If A and B are nonempty subset of X we define $d(A,B) = \sup_{x \in A} d(x,B)$.

Obviously $0 \leq d(A,B) < \infty$. Let A, B, C be nonempty subset of X. let $x \in A$ and $\epsilon > 0$. Choose $z \in C \ni d(x,z) < d(x,C) + \epsilon$, Now

$$d(x,B) \leq d(x,z) + d(z,B) \leq d(x,C) + d(z,B) + \epsilon.$$

since $z \in C$ we have $d(z,B) \leq d(C,B)$. Therefore

$$d(x,B) \leq d(x,C) + d(C,B).$$

Taking the supremum over $x \in A$, we obtain

$$d(A,B) \leq d(A,C) + d(C,B).$$

Now we define Hausdorff metric H by

$$H(A,B) = \max\{d(A,B), d(B,A)\}.$$

ie $$H(A,B) = \max\left\{\sup_{x \in A} d(x,B), \sup_{y \in B} d(y,A)\right\}$$

Remark: ① $H(A,B)$ is a Complete metric space if (X,d) is Complete.

② The distance between two sets (call them X and Y) is to define distance as the Larger of the following two quantities. The maximum distance from all points in X to Their nearest point in Y, and the maximum distance from all points in Y to Their nearest point in X.

Definition Multivalued function maps points in a vector space say X to a subset of another vector space Y ie $T : X \longrightarrow 2^Y$.

We say that $x^* \in X$ is a fixed point of $T: X \longrightarrow 2^X$ if $x^* \in Tx^*$.

Existence of fixed points for multivalued function was studid by Kakutani (1941) in finite dimension it has found important application in Control theory, game Theory and Mathematical economic.

Definition: let (X, d) be a metric space and $F: X \longrightarrow CB(X)$, F is said to be multivalued contraction mapping $\Leftrightarrow H(Fx, Fy) \leq k\, d(x, y)$ $\forall x, y \in X$, $0 \leq k < 1$ where $CB(X)$ denote the set of all nonempty closed and bounded subset of X.

Remark (1) let $A, B \in CB(X)$ and $a \in A$ if $\epsilon > 0$. Then $\exists\, b \in B \ni d(a, b) \leq H(A, B) + \epsilon$.

i.e. $d(a, b) \leq p\, H(A, B)$ where $p > 1$

(2) If $A, B \in CB(X)$, $CB(X)$ = The set of all nonempty compact subset of X. Then

$$d(a, b) \leq H(A, B).$$

Example let $X = [0, 1]$ and $f: X \longrightarrow X \ni$

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & \text{if } 0 \leq x < \frac{1}{2} \\ -\frac{1}{2}x + 1 & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

define $F: X \longrightarrow 2^X$ by $F(x) = \left\{ \{0\} \cup \{f(x)\} \right\}$ $\forall x \in X$. Then F is a multivalued contraction mapping and the set of all fixed points of F is $\{0, 1, 2/3\}$. because if $x \in F(x) \Longrightarrow$

$x \in \{0\} \cup \{f(x)\} \Rightarrow x = 0$ or $x \in f(x) = \frac{1}{2}x + \frac{1}{2}$ if

$0 \leq x < \frac{1}{2} \Rightarrow x = 1$

$x \in f(x) = -\frac{1}{2}x + 1$ if $\frac{1}{2} \leq x \leq 1 \Rightarrow x = 2/3$

Nadler's fixed point theorem: Let (X, d) be a complete metric space if $F: X \longrightarrow CB(X)$ is a multivalued contraction mapping Then F has a fixed point.

proof: We may assume $K > 0$ and choose $x_0 \in X$, let $x_1 \in Fx_0$ since $Fx_0, Fx_1 \in CB(X)$

$d(x_1, x_2) \leq \rho H(Fx_0, Fx_1) \leq \rho K\, d(x_0, x_1)$ where $\rho = \frac{1}{\sqrt{K}}$. We get $x_{i+1} \in Fx_i \ni$

$$d(x_i, x_{i+1}) \leq \rho H(Fx_{i-1}, Fx_i)$$
$$\leq (\rho K)^i d(x_0, x_1) \quad \forall i \geq 1$$

$$d(x_i, x_{i+j}) \leq d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + \dots + d(x_{i+j-1}, x_{i+j})$$
$$\leq \left[(\rho K)^i + (\rho K)^{i+1} + \dots + (\rho K)^{i+j-1} \right] d(x_0, x_1)$$
$$\leq (\rho K)^i \frac{(1 - (\rho K)^i)}{1 - \rho K} d(x_0, x_1)$$

as $\rho K = \frac{1}{\sqrt{K}} K = \sqrt{K} < 1$