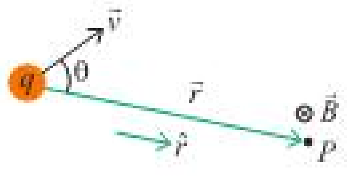
Chapter 7

Magnetic Field

7.1 Magnetic Field

A *moving charge* $\begin{cases} \text{experiences magnetic force in B-field.} \\ \text{can generate B-field.} \end{cases}$



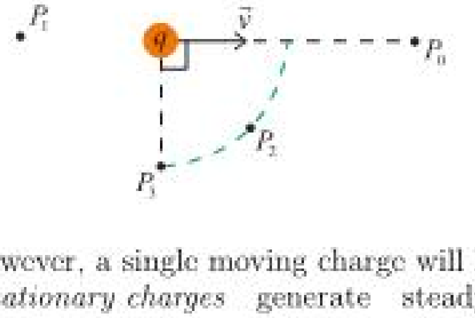
Magnetic field $\vec{B}$ due to moving point charge:

$$\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \cdot \frac{q\vec{v} \times \vec{r}}{r^3}$$

where $\mu_0 = 4\pi \times 10^{-7}$ Tm/A (N/A^2)

Permeability of free space (Magnetic constant)

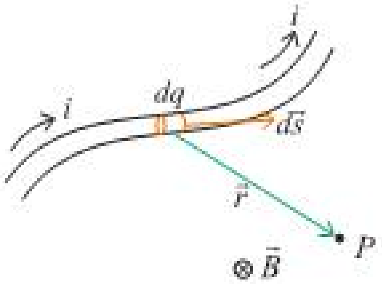
$$|\vec{B}| = \frac{\mu_0}{4\pi} \cdot \frac{qv \sin\theta}{r^2} \quad \begin{cases} \textit{maximum} & \text{when } \theta = 90^\circ \\ \textit{minimum} & \text{when } \theta = 0^\circ/180^\circ \end{cases}$$



$\vec{B}$ at $P_0 = 0 = \vec{B}$ at P_1
$\vec{B}$ at $P_2 < \vec{B}$ at P_3

However, a single moving charge will NOT generate a steady magnetic field.

stationary charges	generate	steady E-field.
steady currents	generate	steady B-field.



Magnetic field at point P can be obtained by *integrating* the contribution from individual current segments. (**Principle of Superposition**)

$$\therefore \quad d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{dq\,\vec{v} \times \hat{r}}{r^2}$$

Notice: $\quad dq\,\vec{v} = dq \cdot \dfrac{d\vec{s}}{dt} = i\,d\vec{s}$

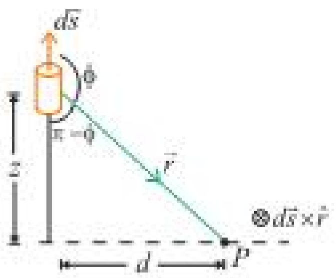
$$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \cdot \frac{i\,d\vec{s} \times \hat{r}}{r^2}} \quad \textbf{Biot-Savart Law}$$

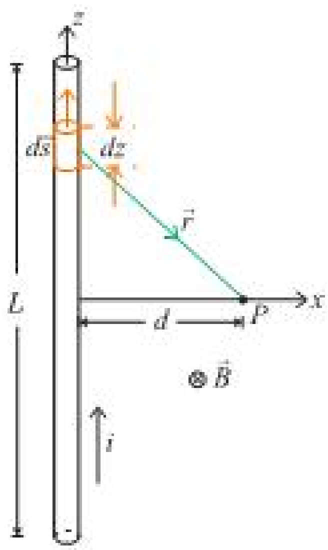
For current around a whole circuit:

$$\vec{B} = \int_{\substack{\text{entire}\\ \text{circuit}}} d\vec{B} = \int_{\substack{\text{entire}\\ \text{circuit}}} \frac{\mu_0}{4\pi} \cdot \frac{i\,d\vec{s} \times \hat{r}}{r^2}$$

Biot-Savart Law is to *magnetic field* as
Coulomb's Law is to *electric field*.
Basic element of E-field: *Electric charges* dq
Basic element of B-field: *Current element* $i\,d\vec{s}$

Example 1 : Magnetic field due to straight current segment





$$\begin{aligned}\therefore\ |d\vec{s}\times\hat{r}| &= dz\sin\phi \\ &= dz\sin(\pi-\phi) \qquad \text{(Trigonometry Identity)} \\ &= dz\cdot\frac{d}{r} = \frac{d\cdot dz}{\sqrt{d^2+z^2}}\end{aligned}$$

$$dB = \frac{\mu_0}{4\pi}\cdot\frac{i\,dz}{r^2}\cdot\frac{d}{r} = \frac{\mu_0 i}{4\pi}\cdot\frac{d}{(d^2+z^2)^{3/2}}\,dz$$

$$\therefore\quad B = \int_{-L/2}^{L/2} dB = \frac{\mu_0 i d}{4\pi}\int_{-L/2}^{+L/2}\frac{dz}{(d^2+z^2)^{3/2}}$$

$$\begin{aligned}B &= \frac{\mu_0 i}{4\pi d}\cdot\frac{z}{(z^2+d^2)^{1/2}}\Big|_{-L/2}^{+L/2} \\ B &= \frac{\mu_0 i}{4\pi d}\cdot\frac{L}{(\frac{L^2}{4}+d^2)^{1/2}}\end{aligned}$$

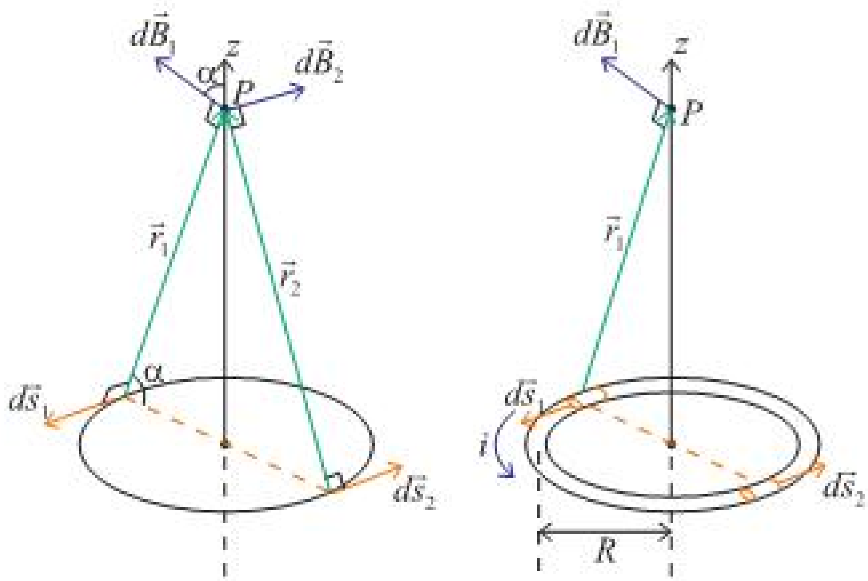
Limiting Cases : When $L \gg d$ (B-field due to long wire)

$$\left(\frac{L^2}{4}+d^2\right)^{-1/2} \approx \left(\frac{L^2}{4}\right)^{-1/2} = \frac{2}{L}$$

$$\therefore\quad \boxed{B = \frac{\mu_0 i}{2\pi d}\ ;\quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

Recall : $E = \dfrac{\lambda}{2\pi\epsilon_0 d}$ for an infinite long line of charge.

