Chapter 5

Capacitance and DC Circuits

5.1 Capacitors

A **capacitor** is a system of *two conductors* that carries *equal and opposite charges*. A capacitor *stores charge and energy* in the form of electro-static field.

We define **capacitance** as

$$\boxed{C = \frac{Q}{V}} \qquad \text{Unit: Farad(F)}$$

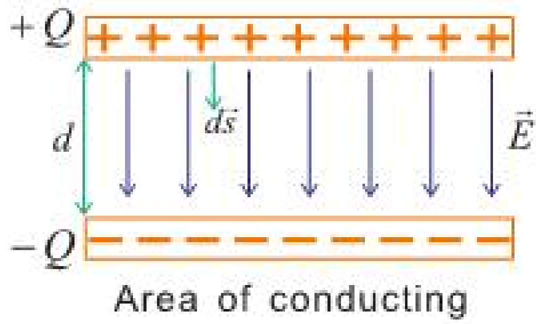
where

$$\begin{aligned} Q &= \text{Charge on one plate} \\ V &= \text{Potential difference between the plates} \end{aligned}$$

Note: The C of a capacitor is a *constant* that depends only on its shape and material.
`i.e.` If we increase V for a capacitor, we can increase Q stored.

5.2 Calculating Capacitance

5.2.1 Parallel-Plate Capacitor



(1) Recall from Chapter 3 note,

$$|\vec{E}| = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

(2) Recall from Chapter 4 note,

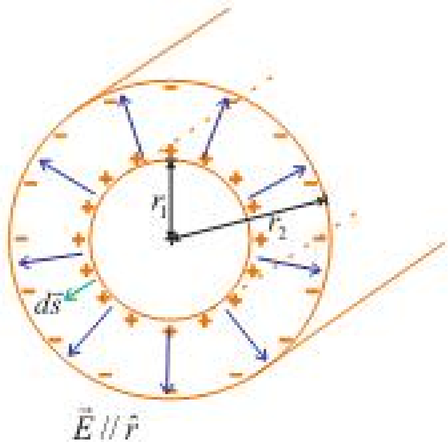
$$\Delta V = V_+ - V_- = -\int_-^+ \vec{E} \cdot d\vec{s}$$

Again, notice that this integral is independent of the path taken.
$\therefore$ We can take the path that is parallel to the $\vec{E}$-field.

$$\begin{aligned} \therefore \quad \Delta V &= \int_+^- \vec{E} \cdot d\vec{s} \\ &= \int_+^- E \cdot ds \\ &= \frac{Q}{\epsilon_0 A} \underbrace{\int_+^- ds}_{\text{Length of path taken}} \\ &= \frac{Q}{\epsilon_0 A} \cdot d \end{aligned}$$

$$(3) \ \therefore \quad \boxed{C = \frac{Q}{\Delta V} = \frac{\epsilon_0 A}{d}}$$

5.2.2 Cylindrical Capacitor



Consider two concentric cylindrical wire of innner and outer radii r_1 and r_2 respectively. The length of the capacitor is L where $r_1 < r_2 \ll L$.

(1) Using Gauss' Law, we determine that the E-field between the conductors is (`cf`. Chap3 note)

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \cdot \frac{\lambda}{r}\,\hat{r} = \frac{1}{2\pi\epsilon_0} \cdot \frac{Q}{Lr}\,\hat{r}$$

where λ is charge per unit length

(2)

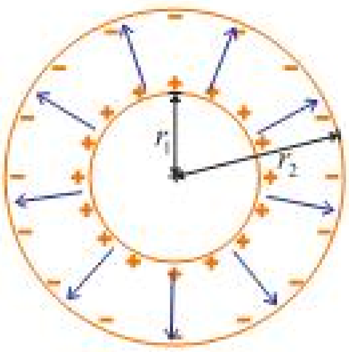
$$\Delta V = \int_{+}^{-} \vec{E} \cdot d\vec{s}$$

Again, we choose the path of integration so that $d\vec{s} \parallel \hat{r} \parallel \vec{E}$

$$\therefore \quad \Delta V = \int_{r_1}^{r_2} E\,dr = \frac{Q}{2\pi\epsilon_0 L} \underbrace{\int_{r_1}^{r_2} \frac{dr}{r}}_{\ln(\frac{r_2}{r_1})}$$

$$\therefore \quad \boxed{C = \frac{Q}{\Delta V} = 2\pi\epsilon_0 \frac{L}{\ln(r_2/r_1)}}$$

5.2.3 Spherical Capacitor



$\vec{E}\,//\,\hat{r}$

Choose $d\vec{s}\,//\,\hat{r}$

For the space between the two conductors,

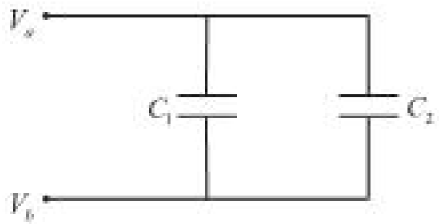
$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}; \qquad r_1 < r < r_2$$

$$\begin{aligned} \Delta V &= \int_{+}^{-} \vec{E} \cdot d\vec{s} \\ \text{Choose } d\vec{s} \parallel \hat{r} \qquad &= \int_{r_1}^{r_2} \frac{1}{4\pi\epsilon_0} \cdot \frac{Q}{r^2}\,dr \\ &= \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_1} - \frac{1}{r_2}\right] \end{aligned}$$

$$\boxed{C = 4\pi\epsilon_0 \left[\frac{r_1 r_2}{r_2 - r_1}\right]}$$

5.3 Capacitors in Combination

(a) Capacitors in Parallel



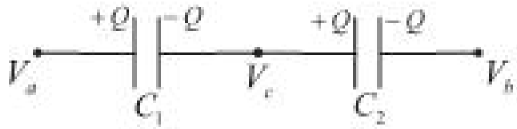
In this case, it's the *potential difference* $V = V_a - V_b$ that is the same across the capacitor.

<u>BUT</u>: *Charge* on each capacitor different

$$\begin{aligned} \text{Total charge} \quad Q &= Q_1 + Q_2 \\ &= C_1 V + C_2 V \\ Q &= \underbrace{(C_1 + C_2)}_{\text{Equivalent capacitance}} V \end{aligned}$$

$\therefore$ For capacitors in parallel: $\boxed{C = C_1 + C_2}$

(b) Capacitors in Series



The *charge across capacitors* are the same.

<u>BUT</u>: *Potential difference* (P.D.) across capacitors different

$$\begin{aligned} \Delta V_1 &= V_a - V_c = \frac{Q}{C_1} \qquad \text{P.D. across } C_1 \\ \Delta V_2 &= V_c - V_b = \frac{Q}{C_2} \qquad \text{P.D. across } C_2 \end{aligned}$$

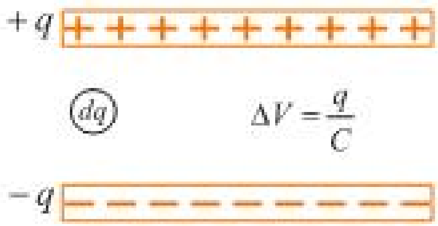
$\therefore$ Potential difference

$$\begin{aligned} \Delta V &= V_a - V_b \\ &= \Delta V_1 + \Delta V_2 \\ \Delta V &= Q\,(\frac{1}{C_1} + \frac{1}{C_2}) = \frac{Q}{C} \end{aligned}$$

where C is the **Equivalent Capacitance**

$$\therefore \quad \boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}}$$

5.4 Energy Storage in Capacitor



In charging a capacitor, *positive charge* is being moved from the *negative plate* to the *positive plate*.
$\Rightarrow$ NEEDS WORK DONE!

