Fixed Point Theorems

Definition: Let M be a non-empty set and $T: M \to M$ be a self map (single valued map). We say that $x \in M$ is a fixed point of T if $T(x) = x$.

Examples: (1) let $T: R \to R$ and $T(x) = x^2 - x$

solution: $T(x) = x \Rightarrow x^2 - x = x \Rightarrow x^2 - 2x = 0$

$x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$ (non Unique solution).

(2) let $T: R \to R$ and $T(x) = x^2 + 5x + 4$

$\Rightarrow T(x) = x = x^2 + 5x + 4 \Rightarrow x^2 + 4x + 4 = 0$

$\Rightarrow (x+2)^2 = 0 \Rightarrow x = -2$ (Unique solution)

(3) let $T: R \to R$ and $T(x) = x + 2 \Rightarrow$

The set of all fixed points of T equal ϕ

(no solution)

We study here condition under which:

(1) A given mapping T has a fixed point,

(2) The fixed point is unique and it can be obtained through an iterative process.

Note that every fixed point of a mapping $T: X \to X$ is also a fixed point of T^n for every positive integer n.

Example let $X = (0, \frac{1}{4}]$ with d the Euclidean metric and $f: X \to X$ given by $f(x) = x^3$. Then f is a contraction mapping and that f has no fixed point.

(H.W) ① Let $X = [1, \infty)$ with d the Euclidean metric and $T : X \longrightarrow X$ be given by $T(x) = x + \frac{1}{x}$. Show that for any distinct $x, y \in X$ $d(Tx, Ty) < d(x, y)$ and that f has no fixed point in X.

we note that T is not Contraction but T is contractive which is weaker than a contraction mapping.

② Let $X = [1, 2] \cap \mathbb{Q}$ with the Euclidean metric and Let $f : X \longrightarrow X$ be the mapping define by $f(x) = -\frac{1}{4}(x^2 - 2) + x$. Prove That f is a Contraction mapping and f has no fixed point in X.

③ let $f : [a, b] \longrightarrow [a, b]$ be differentiable over $[a, b]$. Show that f is a Contraction mapping $\Leftrightarrow$ $\exists$ a number $K < 1 \ni \forall x \in (a, b)$ $|f'(x)| \leq K < 1$

The next result is know as Banach's contraction principle, being due to Banach (1922).

Theorem (Fixed point theorem).

let (X, d) be a complete metric space and $T : X \longrightarrow X$ a Contraction mapping. Then T has a unique fixed point p, and This fixed point Can obtain as The Limit of every sequence generated by the iteration $x_{n+1} = T(x_n)$, x_0 an arbitrary element of X.

<u>Theorem</u>:- If X is a Complete metric space and $T: X \longrightarrow X$ is a mapping such that T^n is contraction for some positive integer n, Then T has a unique fixed point.

<u>proof</u> let $p \in X$ be the unique fixed point of T^n Then, $T(p) = T(T^n(p)) = T^{n+1}(p) = T^n(Tp)$
$\Rightarrow T(p)$ is a fixed point of T^n, by uniquness, we have $p = T(p)$ i.e p is also a fixed point of T. To prove the uniquness of the fixed point of T, let q be some other fixed point of T. Then $q = T(q) = T(Tq) = \cdots = T^n(q)$ i.e q is also a fixed point of T^n. Hence $p = q$.

<u>Remark</u>: As a final introunductory note, we show that by being able to solve the fixed point problem $f(x) = x$, we can solve other problems which donot immediately look like fixed point problems. For examples.

① If we seek a point $x \ni f(x) = 0$. We can find such a point by solving fixed point problem $g(x) = x$ where $g(x) = f(x) + x$

② If we seek a point $x \ni f(x) = y$. We can find such x by solving $g(x) = x$, where $g(x) = x + f(x) - y$.