$$XT_t = kTX_{xx} \Rightarrow \frac{T_t}{kT} = \frac{X_{xx}}{X} \Rightarrow \frac{T_t}{kT} = \frac{X_{xx}}{X} = -\lambda \quad ,\lambda \textit{ is separationconstat}$$

Since u is not the trivial solution $u = 0$, it follows that $X(0) = X(l) = 0$

$$\Rightarrow \frac{d^2X}{dx^2} = -\lambda X,\ 0 < x < l \qquad with \quad X(0) = X(l) = 0$$ " called an *eigenvalue problem*"

$$\frac{dT}{dt} = -\lambda kT,\ t > 0$$

A nontrivial solution of this system is called an *eigenfunction* of the problem with an *eigenvalue* λ. its general solution of an *eigenvalue problem* (which depends on λ) has the following
form:

1- If $\lambda < 0$ then $X(x) = \alpha e^{\sqrt{-\lambda}x} + \beta e^{-\sqrt{-\lambda}x}$
2- If $\lambda = 0$ then $X(x) = \alpha + \beta x$
3- If $\lambda > 0$ then $X(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$

where α, β are arbitrary real numbers. The corresponding eigenfunctions are

$$X_n(x) = \sin(\frac{n\pi x}{l}) \quad ,\lambda = \left(\frac{n\pi}{l}\right)^2,\ n = 1,2,3,...$$

The general solution of second ODE has the form $T_n(t) = B_n e^{-k\left(\frac{n\pi}{l}\right)^2 t},\ n = 1,2,3,...$
The superposition principle implies that any linear combination

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{l}) e^{-k\left(\frac{n\pi}{l}\right)^2 t} \quad ,$$

of separated solutions is also a solution of the heat equation that satisfies the Dirichlet boundary conditions.

From the initial condition we have $f(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi x}{l})$,this is Fourier sine series; Thus ;its coefficient has the form $B_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx$.

Ex: in the above example if $k = 1, l = \pi, f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ \pi - x & \pi/2 \le x \le \pi \end{cases}$,then find $u(x,t)$.

Ex: solve the following problems

1- $u_{tt} - c^2 u_{xx} = 0 \quad 0 \le x \le l, t > 0$
$u(x,0) = f(x), u_t(x,0) = g(x) \quad 0 \le x \le l$
$u_x(0,t) = u_x(l,t) = 0, \quad t \ge 0$

2- $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0 \quad 0 \le r \le b, 0 \le \theta \le \pi$
$u(r,0) = 0, u(r,\pi) = 0 \quad 0 \le r \le b$
$u(1,\theta) = 0, u(b,\theta) = u_0 \quad 0 \le \theta \le \pi$

3-consider the displacement in a stretched string upon which an external force per unit length acts, parallel to the y axis. Let that the force be proportional to the distance from one end, and let the initial displacement and velocity be zero. Units for x and t can be chosen.

4- let $u(x,y,t)$ denote the transverse displacement at each point (x,y) at time t in a membrane stretched across a rigid square frame in the $xy-$plane. We can select the origin and the point (π,π) as the ends of the diagonal of the frame, and let that the membrane is released at rest with a given initial displacement $f(x,y)$ which is continuous and vanishes on the boundary of the square.

Laplace transformation: Laplace transformation for the function $u(x,t)$ with respect to t is $\{u(x,t)\} = \int_0^\infty u(x,t)e^{-st}dt = U(x,s), \quad t>0$

By using this definition ,we can found the Laplace transforms for $u(x,t)$ derivatives as;

$$\{u_t(x,t)\} = \int_0^\infty u_t(x,t)e^{-st}dt = sU(x,s) - u(x,0,$$

$$\{u_{tt}(x,t)\} = \int_0^\infty u_{tt}(x,t)e^{-st}dt = s^2U(x,s) - su(x,0) - u_t(x,0),$$

$$\vdots$$

Note: other derivatives in the PDE is transforms into ordinary differentional(e.g

$$\{u_x(x,t)\} = \int_0^\infty u_x(x,t)e^{-st}dt = \frac{dU(x,s)}{dx}, \quad \{u_{xx}(x,t)\} = \int_0^\infty u_{xx}(x,t)e^{-st}dt = \frac{d^2U(x,s)}{dx^2}, \cdots$$

Example : Solve the following problem

$$u_t - ku_{xx} = 0 \quad 0 \le x \le l, t>0$$

$$u(0,t) = u_0 \quad t \ge 0$$

$$u(x,0) = 0, \quad 0 \le x \le l \quad \text{and} \qquad u(x,t) \text{ bounded,}$$