Example 2 : A circular current loop



Notice that for every current element $id\vec{s}_1$, generating a magnetic field $d\vec{B}_1$ at point P, there is an opposite current element $id\vec{s}_2$, generating B-field $d\vec{B}_2$ so that

$$d\vec{B}_1 \sin\alpha = -d\vec{B}_2 \sin\alpha$$

$\therefore$ Only vertical component of B-field needs to be considered at point P.

$$dB = \frac{\mu_0}{4\pi} \cdot \frac{i\, ds\, \sin\overbrace{90^\circ}^{\because d\vec{s} \perp \vec{r}}}{r^2}$$

$\therefore$ B-field at point P:

$$B = \underbrace{\int}_{\substack{\text{around}\\ \text{circuit}}} dB \underbrace{\cos\alpha}_{\substack{\text{consider vertical}\\ \text{component}}}$$

$$\begin{aligned}
\therefore\quad B &= \int_0^{2\pi} \frac{\mu_0 i \cos\alpha}{4\pi r^2} \cdot \underbrace{ds}_{Rd\theta} \\
&= \frac{\mu_0 i}{4\pi} \cdot \frac{R}{r^3} \underbrace{\int_0^{2\pi} ds}_{\substack{\text{Integrate around circum-}\\ \text{ference of circle} = 2\pi R}} \\
\therefore\quad B &= \frac{\mu_0 i R^2}{2r^3}
\end{aligned}$$

$$\boxed{B = \frac{\mu_0 i R^2}{2(R^2+z^2)^{3/2}}\ ; \quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

Limiting Cases :

(1) B-field at center of loop:

$$z = 0 \quad \Rightarrow \quad \boxed{B = \frac{\mu_0 i}{2R}}$$

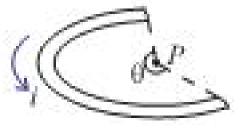
(2) For $z \gg R$,

$$B = \frac{\mu_0 i R^2}{2z^3\left(1+\frac{R^2}{z^2}\right)^{3/2}} \approx \frac{\mu_0 i R^2}{2z^3} \propto \frac{1}{z^3}$$

Recall E-field for an electric dipole: $E = \dfrac{p}{4\pi\epsilon_0 x^3}$

$\therefore$ A circular current loop is also called a **magnetic dipole**.

(3) A current arc:



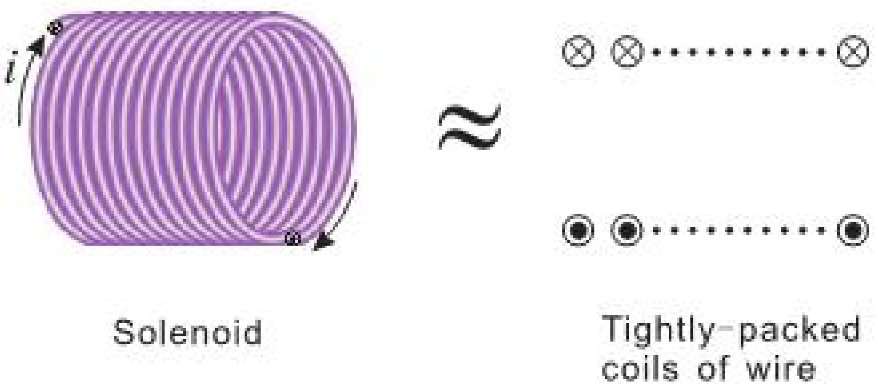
$$B = \int_{\substack{\text{around}\\\text{circuit}}} dB \underbrace{\cos\alpha}_{\substack{z=0 \Rightarrow\\ \alpha=0 \text{ here.}}}$$

$$= \frac{\mu_0 i}{4\pi} \cdot \underbrace{\frac{R}{r^3}}_{\substack{R=r\\ \text{when } \alpha=0}} \cdot \overbrace{\int_0^\theta \underbrace{ds}_{Rd\theta}}^{R\theta = \text{length of arc}}$$

$$B = \frac{\mu_0 i\,\theta}{4\pi R}$$

Example 3 : Magnetic field of a solenoid

Solenoid is used to produce a *strong and uniform* magnetic field inside its coils.



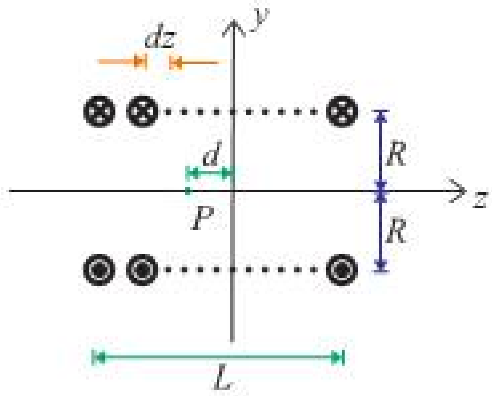
Consider a solenoid of length L consisting of N turns of wire.

Define: n = Number of turns per unit length $= \dfrac{N}{L}$

Consider B-field at distance d from the center of the solenoid:

For a segment of length dz, number of current turns $= n dz$

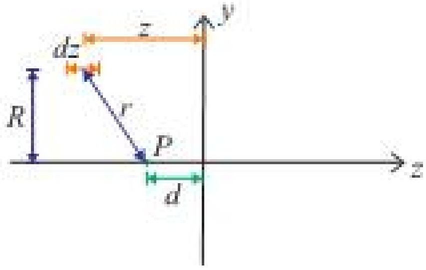
$\therefore$ Total current $= ni\,dz$



Using the result from one coil in Example 2, we get B-field from coils of length dz at distance z from center:

$$dB = \frac{\mu_0 (ni\,dz) R^2}{2r^3}$$

However $r = \sqrt{R^2 + (z-d)^2}$



$$\therefore \quad B = \int_{-L/2}^{+L/2} dB \qquad \text{(Integrating over the entire solenoid)}$$

$$= \frac{\mu_0 n i R^2}{2} \int_{-L/2}^{+L/2} \frac{dz}{[R^2 + (z-d)^2]^{3/2}}$$

$$B = \frac{\mu_0 n i}{2} \left[\frac{\frac{L}{2} + d}{\sqrt{R^2 + (\frac{L}{2} + d)^2}} + \frac{\frac{L}{2} - d}{\sqrt{R^2 + (\frac{L}{2} - d)^2}} \right]$$

along negative z direction

Ideal Solenoid :

$$L \gg R$$

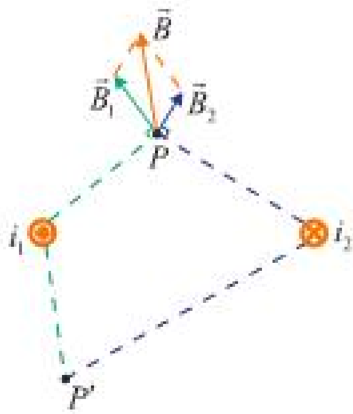
$$\text{then} \quad B = \frac{\mu_0 n i}{2} [1 + 1]$$

$$\therefore \quad \boxed{B = \mu_0 n i\,; \quad \text{direction of B-field determined from } \textit{right-hand screw rule}}$$

