<u>Metric spaces:</u> Let M be a nonempty set a mapping $d: M \times M \longrightarrow R^+$ is called metric or distance on M provided that

(d_1) $d(x,y) = 0 \Longleftrightarrow x = y$

(d_2) $d(x,y) = d(y,x) \quad \forall\, x, y \in M$

(d_3) $d(x,z) \leq d(x,y) + d(y,z) \quad \forall\, x,y,z \in M$

A set M with a metric d is called metric space and is denoted by (M, d).

<u>Example</u> (1) let $M = R$; $d(x,y) = |x-y| \quad \forall x,y \in R$ is a metric on R where $|\cdot|$ denotes the absolute value.

(2) let $M = R^n$; $d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$

$\forall\, x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in R^n$

is a metric on R^n, called the Euclidean metric. The next two mappings

$$\delta(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$\rho(x,y) = \max_{1 \leq i \leq n} |x_i - y_i| \quad \forall\, x,y \in R^n$$

are also metrics on R^n

(3) Let $M = \{ f : [a,b] \to \mathbb{R} \; ; \; f \text{ is continuous} \}$

we define $d : M \times M \longrightarrow \mathbb{R}^+$ by

$$d(f, g) = \max | f(x) - g(x) | \quad \forall \; f, g \in M$$

Then d is a metric on M (The Tschebysev metric). The metric (M, d) is usualy denoted by $C[a,b]$.

<u>Remark 1</u>- Two metric d_1 and d_2 are equivalent

if there exist two constants $w > 0$, $W > 0$ $\ni$

$$w \, d_1(x, y) \le d_2(x, y) \le W \, d_1(x, y) \quad \forall \, x, y \in M$$

<u>Definitions</u> (1) A sequence $\{x_n\}$ of points in a metric space (M, d) is said to converges to $p \in M$ and we write $x_n \to p$ as $n \to \infty$.

if $\forall \, \epsilon > 0$ there is an integer $N \ni d(x_n, p) < \epsilon$, whenever $n \geq N$. IF there is no such p in M. The sequence $\{x_n\}$ is said to diverge.

(2) A sequence $\{x_n\}$ is said to a Cauchy sequence if it is satisfies the following condition

$\forall \, \epsilon > 0 \; \exists$ an integer $N \ni d(x_n, x_m) < \epsilon$

Whenever $n \geq N$ and $m \geq N$

<u>Notethat</u> In a metric space any Convergent sequence is a Cauchy sequence too. but the reverse is not generaly true.

<u>Definition</u>: A metric space (M, d) is called Complete if any Cauchy sequence in M is convergent

<u>Examples</u> (1) The following are complete metric spaces $(R, |\cdot|)$, (R^n, d), (R^n, δ), (R^n, ρ) and $C[a, b]$.

(2) If Q denote the irrationals in R, Then $(Q, |\cdot|)$ is not a Complete metric space.

<u>Definition</u> let (M, d) be a metric space.

A mapping $T: M \longrightarrow M$ is called

(1) Lipschitzian if $\exists\ L > 0 \ni$
$d(Tx, Ty) \leq L\, d(x, y) \quad \forall\ x, y \in M$

(2) nonexpansive if $d(Tx, Ty) \leq d(x, y)$
$\forall\ x, y \in M$ ie $L = 1$

(3) Contractive if $d(Tx, Ty) < d(x, y)$
$\forall\ x, y \in M, x \neq y$

(4) Contraction if $d(Tx, Ty) \leq \alpha\, d(x, y)$
$\alpha \in (0, 1]$, $x, y \in M$

(5) Isometry if $d(Tx, Ty) = d(x, y)$
$\forall\ x, y \in M$

Note that Contraction $\Rightarrow$ Contractive $\Rightarrow$ nonexpansive $\Rightarrow$ Lipschitz and all such mapping are Continuous.