Write $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$. We claim that w is a solution of a second-order equation of the same type. Using the chain rule one finds that

$$
\begin{aligned}
u_x &= w_\xi \xi_x + w_\eta \eta_x, \\
u_y &= w_\xi \xi_y + w_\eta \eta_y, \\
u_{xx} &= w_{\xi\xi}\xi_x^2 + 2w_{\xi\eta}\xi_x\eta_x + w_{\eta\eta}\eta_x^2 + w_\xi \xi_{xx} + w_\eta \eta_{xx}, \\
u_{xy} &= w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + w_{\eta\eta}\eta_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}, \\
u_{yy} &= w_{\xi\xi}\xi_y^2 + 2w_{\xi\eta}\xi_y\eta_y + w_{\eta\eta}\eta_y^2 + w_\xi \xi_{yy} + w_\eta \eta_{yy},
\end{aligned}
$$

Substituting these formulas in (10), we see that w is satisfies the linear equation

$$\ell[w] = Aw_{\xi\xi} + Bw_{\xi\eta} + Cw_{\eta\eta} + Dw_\xi + Ew_\eta + Fw = G \qquad (11)$$

Where

$$
\begin{aligned}
A(\xi,\eta) &= a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \\
B(\xi,\eta) &= a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y, \\
C(\xi,\eta) &= a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2,
\end{aligned}
$$

Note: we do not need to compute the coefficients of the lower-order derivatives (D,E,F)

An elementary calculation shows that these coefficients satisfy the following matrix equation:

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix},$$

Denote by J the Jacobian of the transformation. Taking the determinant of the two sides of the above matrix equation, we find

$$-\delta(\ell) = AC - B^2 = J^2(ac - b^2) = -J^2\delta(L)$$

Therefore, the type of the equation is invariant under nonsingular transformations.

Definition 3.4 The *canonical form of hyperbolic, parabolic and elliptic equations are*

$$
\begin{aligned}
\ell[w] &= w_{\xi\eta} + \ell_1[w] = G(\xi,\eta) \\
\ell[w] &= w_{\xi\xi} + \ell_1[w] = G(\xi,\eta) \\
\ell[w] &= w_{\xi\xi} + w_{\eta\eta} + \ell_1[w] = G(\xi,\eta)
\end{aligned}
$$

where ℓ_1 is a first-order linear differential operator, and G is a function.

<u>Theorem</u>: *Suppose that (10) is hyperbolic in a domain D. There exists a coordinate system* (ξ, η) *in which the equation has the canonical form*

$$w_{\xi\eta} + \ell_1[w] = G(\xi,\eta)$$

where $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, ℓ_1 is a first-order linear differential operator, and G is a function *which depends on (10)*.

Proof:
Without loss of generality, we may assume that $a(x, y) \neq 0$ for all $(x, y) \in D$. We need to find two functions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ such that

$$A(\xi,\eta) = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$$

$$C(\xi,\eta) = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$$

The equation that was obtained for the function η is actually the same equation as for ξ; therefore, we need to solve only one equation. It is a first-order equation that is not quasilinear; but as a quadratic form in ξ it is possible to write it as a product of two linear terms

$$\frac{1}{a}[a\xi_x + (b - \sqrt{b^2 - ac})\xi_y][a\xi_x + (b + \sqrt{b^2 - ac})\xi_y] = 0$$

Therefore, we need to solve the following linear equations:

$$a\xi_x + (b - \sqrt{b^2 - ac})\xi_y = 0 \qquad (12\#)$$

$$a\xi_x + (b + \sqrt{b^2 - ac})\xi_y = 0 \qquad (13\#\#)$$

In order to obtain a nonsingular transformation ($\xi(x, y)$, $\eta(x, y)$) we choose ξ to be a solution of (12#) and η to be a solution of (13##).The characteristic equations for (#) are

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = b + \sqrt{b^2 - ac}, \quad \frac{d\xi}{dt} = 0$$

Therefore,ξ is constant on each characteristic. The characteristics are solutions of the equation

$$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a} \qquad (14\$)$$

The function η is constant on the characteristic determined by

$$\frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a} \qquad (15\$\$)$$

Definition : The solutions of ($) and ($$) are called the two families of the *characteristics* (or *characteristic projections*) of the equation $L[u] = g$.

Example: Consider the *Tricomi equation:*

$$u_{xx} + xu_{yy} = 0 \quad x < 0$$

Find a mapping $q=q(x, y)$, $r=r(x, y)$ that transforms the equation into its canonical form, and present the equation in this coordinate system.

$$\frac{dy}{dx} = \pm\sqrt{x} \Rightarrow q(x,y) = \frac{3}{2}y + (-x)^{\frac{3}{2}} \quad \text{and} \quad r(x,y) = \frac{3}{2}y - (-x)^{\frac{3}{2}}$$

Define $v(q, r) = u(x, y)$. By the chain rule

$$u_x = -\frac{3}{2}(-x)^{\frac{1}{2}}v_q + \frac{3}{2}(-x)^{\frac{1}{2}}v_r \qquad u_y = \frac{3}{2}(v_q + v_r)$$

$$u_{xx} = -\frac{9}{4}xv_{qq} - \frac{9}{4}xv_{rr} + \frac{9}{2}xv_{qr} + \frac{3}{4}(-x)^{\frac{-1}{2}}(v_q - v_r)$$

$$u_{yy} = -\frac{9}{4}(v_{qq} + 2v_{qr} + v_{rr})$$

Substituting these expressions into the Tricomi equation we obtain

$$u_{xx} + xu_{yy} = -9(q-r)^{\frac{2}{3}}(v_{qr} + \frac{v_q - v_r}{6(q-r)}) = 0$$

Ex: Consider the equation

$$u_{xx} - 2\sin(x)u_{xy} - \cos(2x)u_{yy} - \cos(x)u_y = 0$$

Find a coordinate system $s = s(x, y)$, $t = t(x, y)$ that transforms the equation into its canonical form. Show that in this coordinate system the equation has the form $v_{st} = 0$, and find the general solution.

<u>Theorem</u>: *Suppose that (10) is parabolic in a domain D. There exists a coordinate system (ξ, η) in which the equation has the canonical form* $w_{\xi\xi} + \ell_1[w] = G(\xi,\eta)$ *where* $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, ℓ_1 is a first-order linear differential operator, and G is a function *which depends on (10).*

Proof:

Ex : Prove that the equation $x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} + xu_x + yu_y = 0$ is parabolic and find its canonical form; find the general solution on the half-plane $x > 0$.

<u>Theorem</u>: *Suppose that (10) is elliptic in a domain D. There exists a coordinate system (ξ, η) in which the equation has the canonical form* $w_{\xi\xi} + w_{\eta\eta} + \ell_1[w] = G(\xi,\eta)$ *where* $w(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta))$, ℓ_1 is a first-order linear differential operator, and G is a function *which depends on (10).*

Proof:

Ex: Consider the *Tricomi equation:* $u_{xx} + xu_{yy} = 0 \quad x > 0$

Find a mapping $q=q(x, y)$, $r = r(x, y)$ that transforms the equation into its canonical form, and present the equation in this coordinate system.