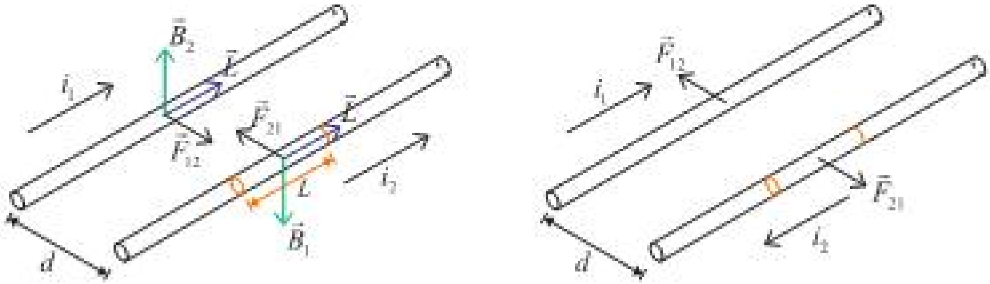
Question : What is the B-field at the end of an ideal solenoid? $B = \frac{\mu_0 n i}{2}$

7.2 Parallel Currents

Magnetic field at point P $\vec{B}$ due to two currents i_1 and i_2 is the *vector sum* of the $\vec{B}$ fields $\vec{B}_1$, $\vec{B}_2$ due to individual currents. **(Principle of Superposition)**



Force Between Parallel Currents :



Consider a segment of length L on i_2 :

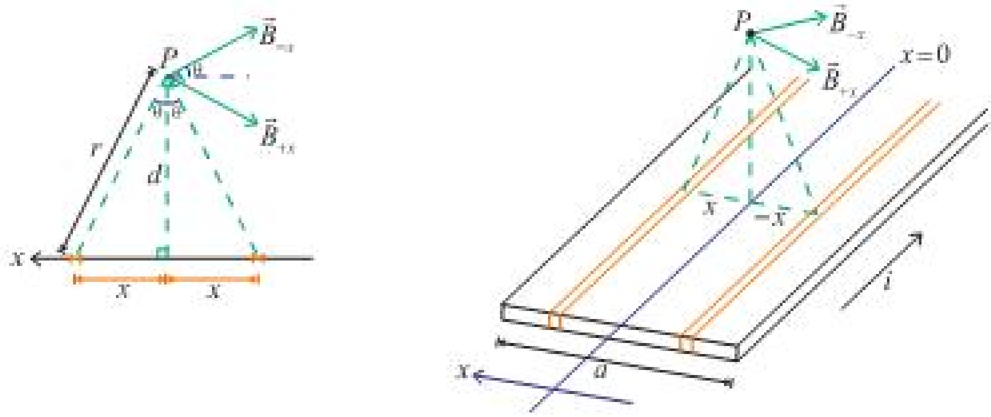
$$\vec{B}_1 = \frac{\mu_0 i_1}{2\pi d} \quad \text{(pointing down)} \qquad\qquad \vec{B}_2 = \frac{\mu_0 i_1}{2\pi d} \quad \text{(pointing up)}$$

Force on i_2 coming from i_1:

$$|\vec{F}_{21}| = i_2 \vec{L} \times \vec{B}_1 = \frac{\mu_0 L i_1 i_2}{2\pi d} = |\vec{F}_{12}| \qquad \textit{(Def'n of ampere, A)}$$

$\therefore$ *Parallel currents attract, anti-parallel currents repel.*

Example : Sheet of current



Consider an infinitesimal wire of width dx at position x, there exists another element at $-x$ so that vertical $\vec{B}$-field components of $\vec{B}_{+x}$ and $\vec{B}_{-x}$ cancel.

$\therefore$ Magnetic field due to dx wire:

$$dB = \frac{\mu_0 \cdot di}{2\pi r} \qquad \text{where } di = i\Big(\frac{dx}{a}\Big)$$

$\therefore$ Total B-field (*pointing along* $-x$ *axis*) at point P:

$$B = \int_{-a/2}^{+a/2} dB \cos\theta = \int_{-a/2}^{+a/2} \frac{\mu_0 i}{2\pi a} \cdot \frac{dx}{r} \cdot \cos\theta$$

Variable transformation (Goal: change r, x to d, θ, then integrate over θ):

$$\begin{cases} d = r\cos\theta & \Rightarrow \quad r = d\sec\theta \\ x = d\tan\theta & \Rightarrow \quad dx = d\sec^2\theta\, d\theta \end{cases}$$

Limits of integration: $-\theta_0$ to θ_0, where $\tan\theta_0 = \dfrac{a}{2d}$

$$\begin{aligned} \therefore \quad B &= \frac{\mu_0 i}{2\pi a}\int_{-\theta_0}^{\theta_0} \frac{d\sec^2\theta\, d\theta}{d\sec\theta}\cdot\cos\theta \\ &= \frac{\mu_0 i}{2\pi a}\int_{-\theta_0}^{\theta_0} d\theta \\ B &= \frac{\mu_0 i\theta_0}{\pi a} = \frac{\mu_0 i}{\pi a}\tan^{-1}\left(\frac{a}{2d}\right) \end{aligned}$$

Limiting Cases :

(1) $d \gg a$

$$\tan\theta = \frac{a}{2d} \quad \Rightarrow \quad \theta \approx \frac{a}{2d}$$

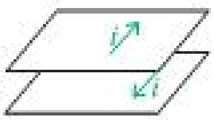
$$\therefore \quad B = \frac{\mu_0 i}{2\pi a}$$ B-field due to infinite long wire

(2) $d \ll a$ (*Infinite sheet of current*)

$$\tan\theta = \frac{a}{2d} \to \infty \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

$$\therefore \quad B = \frac{\mu_0 i}{2a} \qquad \textit{Constant!}$$

Question : Large sheet of opposite flowing currents.



What's the B-field between & outside the sheets?

7.3 Ampère's Law

In our study of electricity, we notice that the **inverse square force law** leads to **Gauss' Law**, which is useful for *finding E-field for systems with high level of symmetry*.
For magnetism, Gauss' Law is simple

$$\oiint_S \qquad \boxed{\oint_S \vec{B}\cdot d\vec{A} = 0}$$

$\because$ There is no magnetic monopole.

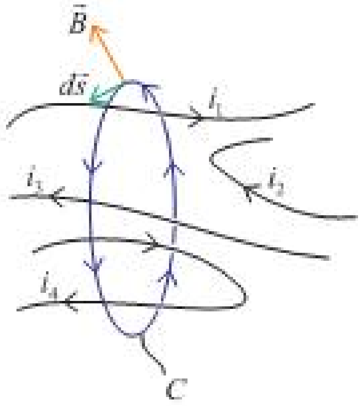
A more useful law for calculating B-field for highly symmetric situations is **the Ampère's Law**:

$$\oint_C \qquad \boxed{\oint_C \vec{B}\cdot d\vec{s} = \mu_0 i}$$

$\oint_C$ = *Line intefral* evaluated around a *closed loop C* **(Amperian curve)**

i = *Net current* that penetrates the area bounded by curve C^* (topological property)

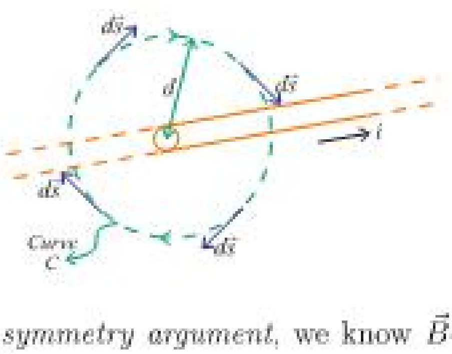
Convention : Use the **right-hand screw rule** to determine the *sign* of current.