Suppose we move charge dq from $-ve$ to $+ve$ plate, *change in potential energy*

$$dU = \Delta V \cdot dq = \frac{q}{C}\,dq$$

Suppose we keep putting in a total charge Q to the capacitor, the *total potential energy*

$$U = \int dU = \int_0^Q \frac{q}{C}\,dq$$

$$\therefore \quad \boxed{U = \frac{Q^2}{2C} = \frac{1}{2}C\Delta V^2} \qquad (\because Q = C\Delta V)$$

The energy stored in the capacitor is stored in the **electric field** between the plates.

Note : In a parallel-plate capacitor, the *E-field is constant between the plates.*

$\therefore$ We can consider the E-field energy

$$\text{density u} = \frac{\text{Total energy stored}}{\text{Total volume with E-field}}$$

$$\therefore \quad u = \frac{U}{\underbrace{Ad}_{\text{Rectangular volume}}}$$

Recall

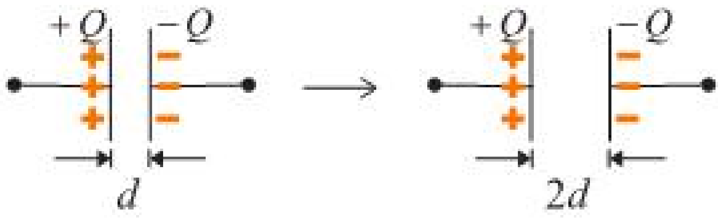
$$\begin{cases} C = \dfrac{\epsilon_0 A}{d} \\ E = \dfrac{\Delta V}{d} \quad \Rightarrow \quad \Delta V = Ed \end{cases}$$

$$\therefore \quad u = \frac{1}{2}\overbrace{(\frac{\epsilon_0 A}{d})}^{C} \cdot \overbrace{(\,Ed\,)^2}^{(\Delta V)^2} \cdot \overbrace{\frac{1}{Ad}}^{\frac{1}{Volume}}$$

$$\boxed{u = \frac{1}{2}\epsilon_0 E^2}$$ Energy per unit volume of the electrostatic field

$\uparrow$
can be generally applied

Example : Changing capacitance



(1) Isolated Capacitor:
Charge on the capacitor plates remains *constant.*
<u>BUT</u>: $C_{new} = \dfrac{\epsilon_0 A}{2d} = \dfrac{1}{2} C_{old}$
$\therefore \quad U_{new} = \dfrac{Q^2}{2C_{new}} = \dfrac{Q^2}{2C_{old}/2} = 2U_{old}$
$\therefore$ In pulling the plates apart, work done $W > 0$

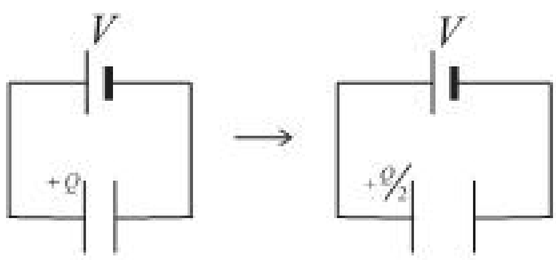
Summary :

$$\begin{array}{rcccccccl} & Q & \rightarrow & Q & & C & \rightarrow & C/2 & \\ (V = \frac{Q}{C}) \Rightarrow & V & \rightarrow & 2V & & E & \rightarrow & E & (E = \frac{V}{d}) \\ \frac{1}{2}\epsilon_0 E^2 = & u & \rightarrow & u & & U & \rightarrow & 2U & (U = u \cdot volume) \end{array}$$

(2) Capacitor connected to a battery:
Potential difference between capacitor plates remains *constant.*
$U_{new} = \dfrac{1}{2} C_{new} \Delta V^2 = \dfrac{1}{2} \cdot \dfrac{1}{2} C_{old} \Delta V^2 = \dfrac{1}{2} U_{old}$
$\therefore$ In pulling the plates apart, work done by battery < 0

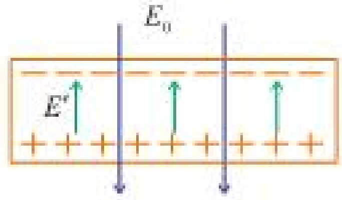
Summary :

$$\begin{array}{ccccccc} Q & \rightarrow & Q/2 & & C & \rightarrow & C/2 \\ V & \rightarrow & V & & E & \rightarrow & E/2 \\ u & \rightarrow & u/4 & & U & \rightarrow & U/2 \end{array}$$



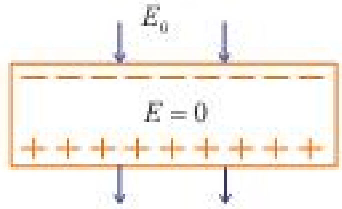
5.5 Dielectric Constant

We first recall the case for a *conductor* being placed in an *external E-field* E_0.



In a conductor, charges are free to move inside so that the *internal E-field* E' set up by these charges

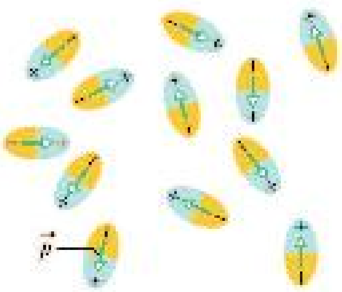
$$E' = -E_0$$

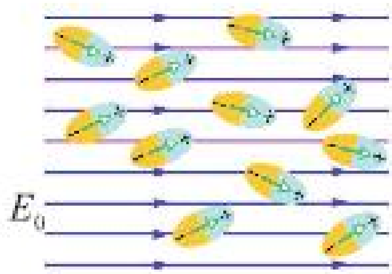


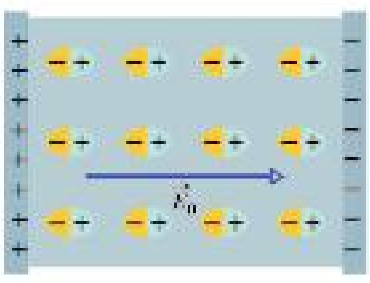
so that E-field inside conductor = 0.

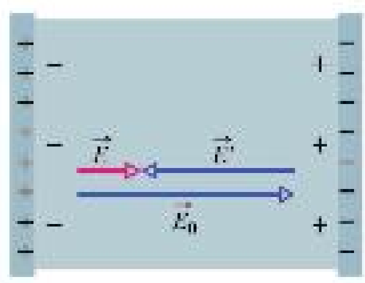
Generally, for **dielectric**, the atoms and molecules behave like a **dipole** in an E-field.

Or, we can envision this so that in the absence of E-field, the *direction of dipole in the dielectric are* randomly distributed.









