau}{2}) - A\delta(t-\frac{\tau}{2})$$, F.T for both sides

$$j2\pi f\, X(f) = F.T\,[A\delta(t+\frac{\tau}{2})] - F.T\,[A\delta(t-\frac{\tau}{2})]$$

Time Shifting property.

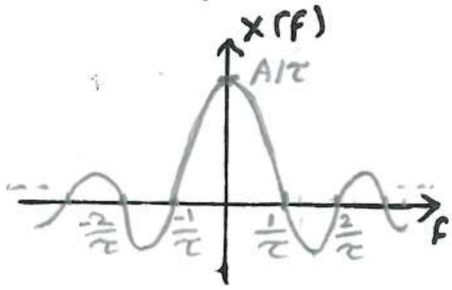
$$j2\pi f\, X(f) = A \times e^{j2\pi f\frac{\tau}{2}} - A \times e^{-j2\pi f\frac{\tau}{2}}$$

$$X(f) = \frac{A}{\pi f} \cdot \frac{e^{j\pi f\tau} - e^{-j\pi f\tau}}{j2}$$

$$X(f) = A\,\frac{\sin \pi f\tau}{\pi f} \times \frac{\tau}{\tau} = A\tau\,\frac{\sin \pi f\tau}{\pi f\tau}$$

$$X(f) = A\tau\,\mathrm{Sinc}(f\tau)$$

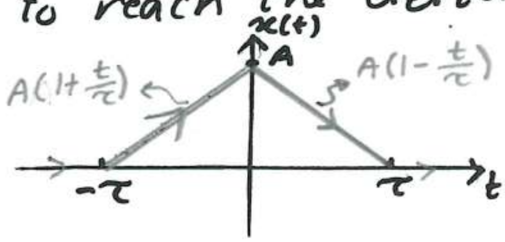
$$\therefore \quad \boxed{A\Pi(\frac{t}{\tau}) \xleftrightarrow{F.T} A\tau\,\mathrm{Sinc}(f\tau)}$$

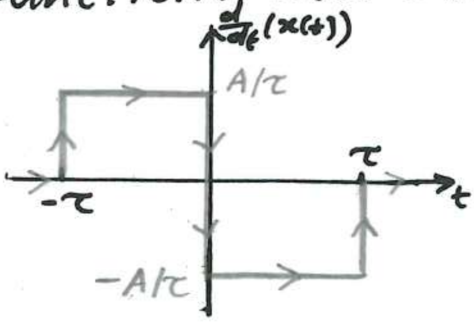


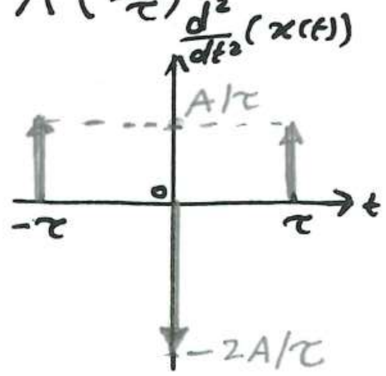
OR $\quad A\Pi(\frac{t}{\tau}) \xleftrightarrow{F.T} A\tau\,\mathrm{Sinc}(\omega\frac{\tau}{2})$

5 Triangular Function $\Lambda(\frac{t}{\tau})$:

The triangular function requires the second derivative to reach the delta function. $x(t) = A\,\Lambda(\frac{t}{\tau})$







$$\frac{d^2}{dt^2}[x(t)] = \frac{A}{\tau}\delta(t+\tau) - 2\frac{A}{\tau}\delta(t) + \frac{A}{\tau}\delta(t-\tau)$$

$$\frac{d^2}{dt^2}[x(t)] = -2\frac{A}{\tau}\delta(t) + \frac{A}{\tau}\delta(t+\tau) + \frac{A}{\tau}\delta(t-\tau)$$