$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0(i_1 - i_3 + i_4 - i_4) = \mu_0(i_1 - i_3)$$

Applications of the Ampere's Law :

(1) Long-straight wire



Construct an Amperian curve of radius d:

By symmetry argument, we know $\vec{B}$-field only has *tangential component*

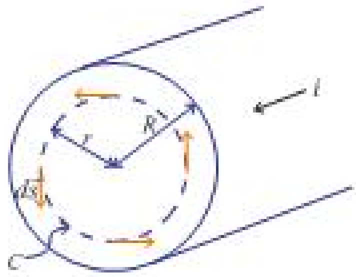
$$\therefore \quad \oint_C \vec{B}\cdot d\vec{s} = \mu_0 i$$

Take $d\vec{s}$ to be the tangential vector around the circular path:

$$\begin{aligned} \therefore \quad \vec{B}\cdot d\vec{s} &= B\,ds \\ B \underbrace{\oint_C ds}_{\substack{\text{Circumference} \\ \text{of circle } = 2\pi d}} &= \mu_0 i \\ \therefore \quad B(2\pi d) &= \mu_0 i \end{aligned}$$

B-field due to long, straight current $\boxed{B = \dfrac{\mu_0 i}{2\pi d}}$ (Compare with 7.1 Example 1)

(2) Inside a current-carrying wire

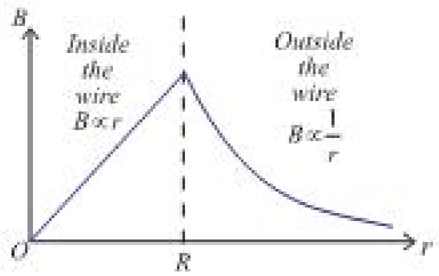


Again, symmetry argument implies that $\vec{B}$ is *tangential* to the Amperian curve and $\vec{B} \to B(r)\hat{\theta}$

Consider an Amperian curve of radius $r(< R)$

$$\oint_C \vec{B}\cdot d\vec{s} = B\oint ds = B(2\pi r) = \mu_0 i_{included}$$

But $i_{included} \propto$ cross-sectional area of C

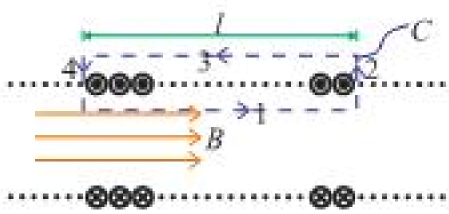


$$\begin{aligned} \therefore \quad \frac{i_{included}}{i} &= \frac{\pi r^2}{\pi R^2} \\ \therefore \quad i_{included} &= \frac{r^2}{R^2}\, i \\ \therefore \quad B &= \frac{\mu_0 i}{2\pi R^2}\cdot r \propto r \end{aligned}$$

Recall: Uniformly charged infinite long rod

(3) Solenoid (Ideal)

Consider the rectangular Amperian curve 1234.



$$\oint_C \vec{B}\cdot d\vec{s} = \int_1 \vec{B}\cdot d\vec{s} + \int_2 \vec{B}\cdot d\vec{s} + \int_3 \vec{B}\cdot d\vec{s} + \int_4 \vec{B}\cdot d\vec{s}$$

$$\int_2 = \int_4 = 0 \quad \because \quad \begin{cases} \vec{B}\cdot d\vec{s} = 0 & \text{inside solenoid} \\ \vec{B} = 0 & \text{outside solenoid} \end{cases}$$

$$\int_3 = 0 \quad \because \quad \vec{B} = 0 \quad \text{outside solenoid}$$

$$\therefore \quad \oint_C \vec{B}\cdot d\vec{s} = \int_1 \vec{B}\cdot d\vec{s} = Bl = \mu_0 i_{tot}$$

But $i_{tot} = \underbrace{nl}_{\text{Number of coils included}} \cdot i$

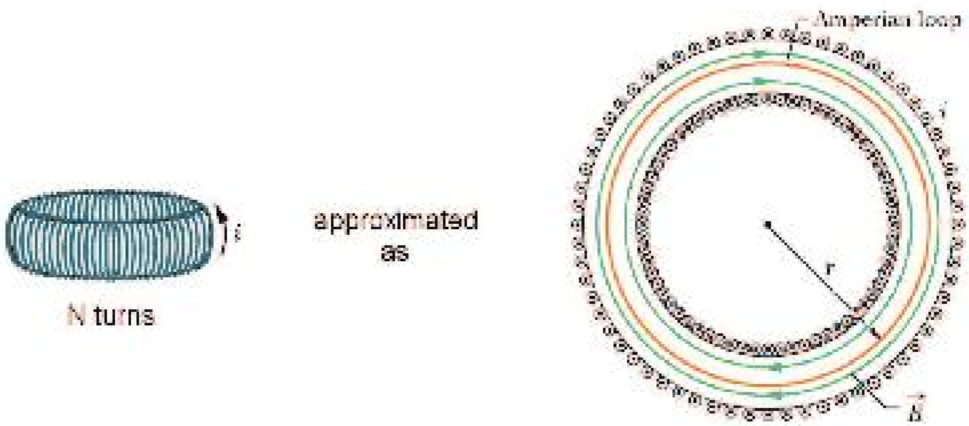
$$\therefore \boxed{B = \mu_0 n i}$$

Note :

(i) The assumption that $\vec{B} = 0$ outside the ideal solenoid is only *approximate.* (Halliday, Pg.763)

(ii) B-field everywhere inside the solenoid is a *constant.* (for ideal solenoid)

(4) Toroid (A *circular solenoid*)



By symmetry argument, the B-field lines form *concentric circles inside the toroid.*

Take Amperian curve C to be a circle of radius r inside the toroid.

$$\oint_C \vec{B}\cdot d\vec{s} = B\oint_C ds = B\cdot 2\pi r = \mu_0(Ni)$$

$$\therefore \quad B = \frac{\mu_0 N i}{2\pi r} \qquad \text{inside toroid}$$

Note :

(i) $B \neq$ constant inside toroid

(ii) Outside toroid:
Take Amperian curve to be circle of radius $r > R$.

$$\oint_C \vec{B} \cdot d\vec{s} = B \oint_C ds = B \cdot 2\pi r = \mu_0 \cdot i_{incl} = 0$$

$$\therefore \quad B = 0$$

Similarly, in the central cavity $B = 0$

7.4 Magnetic Dipole

Recall from §6.4, we define the **magnetic dipole moment** of a rectangular current loop

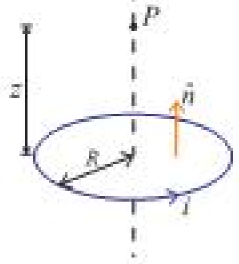
$$\boxed{\vec{\mu} = N i A \hat{n}}$$

where $\hat{n}$ = area unit vector with direction determined by the right-hand rule
N = Number of turns in current loop
A = Area of current loop

This is actually a *general definition* of a magnetic dipole, `i.e.` we use it for current loops of <u>all</u> shapes.

A common and symmetric example: circular current.

