which is a Contradiction.

Now to show T has at most one fixed point

let $x^* = Tx^*$ and $y^* = Ty^*$ Then

$d(Tx^*, Ty^*) = d(x^*, y^*) < d(x^*, y^*) \Rightarrow$

$x^* = y^*$

(H.W): In each of the following Cases, Say whether the sequence $\{a_n\}$ in the metric space (X, d) is ⓐ bounded ⓑ Cauchy ⓒ convergent

(i) $X = \mathbb{C}$ (usual metric), $a_n = e^{in}$

(ii) $X = \mathbb{R}$ (usual metric), $a_n = b_n c_n$ where (b_n) is bounded and (c_n) is Cauchy.

<u>Theorem</u>: Let (X, d) be a complete metric space and $T: X \longrightarrow X$ be a mapping for which $\exists$ a real numbers α, β and γ satisfying $0 \le \alpha < 1$, $0 \le \beta < \frac{1}{2}$ and $\gamma < \frac{1}{2}$ $\ni$ $\forall x, y \in X$ at least one of the following is true:

(Z_1) $d(Tx, Ty) \le \alpha\, d(x, y)$,

(Z_2) $d(Tx, Ty) \le \beta[d(x, Tx) + d(y, Ty)]$

(Z_3) $d(Tx, Ty) \le \gamma[d(x, Ty) + d(y, Tx)]$.

Then T is a picard operator.

[illegible] Zamfirescu mapping.

proof: let $x, y \in X$. At least one of z_1, z_2 or z_3 is true if z_2 holds, then

$$d(Tx,Ty) \le \beta[d(x,Tx) + d(y,Ty)] \le$$

$$\beta[d(x,Tx) + (d(y,x) + d(x,Tx) + d(Tx,Ty))]$$

$$(1-\beta)\, d(Tx,Ty) \le 2\beta\, d(x,Tx) + \beta\, d(x,y)$$

$$\Rightarrow d(Tx,Ty) \le \frac{2\beta}{1-\beta} d(x,Tx) + \frac{\beta}{1-\beta} d(x,y).$$

z_3 holds, Then similarly we get

$$d(Tx,Ty) \le \frac{2\gamma}{1-\gamma} d(x,Tx) + \frac{\gamma}{1-\gamma} d(x,y)$$

Therefore, denoting $\delta = \max\{\frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}, \alpha\}$

we have $0 < \delta < 1$ and Then, for all $x, y \in X$, the following inequality

$$d(Tx,Ty) \le 2\delta\, d(x,Tx) + \delta\, d(x,y) \quad \ldots (*)$$

we will show That T has a unique fixed point

Let $x_0 \in X$ be arbitrary and $\{x_n\}$ be a sequence

$x_n = Tx_{n-1} = \ldots = T^n x_0 \quad n = 0,1,2,\ldots$

The picard iteration associated to T. Then, by

$(*)$ $d(x_{n+1}, x_n) \le \delta\, d(x_n, x_{n-1})$. the rest

of the proof is similar to that of (fixed point Th).

Example (1) Find the roots of $x^2 - 2x - 3 = 0$

Solution: There are several ways to write this into the form $x = f(x)$ and we simply start with some initial guess x_0 and apply the iteration formula $x_n = f(x_{n-1})$

$n = 1, 2, \ldots$

Consider the following $x = \sqrt{2x+3}$ .... (1)

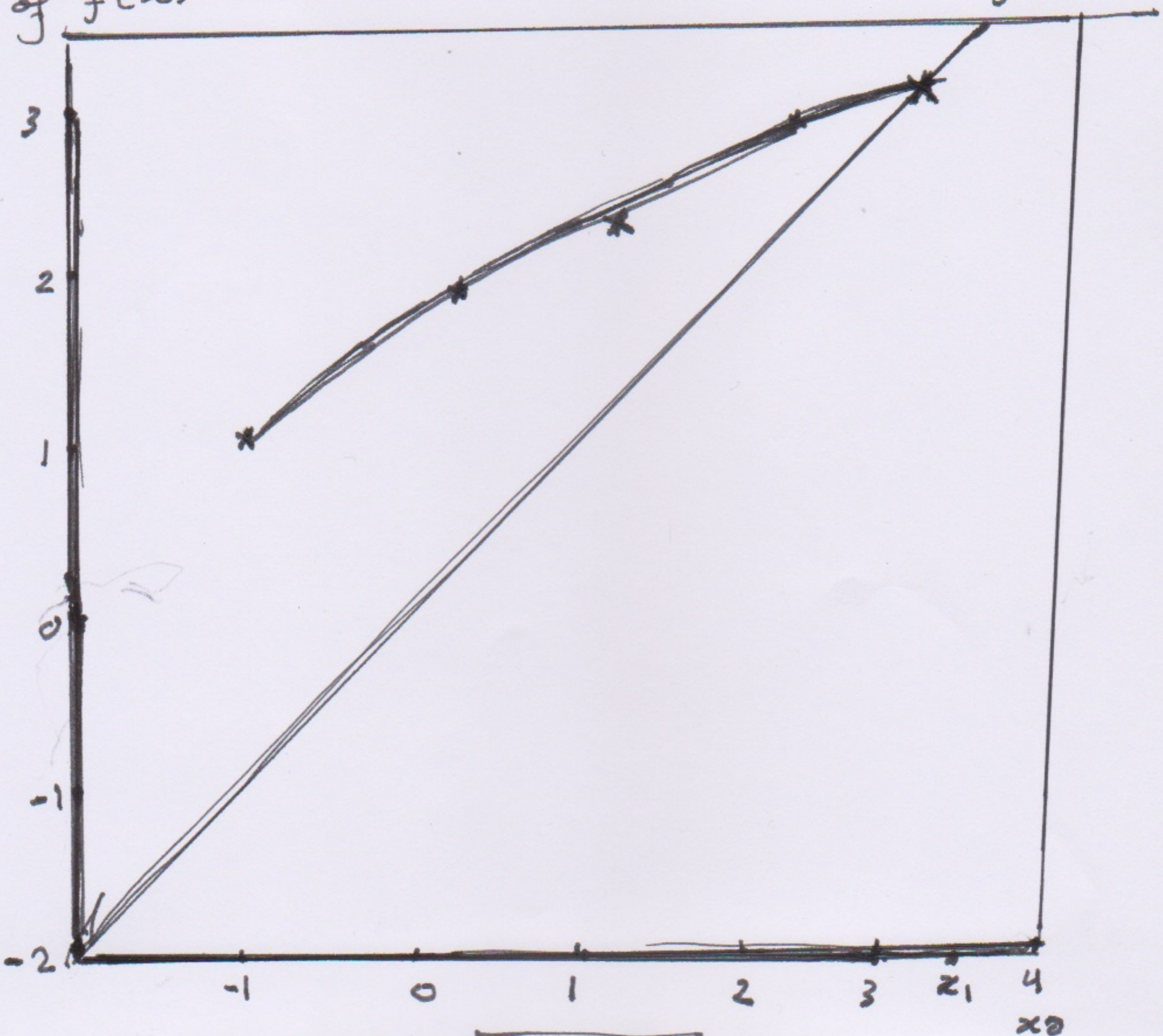
Now suppose we start with initial guess $x_0 = 4$ and iterate using $x_n = \sqrt{2x_{n-1}+3}$ which come from (1) we ge the following sequence of iterates $x_1 = \sqrt{2x_0+3} = \sqrt{2(4)+3} = \sqrt{11} = 3.32$

$x_2 = f(x_1) = f(\sqrt{11}) = f(3.32) = \sqrt{2x_1+3} = \sqrt{2(3.32)+3}$

$= \sqrt{9.63325} = 3.1$

$x_3 = \sqrt{2(3.1)+3} = 3.01144$, $x_4 =$ etc ....

The sequence converges to $x=3$. Observe that the two exact roots for the given problem are at $x=-1$ and $x=3$. The figure below shows a plot of $f(x)$



$f(x) = \sqrt{2x+3}$