The aligned dipoles will generate an *induced E-field* E', where $|E'| < |E_0|$.
We can observe the aligned dipoles in the form of *induced surface charge.*

Dielectric Constant : When a dielectric is placed in an external E-field E_0, the E-field inside a dielectric is *induced.*
E-field in dielectric

$$\boxed{E = \frac{1}{K_e} E_0}$$

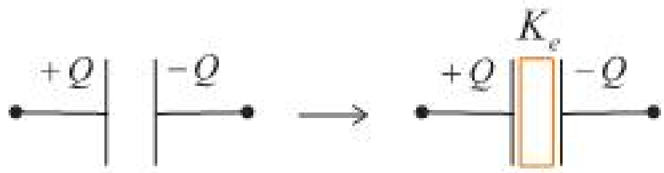
$$K_e \;=\; \textbf{dielectric constant} \quad \geq 1$$

Example :

Vacuum	$K_e = 1$
Porcelain	$K_e = 6.5$
Water	$K_e \sim 80$
Perfect conductor	$K_e = \infty$
Air	$K_e = 1.00059$

5.6 Capacitor with Dielectric

Case I :



Again, the *charge remains constant* after dielectric is inserted.
BUT: $E_{new} = \dfrac{1}{K_e} E_{old}$

$\therefore \quad \Delta V = Ed \quad \Rightarrow \quad \Delta V_{new} = \dfrac{1}{K_e} \Delta V_{old}$

$\therefore \quad C = \dfrac{Q}{\Delta V} \quad \Rightarrow \quad C_{new} = K_e\, C_{old}$

For a parallel-plate capacitor with dielectric:

$$\boxed{C = \frac{K_e \epsilon_0 A}{d}}$$

We can also write $C = \dfrac{\epsilon A}{d}$ in general with

$$\epsilon = K_e \epsilon_0 \qquad \text{(called } \textbf{permittivity of dielectric}\text{)}$$

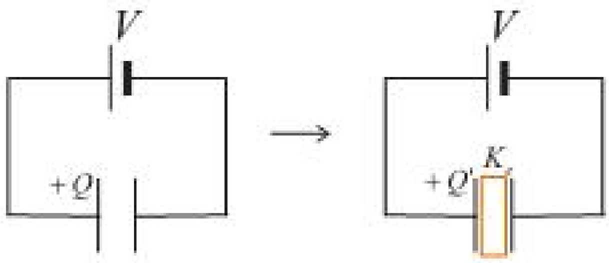
(Recall ϵ_0 = **Permittivity of free space**)

Energy stored $U = \dfrac{Q^2}{2C}$;

$\therefore \quad U_{new} = \dfrac{1}{K_e} U_{old} \quad < \quad U_{old}$

$$\boxed{\therefore \quad \text{Work done in inserting dielectric} < 0}$$

Case II : Capacitor connected to a battery



Voltage across capacitor plates *remains constant* after insertion of dielectric.
In both scenarios, the E-field inside capacitor remains constant
($\because \; E = V/d$)

<u>BUT</u>: How can E-field remain constant?
<u>ANSWER</u>: By having extra charge on capacitor plates.

Recall: For conductors,

$$E = \frac{\sigma}{\epsilon_0} \qquad \text{(Chapter 3 note)}$$

$$\Rightarrow \quad E = \frac{Q}{\epsilon_0 A} \qquad (\sigma = \text{charge per unit area} = Q/A)$$

After insertion of dielectric:

$$E' = \frac{E}{K_e} = \frac{Q'}{K_e \epsilon_0 A}$$

But E-field remains constant!

$$\therefore \quad E' = E \quad \Rightarrow \quad \frac{Q'}{K_e \epsilon_0 A} = \frac{Q}{\epsilon_0 A}$$

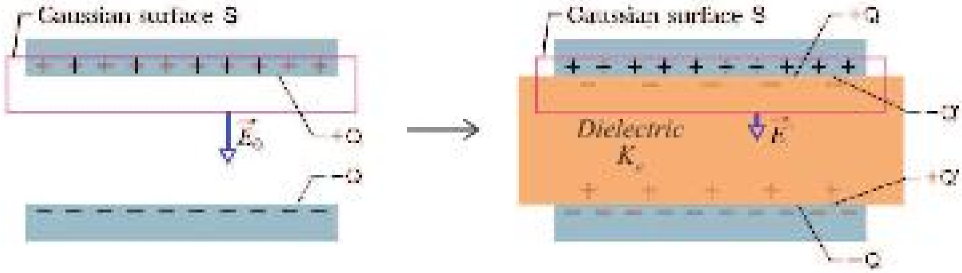
$$\Rightarrow \quad Q' = K_e Q \; > \; Q$$

$\therefore$ Capacitor $C = Q/V \Rightarrow C' \to K_e C$
Energy stored $U = \frac{1}{2}CV^2 \Rightarrow U' \to K_e U$
(i.e. $U_{new} > U_{old}$)

$\therefore$ Work done to insert dielectric > 0

5.7 Gauss' Law in Dielectric

The Gauss' Law we've learned is applicable in *vacuum only*. Let's use the capacitor as an example to examine Gauss' Law in dielectric.



Free charge on plates	$\pm Q$	$\pm Q$
Induced charge on dielectric	0	$\mp Q'$

Gauss' Law

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E_0 = \frac{Q}{\epsilon_0 A} \quad (1)$$

Gauss' Law:

$$\oint_S \vec{E'}\cdot d\vec{A} = \frac{Q-Q'}{\epsilon_0}$$

$$\therefore E' = \frac{Q-Q'}{\epsilon_0 A} \quad (2)$$

However, we define $E' = \dfrac{E_0}{K_e}$ (3)

From (1), (2), (3) $\therefore \dfrac{Q}{K_e\epsilon_0 A} = \dfrac{Q}{\epsilon_0 A} - \dfrac{Q'}{\epsilon_0 A}$

$$\therefore \boxed{\text{Induced charge density } \sigma' = \frac{Q'}{A} = \sigma\Big(1 - \frac{1}{K_e}\Big) < \sigma}$$

where σ is free charge density.

Recall Gauss' Law in Dielectric:

$$\epsilon_0 \oint_S \vec{E'}\cdot d\vec{A} = Q - Q'$$

↑ E-field in dielectric ↑ free charge ↑ induced charge

$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = Q - Q\left[1 - \frac{1}{K_e}\right]$$

$$\Rightarrow \quad \epsilon_0 \oint_S \vec{E}' \cdot d\vec{A} = \frac{Q}{K_e}$$

$$\boxed{\oint_S K_e \vec{E}' \cdot d\vec{A} = \frac{Q}{\epsilon_0}}$$

Gauss' Law in dielectric

Note :

(1) This goes back to the Gauss' Law in vacuum with $E = \dfrac{E_0}{K_e}$ for dielectric

(2) Only *free charges* need to be considered, even for dielectric where there are *induced charges*.

(3) Another way to write:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon}$$

where $\vec{E}$ is E-field in dielectric, $\epsilon = K_e\epsilon_0$ is Permittivity

Energy stored with dielectric:

Total energy stored: $U = \dfrac{1}{2}CV^2$

With dielectric, recall $C = \dfrac{K_e\epsilon_0 A}{d}$

$$V = Ed$$

$\therefore$ Energy stored per unit volume:

$$\boxed{u_e = \frac{U}{Ad} = \frac{1}{2}K_e\epsilon_0 E^2}$$

$$\text{and } u_{dielectric} = K_e u_{vacuum}$$

$\therefore$ More energy is stored per unit volume in dielectric than in vacuum.

5.8 Ohm's Law and Resistance

ELECTRIC CURRENT is defined as the flow of electric charge through a cross-sectional area.

$$\boxed{i = \frac{dQ}{dt}}$$ Unit: Ampere (A) = C/second

Convention :

(1) Direction of current is the direction of *flow of positive charge*.

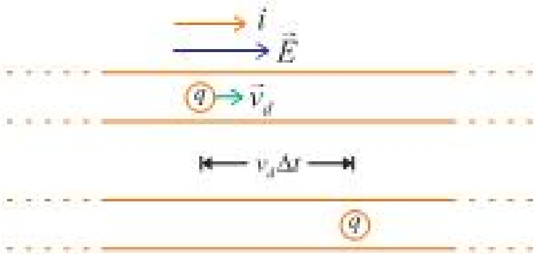
(2) Current is NOT a vector, but the **current density** is a **vector**.