Recall from §7.1 Example 2, magnetic field at point P (height z above the ring)

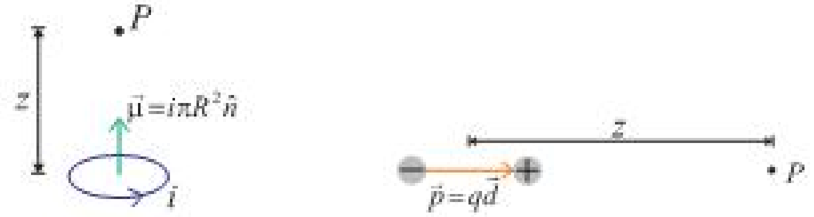


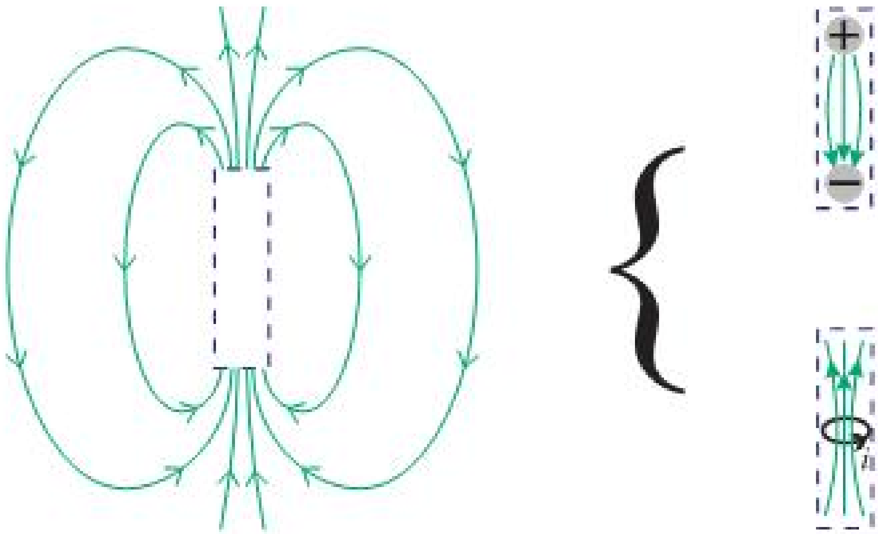
$$\vec{B} = \frac{\mu_0 i R^2 \hat{n}}{2(R^2 + z^2)^{3/2}} = \frac{\mu_0 \vec{\mu}}{2\pi(R^2 + z^2)^{3/2}}$$

At distance $z \gg R$,

$$\vec{B} = \frac{\mu_0 \vec{\mu}}{2\pi z^3} \qquad\qquad \vec{E} = \frac{\vec{p}}{4\pi\epsilon_0 z^3}$$

due to *magnetic dipole* (for $z \gg R$) — due to *electric dipole* (for $z \gg d$)



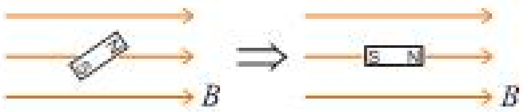


$$\text{Also, notice} \quad \begin{aligned} \vec{\mu} &= \text{magnetic dipole moment} \quad \left[\begin{array}{ll} \text{Unit:} & \text{Am}^2 \\ & J/T \end{array}\right] \\ \mu_0 &= \text{Permeability of free space} \\ &= 4\pi \times 10^{-7}\,\text{Tm/A} \end{aligned}$$

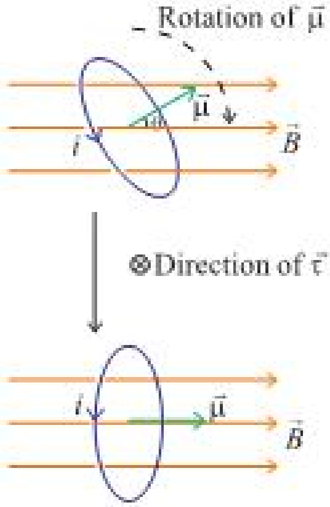
7.5 Magnetic Dipole in A Constant B-field

In the presence of a constant magnetic field, we have shown for a *rectangular current loop*, it experiences a **torque** $\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$. It applies to any magnetic dipole in general.

$\therefore$ External magnetic field aligns the magnetic dipoles.



Similar to electric dipole in a E-field, we can consider the work done in rotating the magnetic dipole. (refer to Chapter 2)



$$dW = -dU, \qquad \text{where } U \text{ is potential energy of dipole}$$

$$\boxed{U = -\vec{\mu} \cdot \vec{B}}$$

Note :

(1) We cannot define the potential energy of a magnetic field in general. However, we can define the potential energy of a magnetic dipole in a constant magnetic field.

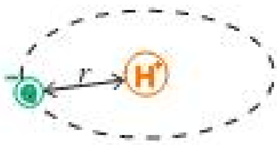
(2) In a *non-uniform external B-field*, the magnetic dipole will *experience a net force (not only net torque)*

7.6 Magnetic Properties of Materials

Recall intrinsic electric dipole in molecules:

Intrinsic dipole (magnetic) in atoms:

In our classical model of atoms, electrons revolve around a positive nuclear.



$$\therefore \quad \text{"Current" } i = \frac{e}{P}, \quad \text{where } P \text{ is period of one orbit around nucleus}$$

$$P = \frac{2\pi r}{v}, \quad \text{where } v \text{ is velocity of electron}$$

$\therefore$ **Orbit magnetic dipole** of atom:

$$\mu = iA = \Big(\frac{ev}{2\pi r}\Big)(\pi r^2) = \frac{erv}{2}$$

Recall: angular momentum of rotation $l = mrv$

$$\therefore \quad \mu = \frac{e}{2m} \cdot l$$

In *quantum mechanics*, we know that

$$l \text{ is quantized, i.e.} \quad l = N \cdot \frac{h}{2\pi}$$

$$\begin{aligned} \text{where } N &= \text{Any positive integer } (1,2,3, \dots) \\ h &= \text{Planck's constant } (6.63 \times 10^{-34} J \cdot s) \end{aligned}$$

$\therefore$ Orbital magnetic dipole moment

$$\mu_l = \underbrace{\frac{eh}{4m\pi}}_{\textbf{Bohr's magneton } \mu_B = 9.27\times 10^{-24} J/T} \cdot N$$

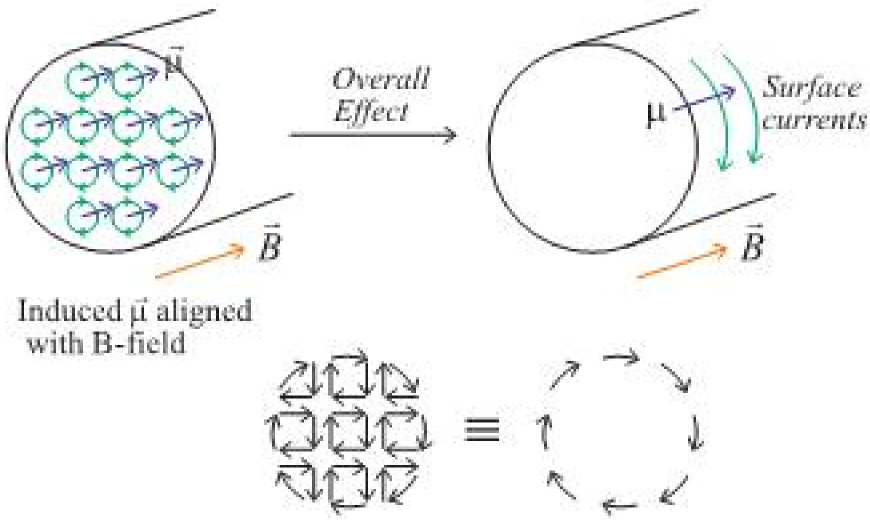
There is <u>another</u> source of intrinsic magnetic dipole moment inside an atom:

Spin dipole moment: coming from the intrinsic "spin" of electrons.

Quantum mechanics suggests that e^- are *always* spinning and it's either an "*up*" spin or a "*down*" spin

$$\mu_e = 9.65 \times 10^{-27} \approx \mu_B$$

So can there be <u>induced magnetic dipole</u>?



