③ The study of dynamical systems often reduce to finding fixed points of iterates of a vector field $f: \mathbb{R}^n \to \mathbb{R}^n$, ie $f^n(x) = x$ $(n \in \mathbb{N})$

④ Questions about the existence and uniqueness of solutions to initial value problems of the form $\dot{x}(t) = (f(t, x(t))$, $x(0) = x_0$ are answed in the affirmative by Banach fixed point theorem.

<u>Theorem</u>: Let $(X, d)$ be a complete metric space and $T: X \to X$ be a mapping for which $\exists\, a \in (0, \frac{1}{2})$ $\ni$ $d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$ ....(*) $\forall\, x, y \in X$. Then $T$ has a unique fixed point.

<u>proof</u>:- let $x_0 \in X$ and $x_n = Tx_{n+1}$ be picard iteration then by (*) $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$

$$\leq a[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]$$

$$= a[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$\Rightarrow d(x_n, x_{n+1}) \leq \frac{a\, d(x_{n-1}, x_n)}{1-a}$$

since $0 < \frac{a}{1-a} < 1$ for $a \in (0, \frac{1}{2})$

we deduce in a similar manner to that proof (Theorem of fixed point) that $\{x_n\}$ is a Cauchy sequence, and hence a convergent sequence $x_0$. let $x^* \in X$ be its limit then we have

$$d(x^*, Tx^*) \leq d(x^*, x_n) + d(x_n, Tx^*)$$

$$\leq d(x^*, x_n) + a\left[ d(Tx_{n-1}, x_{n-1}) + d(x^*, Tx^*)\right]$$

$$\leq \frac{1}{1-a} d(x^*, x_n) + \frac{a}{1-a} d(x_{n-1}, x_n)$$

$$\leq \frac{1}{1-a} d(x^*, x_n) + \left(\frac{a}{1-a}\right)^n d(x_0, x_1)$$

Now letting $n \to \infty$, we obtain $d(x^*, Tx^*) = 0 \Rightarrow x^* = Tx^*$, and therefore $x^*$ is unique fixed point of $T$.

Example ① let $X = R$ and $T: X \to X$, $T(x) = 0$ if $x \in (-\infty, 2]$ and $T(x) = -\frac{1}{2}$ if $x > 2$ Then show that

(i) $T$ is not continuous

(ii) $T$ satisfy (K) with $a = \frac{1}{5}$

(iii) $T$ has a unique fixed point.

solution (H-W)

② use the Contraction Theorem to show how to construct a real sequence converging to the solution of $x^4 - 3x + 1 = 0$ in $[0.3, 0.4]$

write the equation as $x = \frac{x^4+1}{3}$ and define $I = [0.3, 0.4] \ni T(x) = \frac{x^4+1}{3}$ for $x \in I$.

Then $|T'(x)| = \left|\frac{4x^3}{3}\right| = \frac{4}{3}(0.4)^3 < 1$ on $I$

$\Rightarrow T: I \to I$ is a contraction, and $I$ with the usual induced metric is is a Complete metric space. Hence by the contraction mapping Theorem $T$ has a unique fixed point $x^* \in I$ and the sequence $x_0 = 0.3$, $x_{n+1} = Tx_n \ \forall n \in \mathbb{N}$ converge to $x^*$.

(3) show by counter example that the Contraction Theorem ~~is~~ false in general for non-Complete metric spaces.

Solution:: let $T(x) = x^2$, $X = (0, \frac{1}{3}]$ with the usual induced metric Then $T$ is a contraction on $X$ but with no fixed point in $X$.

(4) Give an example of a Complete metric space $(X, d)$ and a map $T: X \to X$ satisfying $d(Tx, Ty) < d(x, y) \ \forall x, y \in X$, $x \neq y$ (contractive condition)) but which has no fixed point. Can such a map have more than one fixed point?

Solution let $X = [0, \infty)$ with the usual metric $d$. Let $T: X \to X$ be defined by $T(x) = x + \frac{1}{1+x} \ \forall x \in X$

Then $T$ satisfy the Contractive condition.

since $|T'(x)| = \left|1 - \frac{1}{(1+x)^2}\right| < 1 \ \forall x \in X$.

but $T$ has no fixed point in $X$ since

$x^* = Tx^* = x^* + \frac{1}{1+x^*} \implies \frac{1}{1+x^*} = 0$