Take Laplace transformation for both sides of PDE with respect to t ,we have $\frac{d^2U(x,s)}{dx^2} = \frac{1}{k}\{sU(x,s) - u(x,0)\} \Rightarrow \frac{d^2U}{dx^2} - \frac{s}{k}U = 0 \Rightarrow U(x,s) = \alpha(s)e^{x\sqrt{\frac{s}{k}}} + \beta(s)e^{-x\sqrt{\frac{s}{k}}}$

From $u(x,t)$ must be bounded $\Rightarrow \alpha(s) = 0$,thus; $U(x,s) = \beta(s)e^{-x\sqrt{\frac{s}{k}}}$

Furthermore, since $u(0,t) = u_0$,we have $U(0,s) = \int_0^\infty u_0 e^{-st}dt = \frac{u_0}{s}$. So, $U(0,s) = \beta(s)$

Solution of ODE is $U(x,s) = \frac{u_0}{s}e^{-x\sqrt{\frac{s}{k}}}$, by taking inverse Laplace transformation we have, $u(x,t) = u_0 \left\{ \frac{e^{-x\sqrt{\frac{s}{k}}}}{s} \right\} = u_0 \, erfc(\frac{x}{2\sqrt{kt}}) = u_0(1 - \int_0^{x/2\sqrt{kt}} e^{-\lambda^2}d\lambda) \quad (\, erfc = 1 - erf \,)$

Ex: Solve the following problem

$$u_t + u_x = x \quad (x,t>0)$$

$$u(x,0)=0, x>0$$
$$u(0,t)=0, t>0$$

Ex: If $u_{tt}=u_{xx}+u_{xxt} \quad (x,t>0)$,and the following data

$u_t(x,0)=0$, $u(0,t)=1$, and $u(x,t)\to 0$ as $x\to\infty$ are satisfying, then

$$u_x(0,t)=-\frac{1}{\sqrt{\pi t}}e^{-t}.$$

Fourier transformation: By using this definition of the complex form of the Fourier series, we can obtain Fourier transformation. Therefore, The Fourier transformation for the function $u(x,t)$ with respect to x is given by

$$F\{u(x,t)\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x,t)e^{-ipx}dx=U(p,t)$$

And the inverse has the form $$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}U(p,t)e^{-ipx}dP$$

By using this definition ,we can found the Fourier sine transforms and Fourier cosine transforms as;

$$F\{u(x,t)\}=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}u(x,t)\sin(px)dx=U_s(p,t)$$

And the inverse has the form $$u(x,t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}U_s(p,t)\sin(px)dP$$

$$F\{u(x,t)\}=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}u(x,t)\cos(px)dx=U_c(p,t)$$

And the inverse has the form $$u(x,t)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}U_c(p,t)\cos(px)dP$$

Also ,we can found the Fourier sine and cosine transforms for the derivatives as;

$$F\{u_x(x,t)\}=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}u_x(x,t)\sin(px)dx=-pU_c(p,t)$$

$$F\{u_x(x,t)\}=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}u_x(x,t)\cos(px)dx=-\sqrt{\frac{2}{\pi}}u(0,t)+pU_s(p,t)$$

$$F\{u_{xx}(x,t)\}=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}u_{xx}(x,t)\sin(px)dx=p\sqrt{\frac{2}{\pi}}u(0,t)-p^2U_s(p,t)$$

$$F\{u_{xx}(x,t)\}=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}u_{xx}(x,t)\cos(px)dx=-\sqrt{\frac{2}{\pi}}u_x(0,t)-p^2U_c(p,t)$$

Note: other derivatives in the PDE is transforms into ordinary differential (e.g

$$F\{u_t(x,t)\} = \frac{dU_s(p,t)}{dt}, \quad F\{u_{tt}(x,t)\} = \frac{d^2U_s(p,t)}{dt^2}, \cdots$$

$$F\{u_t(x,t)\} = \frac{dU_c(p,t)}{dt}, \quad F\{u_{tt}(x,t)\} = \frac{d^2U_c(p,t)}{dt^2}, \cdots$$

Ex: solve the following problem

$$u_t - ku_{xx} = 0 \quad 0 \le x \le l, t > 0$$

$$u(x,0) = 0 \quad x \ge 0$$

$$u(0,t) = a, \quad t > 0$$

$$u \,\&\, u_x \to 0, \quad x \to \infty,$$

Non-linear model of P.D.Es: