Differential quadrature method

Introduction:

In addition to finite difference method, finite elements method and finite volume method there is an efficient discretization technique to obtain accurate numerical solutions. In this technique using a considerably small number of grid points(different point with FDM and FEM),Bellman and his workers (1971, 1972) introduce the method of differential quadrature(DQ) where a partial derivative of a function with respect to a coordinate direction is expressed as a linear weighted sum of all the functional values at mesh points along that direction.The DQ method was initiated from the idea of the integral quadrature(IQ).the key to DQ is to determine the weighting coefficients for the discretization of a derivative of any order .

Bellman et al (1972) use Legendre polynomial to determine the weighting coefficients of the first –order derivative, Civan(1989) improved Bellman approach to determine the weighting coefficients,Quan and Zhang(1989) applied Lagrange interpolated polynomials as test functions,so on.

Concepts and conclusions in DQ:

Differential quadrature method is a numerical method for solving differential equations. It is differs from finite difference method and finite elements method. The derivative along a direction is described into weighting linear combination of functional values at the grid points in differential quadrature method. Because all the information of functional values at the grid points is used in differential quadrature method, it has higher accuracy.

For convenience**,** we assumed that the function $u(x)$ is sufficiently smooth in the interval $[0,1]$, shown in figure (1).

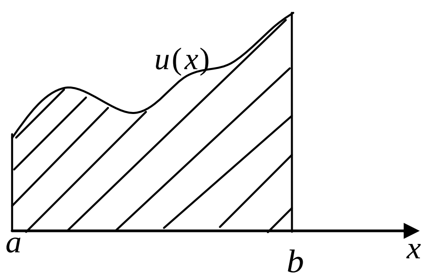

Figure 1- functions u over interval

The integral $\int_a^b u(x)dx$ represents the area under curve $u(x)$ .Thus evaluating the integral is equivalent to the approximation of the area. In general, the integral can be approximated by

$$\int_a^b u(x)dx = w_1u_1 + w_2u_2 + \cdots\cdots\cdots + w_nu_n = \sum_{k=1}^{n} w_ku_k \quad \ldots\ldots\ldots\ldots\ldots(64)$$

Where, $w_1, w_2, \cdots\cdots\cdots, w_n$ are the weighting coefficients, $u_1, u_2, \cdots\cdots, u_n$ are the functional values at the discrete points $a = x_1, x_2, \cdots\cdots, x_n = b$ .equation(64) is called integral quadrature, which uses all the functional values in the whole integral domain to approximate an integral over a finite interval. One of these types of integral Trapezoidal rule, Simpson's rule.

By introducing some grids points $a = x_1 \le x_2 \le ..... \le x_N = b$ in the computational domain, Figure (2). The interval $[0,1]$ is divided into sub-intervals.

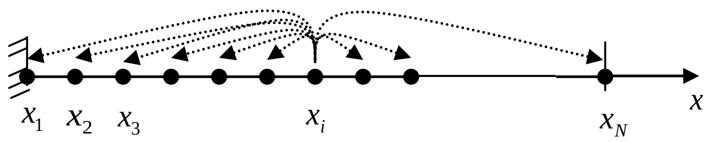

Figure 2- Computational domain stencils.

Assuming that the u_k is a value of function $u(x)$ at $x = x_k$, then the first and second derivatives of $u(x)$ at the grid points x_i is approximated by a linear combination of all functional value as follows;

$$u'(x_i) \cong \sum_{k=1}^{N} C_{ik}^{(1)} u_k \quad , \ \forall i = 1,2,....., N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (65)$$

$$u''(x_i) \cong \sum_{k=1}^{N} C_{ik}^{(2)} u_k \quad , \ \forall i = 1,2,....., N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (66)$$

where $C_{ik}^{(1)}$ and $C_{ik}^{(2)}$ are the weighting coefficients, and N is the number of grid points in the whole domain. Here the weighting coefficients are different at different location points of x_i. Equations (65) and (66) are called differential quadrature. In the application of the differential quadrature formulae (65) and (66), the choice of grid points and the determination of the weighting coefficients are two key factors. Once the grid points are given the weighting coefficients can be determined by using a set of test functions. There are many kinds of test functions that can be used. For example, striz et al (1995) and Shu and xue (1997) used Harmonic function, Shu (1999) used Fourier series expansion, and Guo and Zhong (2004) used the spline function. The polynomial test functions for determining the weighting coefficients are simply reviewed below.

Determination of the weighting coefficients

The calculation of the differential quadrature coefficients can be accomplished by several methods. In most of these methods, test functions $f_l(x), l = 1,2,....,N$, can be chosen such that:

$$u(x) \cong \sum_{l=1}^{N} \phi_l f_l(x) \quad \text{...} \ (67)$$

where, ϕ_l are constants to be determined. However, if the differential quadrature coefficients $C_{ik}^{(1)}$ and $C_{ik}^{(2)}$ are chosen such that the equations are represented as;

$$f_l'(x_i) \cong \sum_{k=1}^{N} C_{ik}^{(1)} f_l(x_k) \quad , \forall i, l = 1,2,....., N \quad \text{...........................} \ (68)$$

$$f_l''(x_i) \cong \sum_{k=1}^{N} C_{ik}^{(2)} f_l(x_k) \quad , \forall i, l = 1,2,....., N \quad \text{...........................} \ (69)$$

A relationship between first- and second- order coefficients can be obtained as:

$$f_k''(x_i) = \sum_{m=1}^{N} C_{im}^{(1)} \sum_{k=1}^{N} C_{ik}^{(1)} f_l(x_k) \cong \sum_{k=1}^{N} \sum_{m=1}^{N} C_{im}^{(1)} C_{mk}^{(1)} f_l(x_k) \quad \text{...............} \ (70)$$

Thus,

$$C_{ik}^{(2)} = \sum_{m=1}^{N} C_{im}^{(1)} C_{mk}^{(1)}, \forall i, k = 1,2,.....N.$$

in matrix notation:

$$\left[C^{(2)}\right] = \left[C^{(1)}\right]^2. \quad \text{...} \ (71)$$

where

$$\left[C^{(1)}\right]=\begin{bmatrix} c_{11}^{(1)} & c_{12}^{(1)} & ..... & c_{1N}^{(1)} \\ c_{21}^{(1)} & c_{22}^{(1)} & ..... & c_{2N}^{(1)} \\ ..... & ..... & ..... & ..... \\ c_{N1}^{(1)} & c_{N2}^{(1)} & ..... & c_{NN}^{(1)} \end{bmatrix}, \quad \left[C^{(2)}\right]=\begin{bmatrix} c_{11}^{(2)} & c_{12}^{(2)} & ..... & c_{1N}^{(2)} \\ c_{21}^{(2)} & c_{22}^{(2)} & ..... & c_{2N}^{(2)} \\ ..... & ..... & ..... & ..... \\ c_{N1}^{(2)} & c_{N2}^{(2)} & ..... & c_{NN}^{(2)} \end{bmatrix}$$

Equation (71) implies that the values of $C_{ik}^{(2)}$ can be determined by two alternative (but equivalent) procedures, i.e. they can be obtained by directly solving equation (69) or by squaring the first –order matrix $\left[C^{(1)}\right]$.One approach for calculating the entries of $\left[C^{(1)}\right]$ and $\left[C^{(2)}\right]$ (Mingle, 1977; Civan and Sliepcevich, 1984; Naadimuthu et al, 1984; Bellman and Roth, 1986) is to use the test functions:

$$f_l(x)=x^{l-1}, l=1,2,....,N \quad \text{……………………………………..} \quad (72)$$

If the polynomials are taken as the test functions, the weighting coefficients ($C_{ik}^{(1)}$ and $C_{ik}^{(2)}$) satisfy the following linear systems

$$\mathbf{V}a_i = Z'(x_k) \quad \text{……………………………………………..} \quad (73)$$

$$\mathbf{V}b_i = Z''(x_k) \quad \text{……………………………………………..} \quad (74)$$

Where

$$a_i = [C_{i1}^{(1)}, C_{i2}^{(1)}, ..., C_{iN}^{(1)}]^{T_r}, \quad b_i = [C_{i1}^{(2)}, C_{i2}^{(2)}, ..., C_{iN}^{(2)}]^{T_r}, \quad Z = [1, x, ..., x^{N-1}]^{T_r}$$

$$\mathbf{V} = \begin{pmatrix} 1 & 1 & 1 & ..... & 1 \\ x_1 & x_2 & x_3 & ..... & x_N \\ x_1^2 & x_2^2 & x_3^2 & ..... & x_N^2 \\ ..... & ..... & ..... & ..... & ..... \\ x_1^{N-1} & x_2^{N-1} & x_3^{N-1} & ...... & x_N^{N-1} \end{pmatrix}$$

Here $\mathbf{V}$ is called Vandrmonde matrix, which is not singular and

$$\det(\mathbf{V}) = \prod_{k=2}^{n}\prod_{j=1}^{k-1}(x_k - x_j) \neq 0$$

Although the weighting coefficients can be determined by solving the linear system (37), the matrix $\mathbf{V}$ is highly –ill conditioned as N is large. In order to overcome this difficulty the Legendre interpolation polynomial are used by Bellman et al (1972).the formulations of the weighting coefficients are givens as follows

$$\left.\begin{aligned} C_{ik}^{(1)} &= \frac{L_N^{(1)}(x_i)}{(x - x_k)L_N^{(1)}(x_k)} \\ C_{ii}^{(1)} &= \frac{1 - 2x_i}{2x_i(x_i - 1))} \end{aligned}\right\} \qquad (75)$$

where $L_N(x)$ and $L_N^{(1)}(x)$ are the Legendre polynomial of degree N and its first order derivative respectively.

Although we can determine the weighting coefficients for the second order derivatives by solving a system (74), the matrices are also highly-ill-conditioned. By using the Lagrange interpolation polynomials as the test function the weighting coefficients of second order derivatives are given by Quan and Chang as follows

$$C_{ik}^{(2)} = \frac{2}{x_k - x_i}\left(\prod_{l=1,l\neq i,k}^{N} \frac{x_i - x_k}{x_k - x_l}\right)\left(\prod_{j=1,j\neq i,k}^{N} \frac{1}{x_i - x_j}\right), \; for \;\; i \neq k \qquad (76)$$

$$C_{ii}^{(2)} = 2\sum_{l=1,l\neq i}^{N-1}\left(\frac{1}{x_i - x_l}\left(\sum_{j=1,j\neq i}^{N} \frac{1}{x_i - x_j}\right)\right) \qquad (77)$$

The recurrence formula to compute the weighting coefficients for mth order derivatives are given by Shu's as follows

$$C_{ik}^{(m)} = m\left(C_{ik}^{(1)}C_{ii}^{(m-1)} - \frac{C_{ik}^{(m-1)}}{(x_i - x_k)}\right), \; for \;\; i,k = 1,2,...N; 2 \leq m \leq N-1 \qquad (78)$$

$$\sum_{k=1}^{N} C_{ik}^{(m)} = 0 \;\; \text{or} \;\; C_{ii}^{(m)} = -\sum_{k=1,k\neq i}^{N} C_{ik}^{(m)} \qquad (79)$$

In similarly way with equation (71), a relationship between first- and high- order coefficients can be obtained,

$$\left[C^{(m)}\right] = \left[C^{(1)}\right] \cdot \left[C^{(m-1)}\right] = \left[C^{(m-1)}\right] \cdot \left[C^{(1)}\right], \;\; m = 2,3,....., N-1 \qquad (80)$$