$\vec{j}$ = charge flow per unit time per unit area

$$\boxed{i = \int \vec{j} \cdot d\vec{A}}$$

Drift Velocity :

Consider a current i flowing through a cross-sectional area A:



$\therefore$ In time Δt, total charges passing through segment:

$$\Delta Q = q \underbrace{A(V_d \Delta t)}_{\substack{\text{Volume of charge} \\ \text{passing through}}} n$$

where q is charge of the current carrier, n is density of charge carrier per unit volume

$$\therefore \quad \text{Current:} \quad \boxed{i = \frac{\Delta Q}{\Delta t} = nqAv_d}$$

$$\text{Current Density:} \quad \boxed{\vec{j} = nq\vec{v}_d}$$

Note : For metal, the charge carriers are the free electrons inside.
$\therefore$ $\vec{j} = -ne\vec{v}_d$ for metals
$\therefore$ Inside metals, $\vec{j}$ and $\vec{v}_d$ are in *opposite direction*.

We define a general property, **conductivity** ($\boldsymbol{\sigma}$), of a material as:

$$\boxed{\vec{j} = \boldsymbol{\sigma}\vec{E}}$$

Note : In general, σ is NOT a constant number, but rather a *function of position and applied E-field.*

A more commonly used property, **resistivity** ($\boldsymbol{\rho}$), is defined as $\rho = \dfrac{1}{\sigma}$

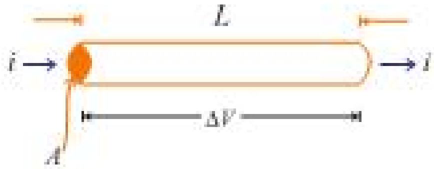
$$\therefore \quad \vec{E} = \rho\vec{j}$$

Unit of ρ : Ohm-meter (Ωm)
where Ohm (Ω) = Volt/Ampere

<u>OHM'S LAW:</u>
Ohmic materials have resistivity that are *independent of the applied electric field.*
`i.e.` metals (in not too high E-field)

Example :

Consider a **resistor** (ohmic material) of length L and cross-sectional area A.



$\therefore$ Electric field inside conductor:

$$\Delta V = \int \vec{E} \cdot d\vec{s} = E \cdot L \quad \Rightarrow \quad E = \frac{\Delta V}{L}$$

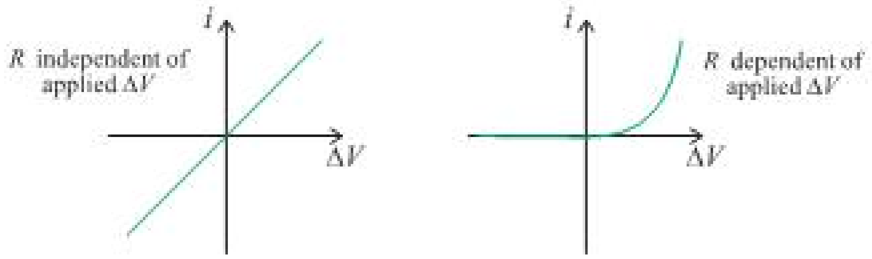
Current density: $j = \dfrac{i}{A}$

$$\therefore \quad \rho = \frac{E}{j}$$
$$\rho = \frac{\Delta V}{L} \cdot \frac{1}{i/A}$$

$$\boxed{\frac{\Delta V}{i} = R = \rho\frac{L}{A}}$$

where R is the **resistance** of the conductor.

Note: $\Delta V = iR$ is NOT a statement of Ohm's Law. It's just a definition for resistance.

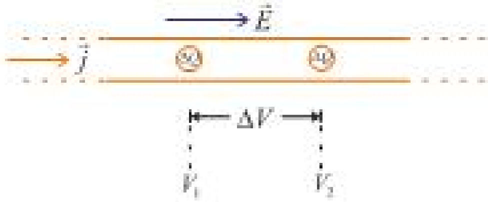


Ohmic material Non-ohmic material

(Read Chap. 29-4 of Halliday Vol 2)

ENERGY IN CURRENT:

Assuming a charge ΔQ enters with potential V_1 and leaves with potential V_2 :



$\therefore$ Potential energy lost in the wire:

$$\begin{aligned}\Delta U &= \Delta Q\, V_2 - \Delta Q\, V_1 \\ \Delta U &= \Delta Q (V_2 - V_1)\end{aligned}$$

$\therefore$ Rate of energy lost per unit time

$$\frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t}(V_2 - V_1)$$

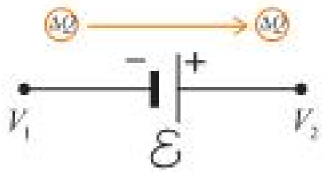
Joule's heating $\boxed{P = i \cdot \Delta V = \text{Power dissipated in conductor}}$

For a resistor R, $P = i^2 R = \dfrac{\Delta V^2}{R}$

5.9 DC Circuits

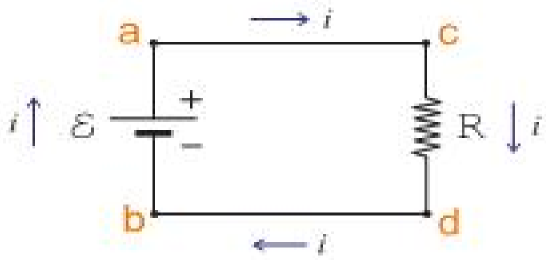
A **battery** is a device that *supplies electrical energy* to maintain a current in a circuit.

In moving from point 1 to 2, electric potential energy increase by $\Delta U = \Delta Q(V_2 - V_1) = \text{Work done by } \mathcal{E}$



Define $\mathcal{E} = \text{Work done/charge} = V_2 - V_1$

Example :



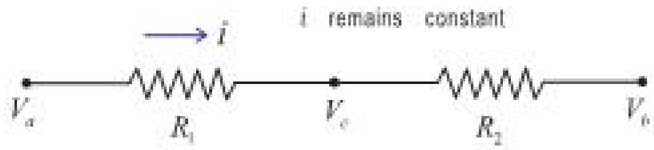
$$\left.\begin{aligned} V_a &= V_c \\ V_b &= V_d \end{aligned}\right\} \quad \text{assuming}^{(1)} \text{ perfect conducting wires.}$$

$$\text{By Definition:} \quad \begin{aligned} V_c - V_d &= iR \\ V_a - V_b &= \mathcal{E} \end{aligned}$$

$$\therefore \quad \mathcal{E} = iR \quad \Rightarrow \quad i = \frac{\mathcal{E}}{R}$$

Also, we have assumed[(2)] zero resistance inside battery.

Resistance in combination :



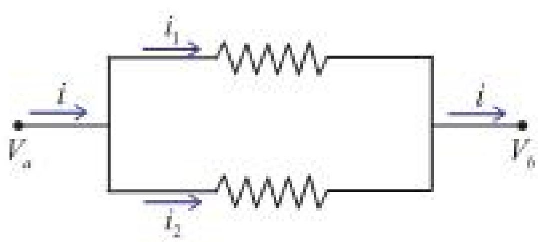
Potential differece (P.D.)

$$\begin{aligned} V_a - V_b &= (V_a - V_c) + (V_c - V_b) \\ &= iR_1 + iR_2 \end{aligned}$$

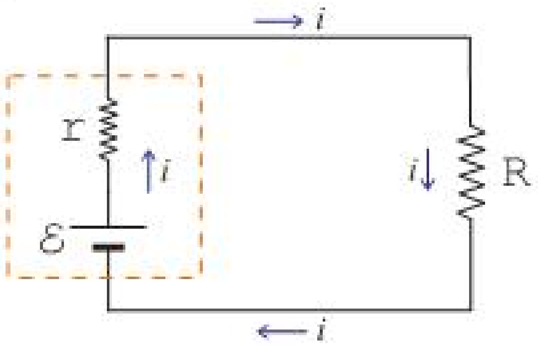
$\therefore$ Equivalent Resistance

$$R = R_1 + R_2 \quad \text{for resistors in series}$$

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{for resistors in parallel}$$



Example :



For real battery, there is an **internal resistance** that we cannot ignore.

$$\therefore \quad \mathcal{E} = i(R+r)$$
$$i = \frac{\mathcal{E}}{R+r}$$

Joule's heating in resistor R :

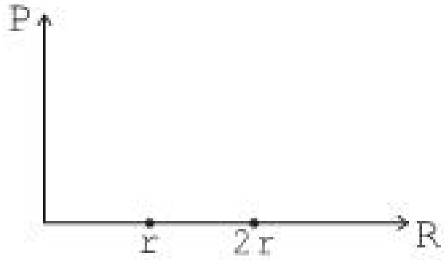
$$P = i \cdot (\text{P.D. across resistor } R)$$
$$= i^2 R$$
$$P = \frac{\mathcal{E}^2 R}{(R+r)^2}$$

Question: What is the value of R to obtain *maximum* Joule's heating?