Recall: for electric field

$$\boxed{E_{dielectric} = K_e E_{vacuum}\ ; \quad K_e \geq 1}$$

For magnetic field in a material:

$$\begin{array}{ccccc} \vec{B}_{net} & = & \vec{B}_0 & + & \vec{B}_M \\ & & \uparrow & & \uparrow \\ & & \text{applied} & & \text{B-field produced} \\ & & \text{B-field} & & \text{by induced dipoles} \end{array}$$

In many materials (except ferromagnets),

$$\vec{B}_M \propto \vec{B}_0$$

Define :

$$\boxed{\vec{B}_M = \chi_m \vec{B}_0}$$

χ_m is a *number* called **magnetic susceptibility**.

$$\begin{aligned} \therefore \quad \vec{B}_{net} &= \vec{B}_0 + \chi_m \vec{B}_0 \\ &= (1 + \chi_m)\,\vec{B}_0 \end{aligned}$$

$$\boxed{\vec{B}_{net} = \kappa_m \vec{B}_0\ ; \quad \kappa_m = 1 + \chi_m}$$

Define : κ_m is a *number* called **relative permeability**.

One more term ......

Define : the **Magnetization** of a material:

$$\vec{M} = \frac{d\vec{\mu}}{dV} \qquad \text{where } \vec{\mu} \text{ is magnetic dipole moment, } V \text{ is volume}$$

(or, the net magnetic dipole moment per unit volume)

In most materials (except ferromagnets),

$$\boxed{\vec{B}_M = \mu_0 \vec{M}}$$

Three types of magnetic materials:

(1) **Paramagnetic**:

$$\begin{array}{c} \kappa_m \geq 1 \\ (\chi_m \geq 0) \end{array}\ ; \qquad \text{induced magnetic dipoles } \textit{aligned} \text{ } \textit{with} \text{ the applied B-field.}$$

`e.g.` Al ($\chi_m \doteq 2.2 \times 10^{-5}$), Mg ($1.2 \times 10^{-5}$), O_2 (2.0×10^{-6})

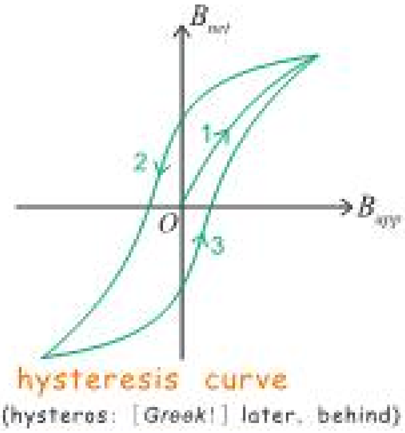
(2) **Diamagnetic**:

$$\begin{array}{c} \kappa_m \leq 1 \\ (\chi_m \leq 0) \end{array} ; \quad \text{induced magnetic dipoles } \textit{aligned opposite with} \text{ the applied B-field.}$$

`e.g.` Cu ($\chi_m \approx -1 \times 10^{-5}$), Ag ($-2.6 \times 10^{-5}$), N_2 (-5×10^{-9})

(3) **Ferromagnetic**:

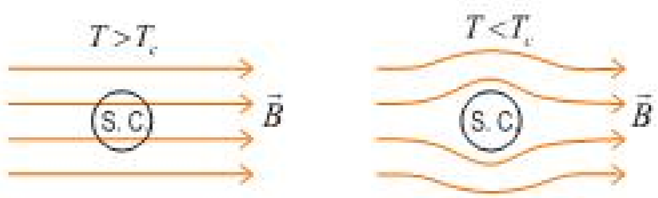
`e.g.` Fe, Co, Ni
Magnetization not linearly proportional to applied field.
$\Rightarrow \quad \dfrac{B_{net}}{B_{app}}$ not a constant (can be as big as $\sim 5000 - 100,000$)



Interesting Case : *Superconductors*

$$\chi_m = -1$$

A perfect diamagnetic.
NO magnetic field inside.



Chapter 8

Faraday's Law of Induction

8.1 Faraday's Law

In the previous chapter, we have shown that *steady electric current* can give *steady magnetic field* because of the symmetry between electricity & magnetism.

We can ask: *Steady magnetic field* can give *steady electric current.* ×

OR *Changing magnetic field* can give *steady electric current.* ✓

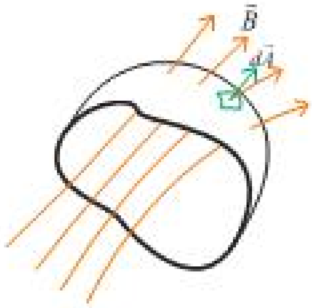
Define :

(1) Magnetic flux through surface S:

$$\boxed{\Phi_m = \int_S \vec{B} \cdot d\vec{A}}$$

Unit of Φ_m : Weber (Wb)

$1\text{Wb} = 1\text{Tm}^2$



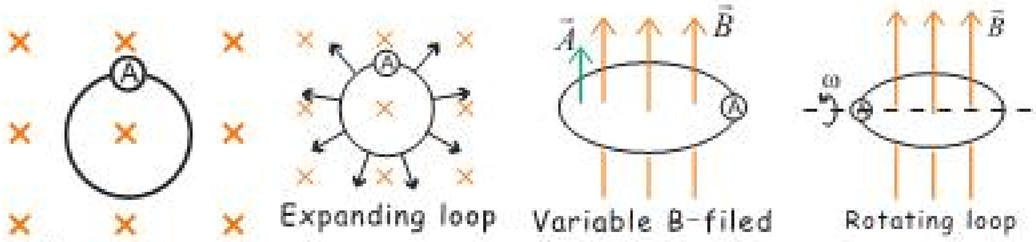
(2) Graphical:

Φ_m = Number of magnetic field lines passing through surface S

Faraday's law of induction:

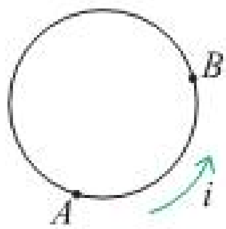
$$\boxed{\text{Induced emf } |\mathcal{E}| = N \left| \frac{d\Phi_m}{dt} \right|}$$

where N = Number of coils in the circuit.



$\vec{B}$ = Constant	$\vec{B}$ = Constant	$\hat{B}$ = Constant	$\vec{B}$ = Constant
$\vec{A}$ = Constant	$\hat{A}$ = Constant	$dB/dt \neq 0$	A = Constant
	$dA/dt \neq 0$	$\vec{A}$ = Constant	$d\hat{A}/dt \neq 0$
$\mathcal{E} = 0$	$\therefore \lvert\mathcal{E}\rvert > 0$	$\therefore \lvert\mathcal{E}\rvert > 0$	$\therefore \lvert\mathcal{E}\rvert > 0$

Note : The *induced emf* drives a current throughout the circuit, similar to the function of a *battery*. However, the difference here is that the induced emf is *distributed throughout the circuit*. The consequence is that *we cannot define a potential difference between any two points in the circuit.*



Suppose there is an *induced current* in the loop, can we define ΔV_{AB}?