By taking F.T for both sides

$$(j2\pi f)^2 X(f) = -2\frac{A}{\tau} + \frac{A}{\tau}e^{j2\pi f\tau} + \frac{A}{\tau}e^{-j2\pi f\tau}$$

$$-(2\pi f)^2 X(f) = -2\frac{A}{\tau}\left[1 - \tfrac{1}{2}\left(e^{j2\pi f\tau} + e^{-j2\pi f\tau}\right)\right]$$

$$(2\pi f)^2 X(f) = 2\frac{A}{\tau}\left[1 - \cos(2\pi f\tau)\right]$$

using the identity $\frac{1}{2}[1-\cos(2\theta)] = \sin^2\theta$

$$(2\pi f)^2 X(f) = 4\frac{A}{\tau}\sin^2(\pi f\tau)$$

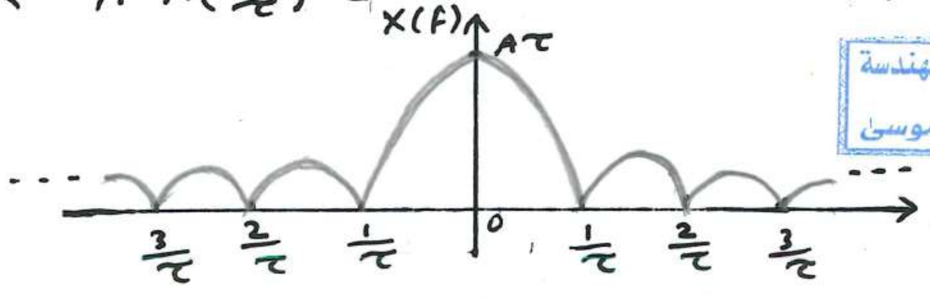
$$X(f) = \cancel{4}\frac{A}{\tau}\frac{\sin^2(\pi f\tau)}{\cancel{2^2}(\pi f)^2} \times \frac{\tau^2}{\tau^2}$$

$$X(f) = A\tau\frac{\sin^2(\pi f\tau)}{(\pi f\tau)^2} = A\tau\left(\frac{\sin(\pi f\tau)}{\pi f\tau}\right)^2$$

$$\therefore\ X(f) = A\tau\,\mathrm{Sinc}^2(f\tau)$$

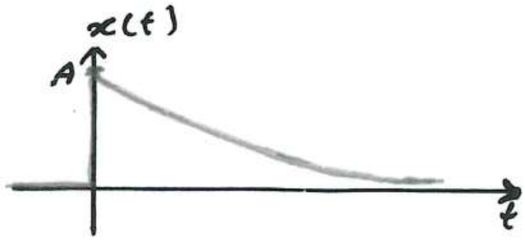
$$\therefore\ \boxed{A\,\Lambda\left(\frac{t}{\tau}\right) \xleftrightarrow{F.T} A\tau\,\mathrm{Sinc}^2(f\tau)}$$

OR $\quad A\,\Lambda\left(\frac{t}{\tau}\right) \xleftrightarrow{F.T} A\tau\,\mathrm{Sinc}^2\left(\omega\frac{\tau}{2}\right)$



[6] Single Sided Exponential:

$$x(t) = A\,e^{-at}\,u(t) \qquad \text{where } a>0$$



$$X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$$

$$X(f) = \int_{0}^{\infty} A e^{-at}\,e^{-j2\pi ft}\,dt$$

$$X(f) = A \int_0^{\infty} e^{-(a+j2\pi f)t}\, dt$$

$$X(f) = A \frac{-1}{a+j2\pi f} \; e^{-(a+j2\pi f)t} \Big|_0^{\infty}$$

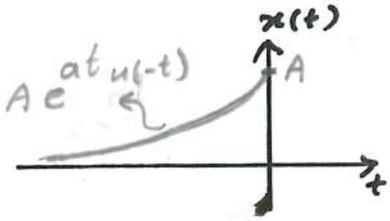
$$X(f) = \frac{-A}{a+j2\pi f} [0 - 1] \Rightarrow X(f) = \frac{A}{a+j2\pi f}$$

$$\therefore \boxed{A\, e^{-at} u(t) \xleftrightarrow{F.T} \frac{A}{a+j2\pi f}}$$