Answer: We want to find R to *maximize* P.

$$\frac{dP}{dR} = \frac{\mathcal{E}^2}{(R+r)^2} - \frac{\mathcal{E}^2\, 2R}{(R+r)^3}$$

$$\text{Setting} \quad \frac{dP}{dR} = 0 \ \Rightarrow \ \frac{\mathcal{E}^2}{(R+r)^3}\left[(R+r) - 2R\right] = 0$$
$$\Rightarrow \ r - R = 0$$
$$\Rightarrow \ R = r$$



ANALYSIS OF COMPLEX CIRCUITS:

KIRCHOFF'S LAWS:

(1) First Law (Junction Rule):
Total current entering a junction equal to the total current leaving the junction.

(2) Second Law (Loop Rule):
The sum of potential differences around a complete circuit loop is zero.

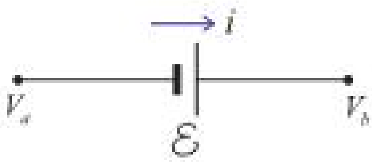
Convention :

(i)

$$V_a > V_b \quad \Rightarrow \quad \text{Potential difference} = -iR$$

i.e. Potential *drops* across resistors

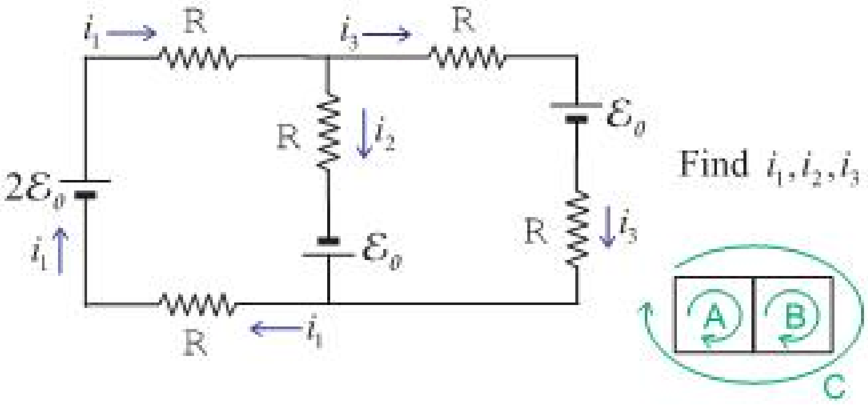
(ii)



$$V_b > V_a \quad \Rightarrow \quad \text{Potential difference} = +\mathcal{E}$$

i.e. Potential *rises* across the negative plate of the battery.

Example :



By junction rule:

$$i_1 = i_2 + i_3 \tag{5.1}$$

By loop rule:

$$\text{Loop A} \Rightarrow 2\mathcal{E}_0 - i_1R - i_2R + \mathcal{E}_0 - i_1R = 0 \tag{5.2}$$
$$\text{Loop B} \Rightarrow -i_3R - \mathcal{E}_0 - i_3R - \mathcal{E}_0 + i_2R = 0 \tag{5.3}$$
$$\text{Loop C} \Rightarrow 2\mathcal{E}_0 - i_1R - i_3R - \mathcal{E}_0 - i_3R - i_1R = 0 \tag{5.4}$$

BUT: (5.4) = (5.2) + (5.3)
General rule: Need only 3 equations for 3 current

$$i_1 = i_2 + i_3 \tag{5.1}$$
$$3\mathcal{E}_0 - 2i_1R - i_2R = 0 \tag{5.2}$$
$$-2\mathcal{E}_0 + i_2R - 2i_3R = 0 \tag{5.3}$$

Substitute (5.1) into (5.2) :

$$3\mathcal{E}_0 - 2(i_2 + i_3)R - i_2R = 0$$
$$\Rightarrow \quad 3\mathcal{E}_0 - 3i_2R - 2i_3R = 0 \tag{5.4}$$

Subtract (5.3) from (5.4), `i.e.` (5.4)−(5.3)

$$3\mathcal{E}_0 - (-2\mathcal{E}_0) - 3i_2R - i_2R = 0$$

$$\Rightarrow \quad \boxed{i_2 = \frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}}$$

Substitute i_2 into (5.3) :

$$-2\mathcal{E}_0 + \Big(\frac{5}{4} \cdot \frac{\mathcal{E}_0}{R}\Big)R - 2i_3R = 0$$

$$\Rightarrow \qquad \boxed{i_3 = -\frac{3}{8} \cdot \frac{\mathcal{E}_0}{R}}$$

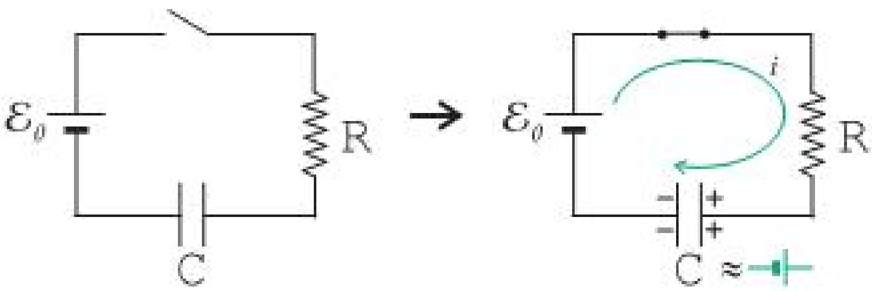
Substitute i_2, i_3 into (5.1) :

$$\boxed{i_1 = \left(\frac{5}{4} - \frac{3}{8}\right)\frac{\mathcal{E}_0}{R} = \frac{7}{8} \cdot \frac{\mathcal{E}_0}{R}}$$

Note: A *negative* current means that it is flowing in *opposite direction* from the one assumed.

5.10 RC Circuits

(A) *Charging* a capacitor with battery:



Using the loop rule:

$$+\mathcal{E}_0 - \underbrace{iR}_{\substack{\text{P.D.}\\ \text{across R}}} - \underbrace{\frac{Q}{C}}_{\substack{\text{P.D.}\\ \text{across C}}} = 0$$

Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *increasing*.

$$\therefore \quad \mathcal{E} = R\overbrace{\frac{dQ}{dt}}^{i} + \frac{Q}{C} \qquad \text{1st order differential eqn.}$$

$$\Rightarrow \qquad \frac{dQ}{\mathcal{E}C - Q} = \frac{dt}{RC}$$

Integrate both sides and use the initial condition:
$t = 0$, Q on capacitor $= 0$

$$\int_0^Q \frac{dQ}{\mathcal{E}C - Q} = \int_0^t \frac{dt}{RC}$$

$$
\begin{aligned}
& -\ln(\mathcal{E}C - Q)\Big|_0^Q = \frac{t}{RC}\Big|_0^t \\
\Rightarrow\quad & -\ln(\mathcal{E}C - Q) + \ln(\mathcal{E}C) = \frac{t}{RC} \\
\Rightarrow\quad & \ln\Big(\frac{1}{1-\frac{Q}{\mathcal{E}C}}\Big) = \frac{t}{RC} \\
\Rightarrow\quad & \frac{1}{1-\frac{Q}{\mathcal{E}C}} = e^{t/RC} \\
\Rightarrow\quad & \frac{Q}{\mathcal{E}C} = 1 - e^{-t/RC} \\
\Rightarrow\quad & \boxed{Q(t) = \mathcal{E}C(1 - e^{-t/RC})}
\end{aligned}
$$