Recall:

$$\Delta V_{AB} = V_A - V_B = iR > 0$$
$$\Rightarrow \quad V_A > V_B$$

Going *anti-clockwise (same as i)*,

If we start from A, going to B, then we get $V_A > V_B$.
If we start from B, going to A, then we get $V_B > V_A$.

$$\boxed{\therefore \text{ We cannot define } \Delta V_{AB} \text{ !!}}$$

This situation is like when we study *the interior of a battery*.

$$\left.\begin{array}{l}\text{A battery}\\ \\ \text{The loop}\end{array}\right\} \begin{array}{l}\text{provides the energy needed to drive the}\\ \text{charge carriers around the circuit by}\end{array} \left\{\begin{array}{l}\text{chemical reactions.}\\ \\ \text{changing magnetic flux.}\end{array}\right.$$

sources of emf *non-electric means*

8.2 Lenz' Law

(1) The flux of the magnetic field due to induced current *opposes* the change in flux that causes the induced current.

(2) The induced current is in such a direction as to *oppose* the changes that produces it.

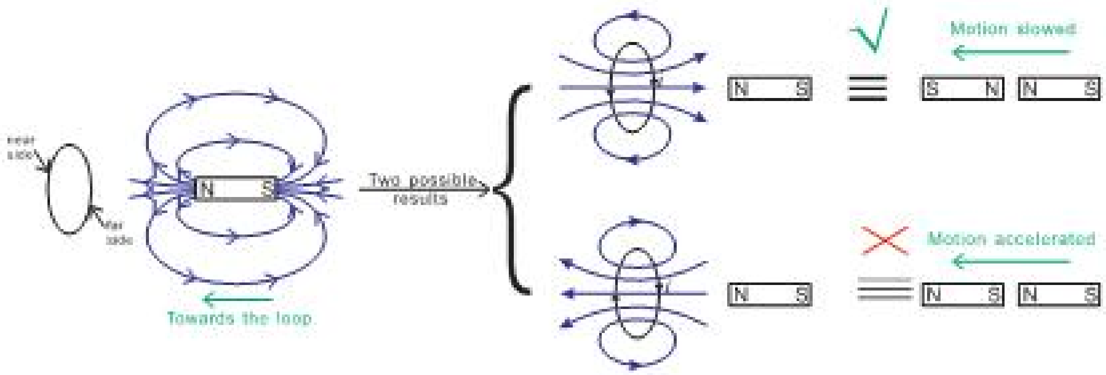
(3) Incorporating Lentz' Law into Faraday's Law:

$$\boxed{\mathcal{E} = -N\frac{d\Phi_m}{dt}}$$

If $\dfrac{d\Phi_m}{dt} > 0$, $\Phi_m \uparrow$ $\Rightarrow$ $\mathcal{E}$ appears $\Rightarrow$ Induced current appears.

$\Rightarrow$ $\vec{B}$-field due to induced current $\Rightarrow$ change in Φ_m $\overset{so\ that}{\Longrightarrow}$ $\Phi_m \downarrow$

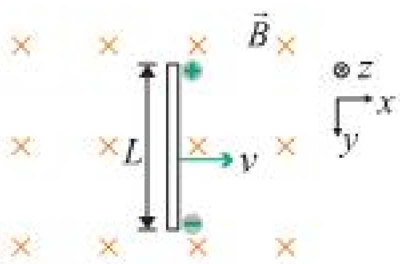
(4) Lenz' Law is a consequence from *the principle of conservation of energy.*



8.3 Motional EMF

Let's try to look at a special case when the *changing magnetic flux* is carried by *motion in the circuit wires.*

Consider a conductor of length L moving with a velocity v in a magnetic field $\vec{B}$.



Hall Effect for the charge carriers in the rod:

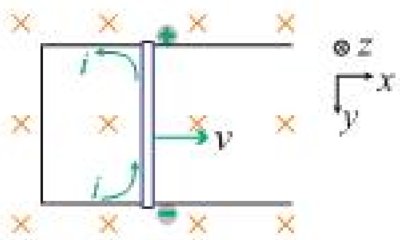
$$\begin{aligned} & \vec{F}_E + \vec{F}_B = 0 \\ \Rightarrow \quad & q\vec{E} + q\vec{v} \times \vec{B} = 0 \qquad \text{(where } \vec{E} \text{ is Hall electric field)} \\ \Rightarrow \quad & \vec{E} = -\vec{v} \times \vec{B} \end{aligned}$$

Hall Voltage inside rod:

$$\begin{aligned} \Delta V &= -\int_0^L \vec{E} \cdot d\vec{s} \\ \Delta V &= -EL \end{aligned}$$

$$\therefore \quad \text{Hall Voltage : } \boxed{\Delta V = vBL}$$

Now, suppose the moving wire *slides without friction* on a stationary U-shape conductor. The motional emf can drive an electric current i in the U-shape conductor.
$\Rightarrow$ Power is dissipated in the circuit.
$\Rightarrow$ $P_{out} = Vi$ (Joule's heating)
(see Lecture Notes Chapter 4)



What is the source of this power?
Look at the forces acting on the conducting rod:

- Magnetic force:

$$\begin{aligned} \vec{F}_m &= i\vec{L} \times \vec{B} \\ F_m &= iLB \quad \text{(pointing left)} \end{aligned}$$

- For the rod to continue to move at constant velocity v, we need to *apply an external force*:

$$\vec{F}_{ext} = -\vec{F}_m = iLB \quad \text{(pointing right)}$$

$\therefore$ *Power* required to keep the rod moving:

$$\begin{aligned} P_{in} &= \vec{F}_{ext} \cdot \vec{v} \\ &= iBLv \\ &= iBL\frac{dx}{dt} \\ &= iB\frac{d(xL)}{dt} \qquad (\ xL = A, \text{ area enclosed by circuit}) \\ &= i\frac{d(BA)}{dt} \qquad (\ BA = \Phi_m, \text{ magnetic flux}) \end{aligned}$$

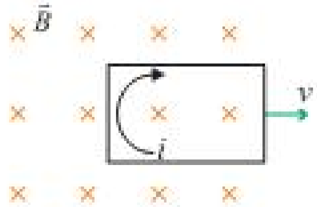
Since energy is not being stored in the system,

$$\begin{aligned} \therefore \quad P_{in} + P_{out} &= 0 \\ iV + i\frac{d\Phi_m}{dt} &= 0 \end{aligned}$$

We "prove" Faraday's Law $\Rightarrow$ $\boxed{V = -\frac{d\Phi_m}{dt}}$

Applications :

(1) Eddy current: moving conductors in presence of magnetic field

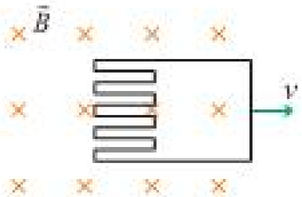


Induced current

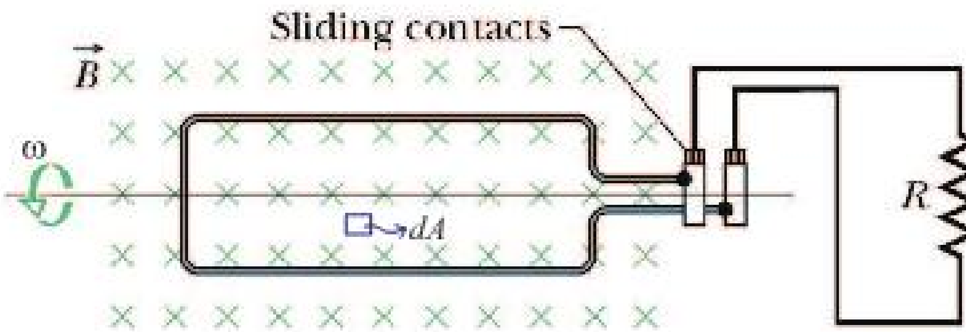
$\Rightarrow$ Power lost in Joule's heating $\left(\frac{\mathcal{E}^2}{R}\right)$