OR $$A\, e^{-at} u(t) \xleftrightarrow{F.T} \frac{A}{a+j\omega}$$

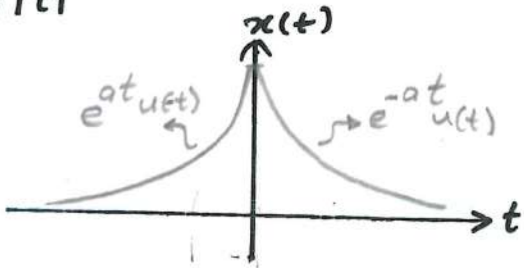
It can be proved that:

$$A\, e^{+at} u(-t) \xleftrightarrow{F.T} \frac{A}{a-j2\pi f}$$



<u>Example:</u> Find F.T of the double-sided exponential:

$$x(t) = e^{-a|t|}$$



<u>Sol.</u>

$$e^{-a|t|} = e^{-at} u(t) + e^{+at} u(-t)$$

$$\therefore X(f) = \frac{1}{a+j2\pi f} + \frac{1}{a-j2\pi f}$$

$$X(f) = \frac{a - j2\pi f + a + j2\pi f}{a^2 + (2\pi f)^2}$$

$$\therefore \quad X(f) = \frac{2a}{a^2 + (2\pi f)^2}$$

7 Sinusoidal Signal:

Let's investigate F.T of $e^{j2\pi f_0 t}$ at first.

$$x(t) = e^{j2\pi f_0 t}$$

$$x(t) = 1 * e^{j2\pi f_0 t}$$

$$X(f) = F.T\left[1 * e^{j2\pi f_0 t}\right]$$ frequency shifting property

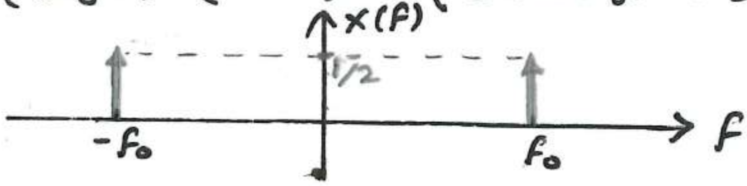
$$\therefore X(f) = \delta(f - f_0)$$

This concept can be generalized to $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$.

$$x(t) = \cos(2\pi f_0 t) = \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

$$\therefore \quad \boxed{\cos(2\pi f_0 t) \xleftrightarrow{F.T} \frac{1}{2}\left(\delta(f-f_0) + \delta(f+f_0)\right)}$$

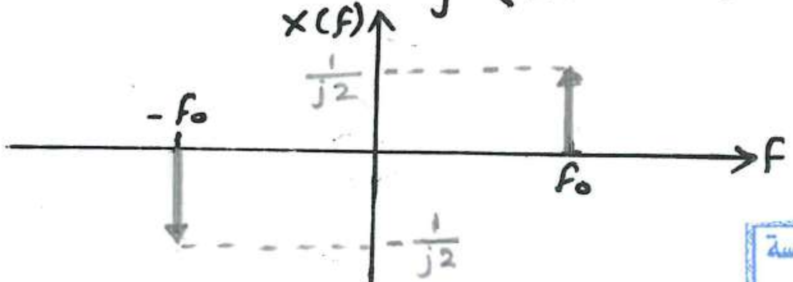
OR $$\cos(\omega_0 t) \xleftrightarrow{F.T} \pi\left(\delta(\omega-\omega_0) + \delta(\omega-\omega_0)\right)$$



$$x(t) = \sin(2\pi f_0 t) = \frac{1}{j2}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)$$

$$\boxed{\sin(2\pi f_0 t) \xleftrightarrow{F.T} \frac{1}{j2}\left(\delta(f-f_0) - \delta(f+f_0)\right)}$$

