to the integral Equation. (15)

The map use is $T : C[a,b] \to C[a,b]$

$T(f) = g$ is given by

$$T(f)(x) = g(x) = y_0 + \int_{x_0}^{x} \phi(t, f(t))\, dt,$$

any solution of the original differential system is a solution to the integral equation which is turn in a fixed point of the map T. It is important to realize that this T may or may not be a contraction. Our theorem, will be apply in the case that T is a contraction so we search for conditions on T ensuring this. assume that ϕ satisfy a Lipschitz condition in the second variable i.e, we assume that $\exists$ a positive $k \ni |\phi(x, y_1) - \phi(x, y_2)| \leq k|y_1 - y_2|$ $\forall x \in [a,b]$ and $\forall y_1, y_2 \in \mathbb{R}$. For $f_1, f_2 \in C[a,b]$ and $x \in [a,b]$ we have,

$$|(Tf_1)(x) - (Tf_2)(x)| = \left| \int_{x_0}^{x} [\phi(t, f_1(t)) - \phi(t, f_2(t))]\, dt \right|$$

$$\leq \int_{x_0}^{x} |\phi(t, f_1(t)) - \phi(t, f_2(t))|\, dt$$

$$\leq \int_{x_0}^{x} K\, |f_1(t) - f_2(t)|\, dt$$

$$\leq \int_{x_0}^{x} K\, d(f_1, f_2)\, dt \leq K(b-a)\, d(f_1, f_2)$$

$$\forall x \in [a,b].$$

$d(Tf_1, Tf_2) \leq K(b-a)\, d(f_1, f_2$

$\Rightarrow$ T is a contraction so long as $K(b-a) < 1$.

Exercise: Consider the system

$$\frac{dx}{dt} = x + t = \varphi(x, t) \quad , \quad x(0) = 0$$

First verify that conditions of Banach contraction mapping. Then use Picard's iteration method to show that $f(t) = e^{t} - t - 1$ is the solution to the system

To do this choose some convenient f_0, define $f_{n+1} = T f_n$ and show that $f_1, f_2, \ldots$ converge to $f(t) = e^{t} - t - 1$.

Multi-valued function

Introduction: Let X be a metric space. If B is a nonempty subset of X we define

$$d(x, B) = \inf_{y \in B} d(x, y)$$

Now To show that $|d(x, B) - d(y, B)| \leq d(x, y$

let $x, y \in X$ if $\epsilon > 0$ choose $z \in B \ni$ $d(y, z) < d(y, B) + \epsilon$. Then $d(x, B) \leq d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + d(y, B) + \epsilon$

Thus we conclude $|d(x, B) - d(y, B)| \leq d(x, y)$.

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