$\Rightarrow$ Extra power input to keep moving

To reduce Eddy currents:



(2) Generators and Motors:
Assume that the circuit loop is *rotating at a constant angular velocity* ω, (Source of rotation, `e.g.` steam produced by burner, water falling from a dam)



Magnetic flux through the loop

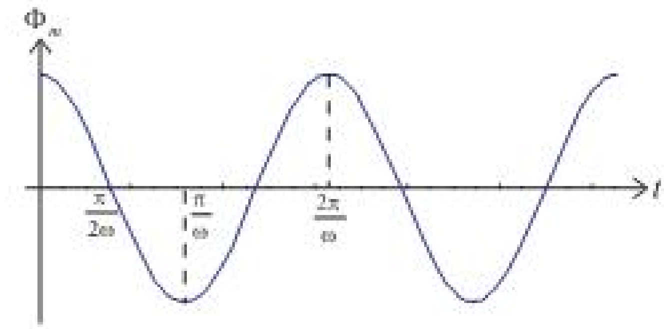
$$\Phi_B = \underset{\text{Number of coils}}{N} \int_{\text{loop}} \vec{B} \cdot d\vec{A} = NBA \underset{\text{changes with time! } \theta = \omega t}{\cos\theta}$$

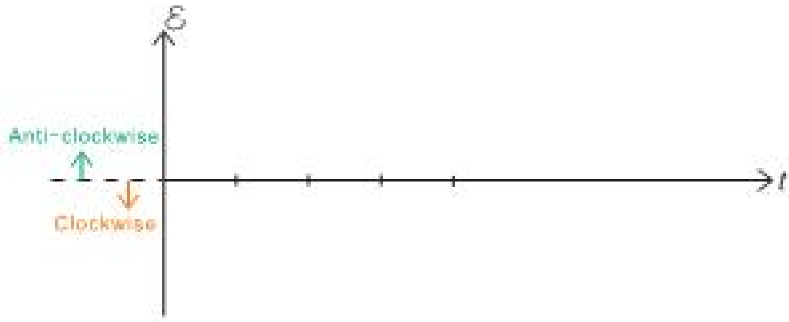
$$\therefore \quad \Phi_B = NBA\cos\omega t$$

$$\text{Induced emf:} \quad \mathcal{E} = -\frac{d\Phi_B}{dt} = -NBA\frac{d}{dt}(\cos\omega t) = NBA\omega\sin\omega t$$

$$\text{Induced current:} \quad i = \frac{\mathcal{E}}{R} = \frac{NBA\omega}{R}\sin\omega t$$

Alternating current (AC) voltage generator





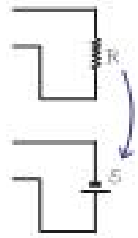
Power has to be provided by the source of rotation to overcome the torque acting on a current loop in a magnetic field.

$$\vec{\tau} = \overbrace{Ni\vec{A}}^{\vec{\mu}} \times \vec{B}$$

$$\therefore \tau = NiAB\sin\theta$$

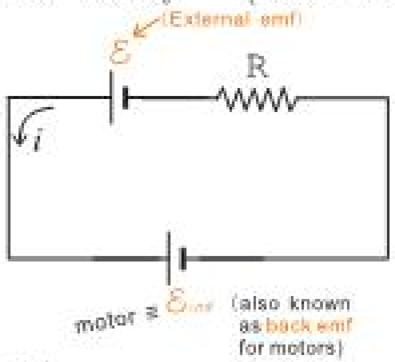
The net effect of the torque is to *oppose* the rotation of the coil.

An *electric motor* is simply a *generator operating in reverse.*
$\Rightarrow$ Replace the load resistance R with a battery of emf $\mathcal{E}$.



With the battery, there is a current in the coil, and it experiences a torque in the B-field.
$\Rightarrow$ Rotation of the coil leads to an induced emf, $\mathcal{E}_{ind}$, in the direction opposite of that of the battery. (Lenz' Law)

$$\therefore \quad i = \frac{\mathcal{E} - \mathcal{E}_{ind}}{R}$$

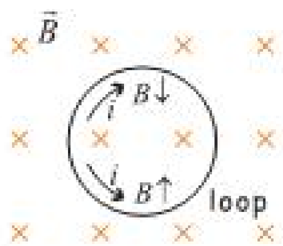


$\Rightarrow$ As motor speeds up, $\mathcal{E}_{ind} \uparrow$, $\quad \therefore\ i \downarrow$
$\therefore$ mechanical power delivered = torque delivered = $NiAB \sin\theta \downarrow$
In conclusion, we can show that

$$\underbrace{P_{electric}}_{\text{Electric power input}} = i^2 R + \underbrace{P_{mechanical}}_{\text{Mechanical power delivered}}$$

8.4 Induced Electric Field

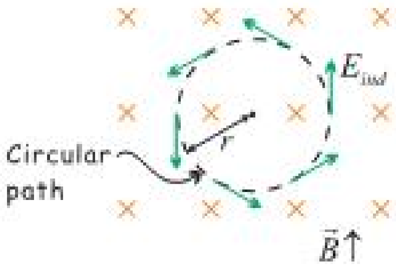
So far we have discussed that a *change* in magnetic flux will lead in an induced emf distributed in the loop, resulting from an induced E-field.



$\therefore \vec{B}\uparrow \Rightarrow$ anti-clockwise current
$\vec{B}\downarrow \Rightarrow$ clockwise current

However, even in the *absence* of the loop (so that there is no induced current), the induced E-field will still accompany a change in magnetic flux.

∴ Consider a circular path in a region with changing magnetic field.



The induced E-field only has tangential components. (`i.e.` radial E-field = 0)
`Why?`
Imagine a point charge q_0 travelling around the circular path.
Work done by induced E-field = $\underbrace{q_0 E_{ind}}_{force} \cdot \underbrace{2\pi r}_{distance}$
Recall work done also equals to $q_0\mathcal{E}$, where $\mathcal{E}$ is induced emf

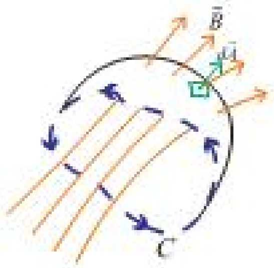
$$\therefore \quad \mathcal{E} = E_{ind} 2\pi r$$

Generally,

$$\mathcal{E} = \oint \vec{E}_{ind} \cdot d\vec{s}$$

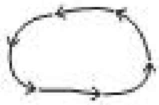
where $\oint$ is line integral around a closed loop, $\vec{E}_{ind}$ is induced E-field, $\vec{s}$ is tangential vector of path.
∴ Faraday's Law becomes

$$\boxed{\oint_C \vec{E}_{ind} \cdot d\vec{s} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{A}}$$



L.H.S. = Integral around a closed loop C
R.H.S. = Integral over a surface bounded by C

Direction of $d\vec{A}$ determined by direction of line integration C (Right-Hand Rule)

"Regular" E-field	Induced E-field
created by charges	created by changing B-field
E-field lines start from $+ve$ and end on $-ve$ charge	E-field lines form closed loops
	
can define electric potential so that we can discuss potential difference between two points	Electric potential cannot be defined (or, potential has no meaning)
$\Downarrow$ Conservative force field	$\Downarrow$ Non-conservative force field

The classification of electric and magnetic effects *depend on the frame of reference of the observer.* `e.g.` For motional emf, observer in the reference frame of the moving loop, will NOT see an induced E-field, just a "regular" E-field.
(Read: Halliday Chap.33-6, 34-7)