OR $$\sin(\omega_0 t) \xleftrightarrow{F.T} \frac{\pi}{j}\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$$



8 Periodic Signals:

* Periodic Signals can be represented by Fourier series.
* Let the periodic signal $x_p(t)$ with period (T_0), so

$$f_0 = \frac{1}{T_0}$$

$$x_p(t) = \sum_{n=-\infty}^{\infty} D_n\, e^{j2\pi n f_0 t}$$

Take F.T for both sides

$$X_p(f) = F.T\left[\sum_{n=-\infty}^{\infty} D_n\, e^{j2\pi n f_0 t}\right]$$

n is an integer number, and D_n is time independent, so F.T does not effect D_n and the summation.

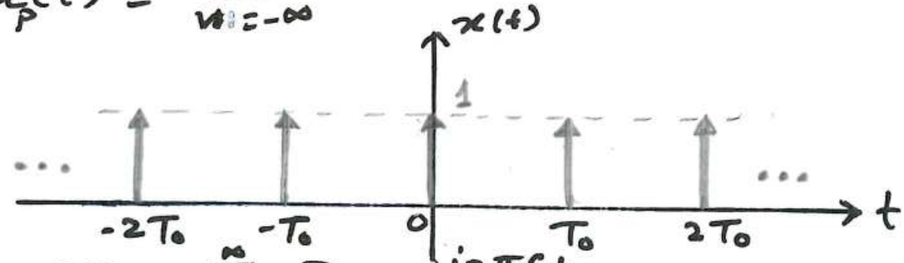
$$X_p(f) = \sum_{n=-\infty}^{\infty} D_n\left[F.T\left[e^{j2\pi n f_0 t}\right]\right]$$

$$\therefore \quad \boxed{X_p(f) = \sum_{n=-\infty}^{\infty} D_n \, \delta(f - nf_0)}$$

Example: Find Fourier transform of the following periodic signal:

$$x_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

Sol.



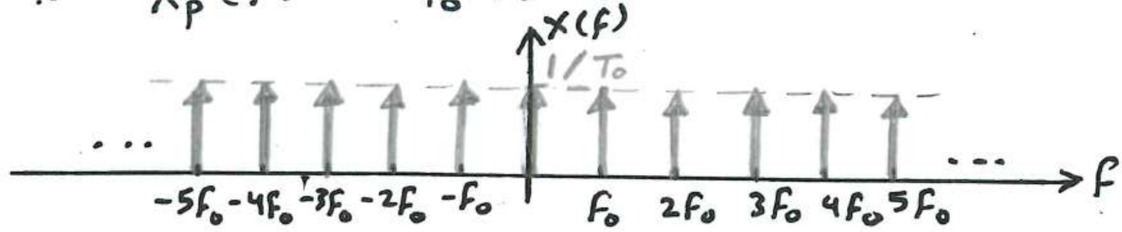
$$x_p(t) = \sum_{n=-\infty}^{\infty} D_n \, e^{j2\pi nf_0 t}$$

$$D_n = \frac{1}{T_0}\int_0^{T_0} \delta(t)\, e^{-j2\pi nf_0 t}\, dt$$

$$\therefore \quad D_n = \frac{1}{T_0}$$

$$\therefore \quad x_p(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{j2\pi nf_0 t}$$

$$\therefore \quad X_p(f) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$



3. Parseval's and Rayleigh's Theorem:

- Parseval's Theorem: The average power of a periodic signal is the sum of the powers in the phasor components of its Fourier series.

$$x(t) = \sum_{n=-\infty}^{\infty} D_n \, e^{j2\pi nf_0 t}$$

$$P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} |x(t)|^2 dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, x^*(t)\, dt$$

$$x^*(t) = \sum_{n=-\infty}^{\infty} D_n^* \, e^{-j2\pi nf_0 t}$$

$$P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) \sum_{n=-\infty}^{\infty} D_n^* \, e^{-j2\pi nf_0 t}\, dt$$