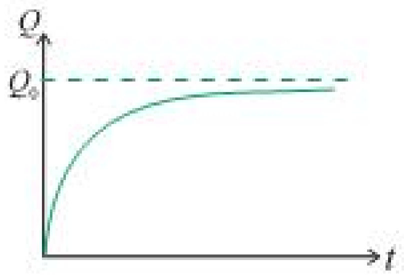
Note: (1) At $t = 0$, $Q(t = 0) = \mathcal{E}C(1 - 1) = 0$

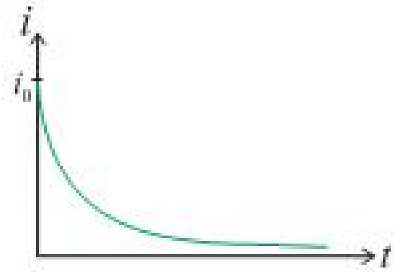
(2) As $t \to \infty$, $Q(t \to \infty) = \mathcal{E}C(1 - 0) = \mathcal{E}C$
= Final charge on capacitor (Q_0)

(3) Current:

$$
\begin{aligned}
i &= \frac{dQ}{dt} \\
&= \mathcal{E}C\Big(\frac{1}{RC}\Big)e^{-t/RC} \\
i(t) &= \frac{\mathcal{E}}{R}e^{-t/RC}
\end{aligned}
$$

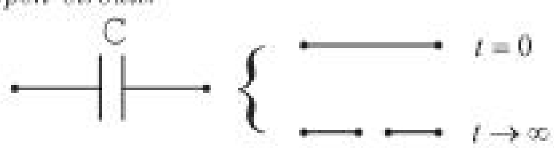
$$
\begin{cases}
i(t=0) = \frac{\mathcal{E}}{R} = \text{Initial current} = i_0 \\
i(t\to\infty) = 0
\end{cases}
$$





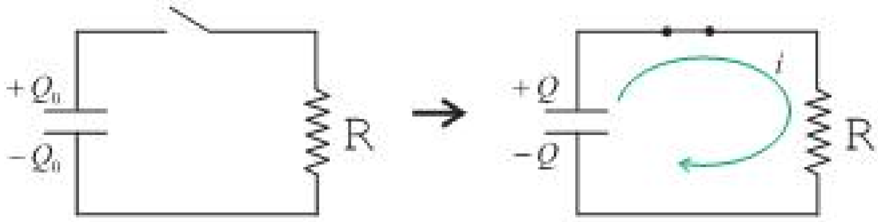
(4) At time = 0, the capacitor acts like *short circuit* when there is *zero charge on the capacitor*.

(5) As time $\to \infty$, the capacitor is *fully charged* and current = 0, it acts like a *open circuit*.



(6) $\tau_c = RC$ is called the **time constant**. It's the time it takes for the charge to reach $(1-\frac{1}{e})\,Q_0 \simeq 0.63Q_0$

(B) *Discharging* a charged capacitor:



Note: Direction of i is chosen so that the current represents the rate at which the charge on the capacitor is *decreasing*.

$$\therefore \quad i = -\frac{dQ}{dt}$$

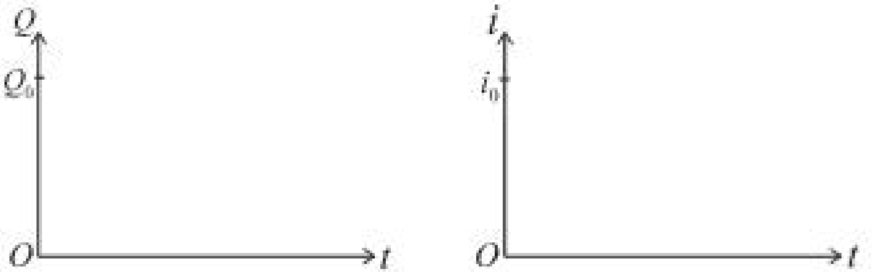
Loop Rule:

$$\begin{aligned} & V_c - iR = 0 \\ \Rightarrow \quad & \frac{Q}{C} + \frac{dQ}{dt}R = 0 \\ \Rightarrow \quad & \frac{dQ}{dt} = -\frac{1}{RC}\,Q \end{aligned}$$

Integrate both sides and use the initial condition:
$t = 0, \quad Q$ on capacitor $= Q_0$

$$\begin{aligned} & \int_{Q_0}^{Q} \frac{dQ}{Q} = -\frac{1}{RC}\int_0^t dt \\ \Rightarrow \quad & \ln Q - \ln Q_0 = -\frac{t}{RC} \\ \Rightarrow \quad & \ln\Big(\frac{Q}{Q_0}\Big) = -\frac{t}{RC} \\ \Rightarrow \quad & \frac{Q}{Q_0} = e^{-t/RC} \\ \Rightarrow \quad & Q(t) = Q_0\, e^{-t/RC} \\ (i = -\frac{dQ}{dt}) \qquad \Rightarrow \quad & i(t) = \frac{Q_0}{RC}\, e^{-t/RC} \\ (\text{At } t=0) \qquad \Rightarrow \quad & i(t=0) = \frac{1}{R}\cdot\underbrace{\frac{Q_0}{C}}_{\text{Initial P.D. across capacitor}} \end{aligned}$$

$$i_0 = \frac{V_0}{R}$$



At $\quad t = RC = \tau \qquad\qquad Q(t = RC) \;=\; \dfrac{1}{e} Q_0 \;\simeq\; 0.37 Q_0$

Chapter 6

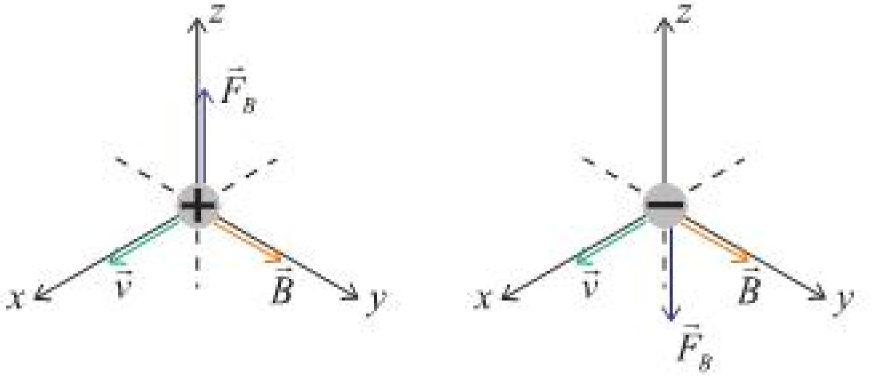
Magnetic Force

6.1 Magnetic Field

For stationary charges, they experienced an **electric force** in an **electric field**. For moving charges, they experienced a **magnetic force** in a **magnetic field**.

Mathematically, $\vec{F}_E = q\vec{E}$ (electric force)
$\vec{F}_B = q\vec{v} \times \vec{B}$ (magnetic force)

Direction of the magnetic force determined from *right hand rule*.



Magnetic field $\vec{B}$: Unit = Tesla (T)
1T = 1C moving at 1m/s experiencing 1N

Common Unit: 1 Gauss (G) = 10^{-4}T ≈ magnetic field on earth's surface

Example: What's the force on a 0.1C charge moving at velocity $\vec{v} = (10\hat{j} - 20\hat{k})ms^{-1}$ in a magnetic field $\vec{B} = (-3\hat{i} + 4\hat{k}) \times 10^{-4}T$

$$\vec{F} = q\vec{v} \times \vec{B}$$

$$\begin{aligned} &= +0.1\,(10\hat{j} - 20\hat{k}) \times (-3\hat{i} + 4\hat{k}) \times 10^{-4} N \\ &= 10^{-5}\,(-30 \cdot -\hat{k} + 40\hat{i} + 60\hat{j} + 0) N \end{aligned}$$

Effects of magnetic field is usually quite small.

$$\begin{aligned} \vec{F} &= q\vec{v} \times \vec{B} \\ |\vec{F}| &= qvB\sin\theta, \quad \text{where } \theta \text{ is the angle between } \vec{v} \text{ and } \vec{B} \end{aligned}$$

$\therefore$ Magnetic force is *maximum* when $\theta = 90^\circ$ (`i.e.` $\vec{v} \perp \vec{B}$)
Magnetic force is *minimum* (0) when $\theta = 0^\circ, 180^\circ$ (`i.e.` $\vec{v} \parallel \vec{B}$)

Graphical representation of B-field: **Magnetic field lines**
Compared with **Electric field lines**:

Similar characteristics :

(1) Direction of E-field/B-field indicated by *tangent* of the field lines.

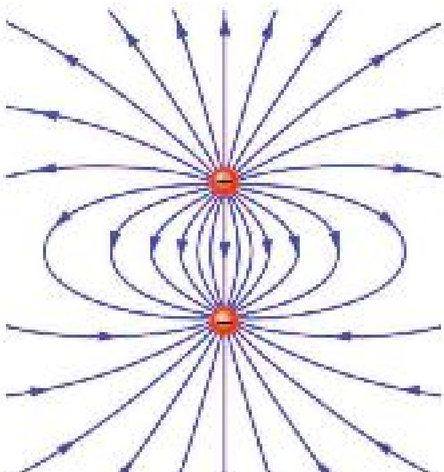
(2) Magnitude of E-field/B-field indicated by *density* of the field lines.

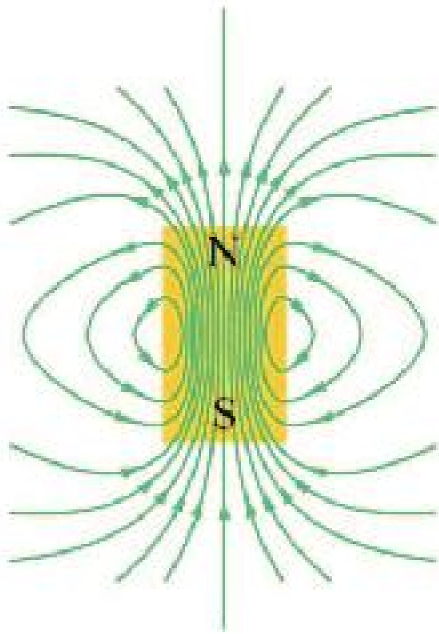
Differeces :

(1) $\vec{F}_E \parallel$ E-field lines; $\vec{F}_B \perp$ B-field lines

(2) E-field line begins at positive charge and ends at negative charge; B-field line forms a closed loop.

Example : Chap35, Pg803 Halliday

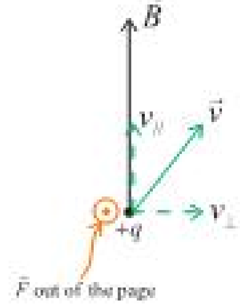




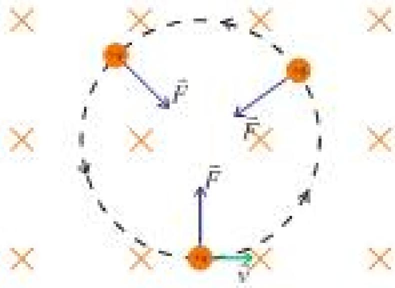
Note: Isolated magnetic monopoles do not exist.

6.2 Motion of A Point Charge in Magnetic Field

Since $\vec{F}_B \perp \vec{v}$, therefore B-field only changes the *direction* of the velocity but not its *magnitude*.



Generally, $\vec{F}_B = q\vec{v} \times \vec{B} = q\, v_{\perp} B$,
$\therefore$ We only need to consider the motion component $\perp$ to B-field.



We have *circular motion*. Magnetic force provides the *centripetal force* on the moving charge particles.

$$\begin{aligned} \therefore \quad F_B &= m\frac{v^2}{r} \\ |q|\, vB &= m\frac{v^2}{r} \\ \therefore \quad r &= \frac{mv}{|q|B} \end{aligned}$$

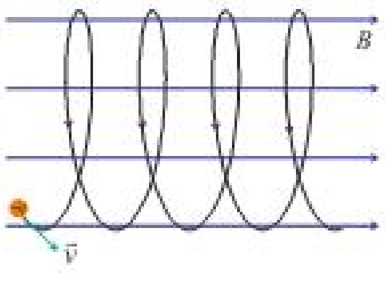
where r is radius of circular motion.

Time for moving around one orbit:

$$\boxed{T = \frac{2\pi r}{v} = \frac{2\pi m}{qB}} \quad \text{Cyclotron Period}$$

(1) Independent of v (non-relativistic)

(2) Use it to measure m/q



Generally, charged particles with constant velocity moves in **helix** in the presence of constant B-field.

Note :

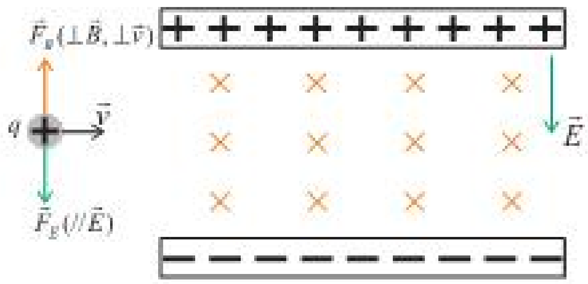
(1) B-field does NO work on particles.

(2) B-field does NOT change K.E. of particles.

Particle Motion in Presence of E-field & B-field:

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}} \quad \text{Lorentz Force}$$

Special Case : $\vec{E} \perp \vec{B}$



When $|\vec{F}_E| = |\vec{F}_B|$

$$qE = qvB$$

$$\Rightarrow \boxed{v = \frac{E}{B}}$$

$\therefore$ For charged particles moving at $v = E/B$, they will pass through the *crossed E and B fields* without vertical displacement.

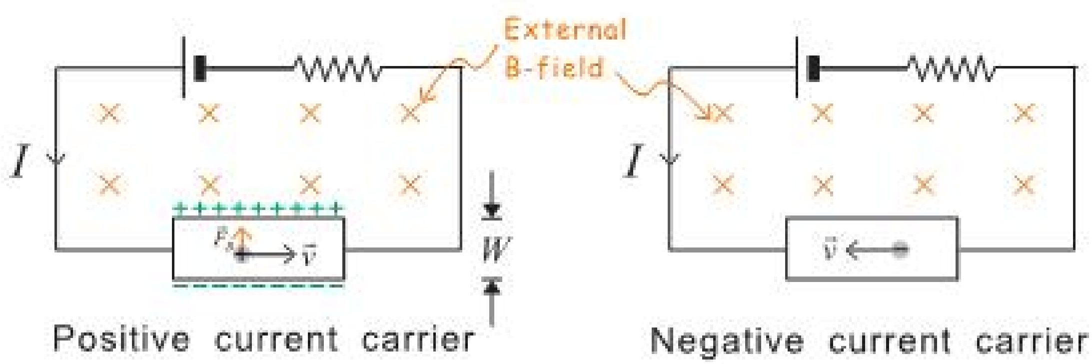
$\Rightarrow$ **velocity selector**

Applications :

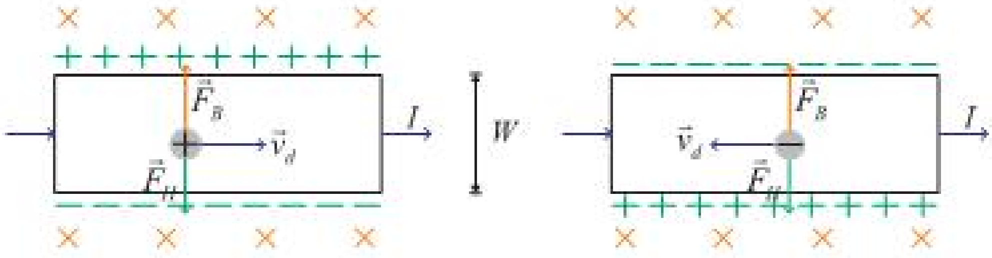
- Cyclotron (Lawrence & Livingston 1934)
- Measuring e/m for electrons (Thomson 1897)
- Mass Spectrometer (Aston 1919)

6.3 Hall Effect

Charges travelling in a conducting wire will be *pushed to one side of the wire by the external magnetic field*. This separation of charge in the wire is called the **Hall Effect**.



The separation will stop when F_B experienced by the current carrier is *balanced* by the force $\vec{F}_H$ caused by the E-field set up by the separated charges.



Define :

$$\Delta V_H = \textbf{Hall Voltage}$$
$$= \text{Potential difference across the conducting strip}$$

$\therefore$ E-field from separated charges: $E_H = \dfrac{\Delta V_H}{W}$
where W = *width* of conducting strip

In equilibrium: $q\vec{E}_H + q\vec{v}_d \times \vec{B} = 0$, where $\vec{v}_d$ is drift velocity

$$\therefore \quad \frac{\Delta V_H}{W} = v_d B$$

Recall from Chapter 5,

$$i = nqAv_d$$

where n is density of charge carrier,
A is cross-sectional area = width $\times$ thickness = $W \cdot t$

$$\therefore \quad \frac{\Delta V_H}{W} = \frac{i}{nqWt} B$$

$$\Rightarrow \quad \boxed{n = \frac{iB}{qt\Delta V_H}}$$

To determine density of charge carriers

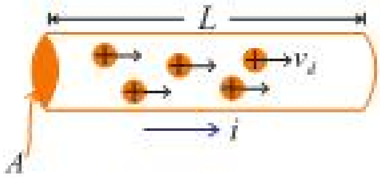
Suppose we determine n for a particular metal ($\therefore q = e$), then we can *measure B-field strength by measuring the Hall voltage*:

$$\boxed{B = \frac{net}{i} \Delta V_H}$$

6.4 Magnetic Force on Currents

Current = many charges moving together

Consider a wire segment, length L, carrying current i in a magnetic field.



$$\text{Total magnetic force} = (\underbrace{q\vec{v}_d \times \vec{B}}_{\text{force on one charge carrier}}) \cdot \underbrace{n\,A\,L}_{\text{Total number of charge carrier}}$$

Recall $i = nqv_dA$

$$\therefore \boxed{\text{Magnetic force on current } \vec{F} = i\vec{L} \times \vec{B}}$$

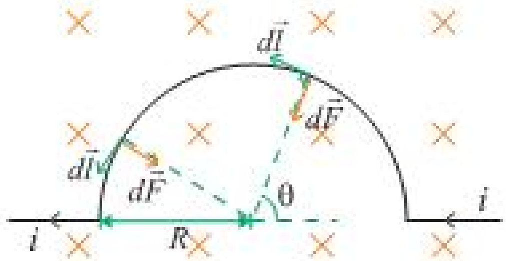
where $\vec{L}$ = Vector of which: $|\vec{L}|$ = length of current segment; direction = direction of current

For an infinitesimal wire segment $d\vec{l}$

$$\boxed{d\vec{F} = i\, d\vec{l} \times \vec{B}}$$

Example 1: Force on a semicircle current loop

$$\begin{aligned} d\vec{l} &= \text{Infinitesimal} \\ &\quad\ \text{arc length element} \perp \vec{B} \\ \therefore \quad dl &= R\, d\theta \\ \therefore \quad dF &= iRB\, d\theta \end{aligned}$$



By symmetry argument, we only need to consider vertical forces, $dF \cdot \sin\theta$

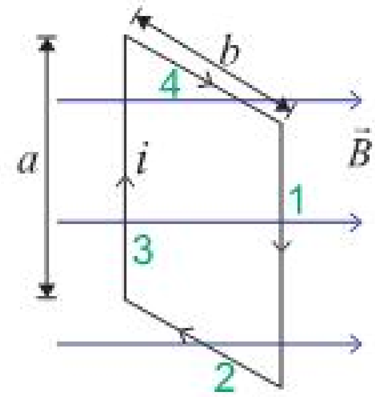
$$\begin{aligned} \therefore \quad \text{Net force } F &= \int_0^{\pi} dF\, \sin\theta \\ &= iRB \int_0^{\pi} \sin\theta\, d\theta \\ F &= 2iRB \quad \text{(downward)} \end{aligned}$$

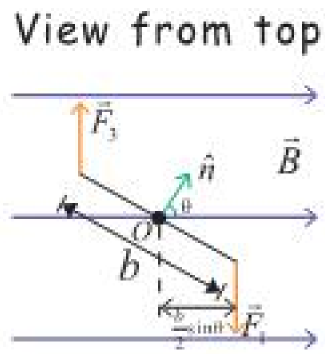
Method 2: Write $d\vec{l}$ in $\hat{i}, \hat{j}$ components

$$
\begin{aligned}
d\vec{l} &= -dl\sin\theta\,\hat{i} + dl\cos\theta\,\hat{j} \\
&= R\,d\theta\,(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}) \\
\vec{B} &= -B\,\hat{k} \quad \text{(into the page)} \\
\therefore\quad d\vec{F} &= i\,d\vec{l} \times \vec{B} \\
&= -iRB\sin\theta\,d\theta\,\hat{j} - iRB\cos\theta\,\hat{i}
\end{aligned}
$$

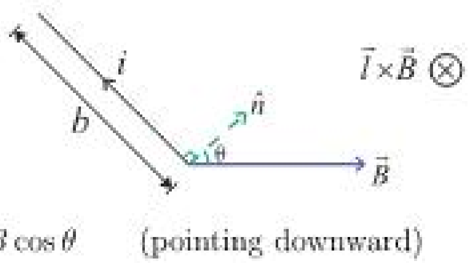
$$
\begin{aligned}
\therefore\quad \vec{F} &= \int_0^\pi d\vec{F} \\
&= -iRB\left[\int_0^\pi \sin\theta\,d\theta\,\hat{j} + \int_0^\pi \cos\theta\,d\theta\,\hat{i}\right] \\
&= -2iRB\,\hat{j}
\end{aligned}
$$

Example 2: Current loop in B-field



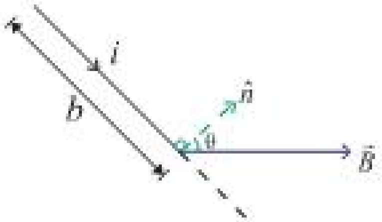


For segment2:



$$F_2 = ibB\sin(90^\circ + \theta) = ibB\cos\theta \qquad \text{(pointing downward)}$$

For segment4:



$$F_2 = ibB\sin(90^\circ - \theta) = ibB\cos\theta \qquad \text{(pointing upward)}$$