Stability of finite difference:

the numerical calculation using finite difference formula are done in digital calculators which have round off error, each point calculated there will be error in the finite result which differs from the exact finite difference formula. For stable solution, there should be no large accumulation of round off error.

Von-Neumann (Fourier-series) method:

In each, mesh point (i,n) there will be round off error e_i^n, the sum of which error the grid points could represent as,

$$E(x,t)=\sum_{i,n}^{M} e_i^n$$

$$=\sum_{i,n}^{M} A_i e^{\sqrt{-1}\beta x} e^{\alpha t} \quad , \quad \beta,\alpha \ \textit{are arbitrary phase constants.}$$

where, $e^{\alpha t}$: growth factor of error with time(amplification factor)

$$E(x,t)=\sum_{i,n}^{M} A_i e^{\sqrt{-1}\beta(i\Delta x)} e^{\alpha(n\Delta t)}$$

If suppose $e^{\alpha t}=\zeta$ is an arbitrary real or complex number, where $\alpha(\Delta t)$ =constant, then

$$E(x,t)=\sum_{i,n}^{M} A_i e^{\sqrt{-1}\beta(i\Delta x)} \zeta^{(n)}$$

If $\zeta>1$, then $\zeta^{(n)}$ will increase with time(i.e. instability), therefore , the require equation to gives the stability condition is

$$|\zeta|\le 1 \qquad \text{………….……….(50)}$$

Simplification: The propagation of error with time is taken for one mesh point rather than the whole mesh

$$\varepsilon_i^n = e^{\sqrt{-1}\beta(i\Delta x)}\zeta^{(n)} \quad \text{(error for single mesh point)} \quad \text{…..…..(51)}$$

Theorem: The error term in each mesh point ε_i^n satisfies the same finite difference formula used to calculate value of u at that mesh point.

Proof: For the explicit finite difference formula (equation (10))

$$\bar{u}_i^{n+1}=(1-2r)\bar{u}_i^n + r(\bar{u}_{i+1}^n+\bar{u}_{i-1}^n)$$

$$\bar{u}_1^{n+1}=(1-2r)\bar{u}_1^n + r(\bar{u}_2^n+\bar{u}_0^n)$$

.

.

$$\bar{u}_M^{n+1}=(1-2r)\bar{u}_M^n + r(\bar{u}_{M+1}^n+\bar{u}_{M-1}^n)$$

$$\Rightarrow \vec{u}^{(n+1)} = A^{(n)}\vec{u}^{(0)}$$

where A is ($M \times M$) bounded matrix, $\vec{u}^{(0)}$ is initial value at $t = 0$. Suppose we introduce an error ε_i^n in initial calculation, we get

$$\vec{\varepsilon}^{(n+1)} = A^{(n)}\vec{\varepsilon}^{(0)} \qquad \text{(Prove that)}$$

Example: Find the stability condition for the explicit finite difference formula.The explicit finite difference formula is

$$u_i^{n+1} = (1-2r)u_i^n + r(u_{i+1}^n + u_{i-1}^n)$$

From the above theorem, we get

$$\varepsilon_i^{n+1} = (1-2r)\varepsilon_i^n + r(\varepsilon_{i+1}^n + \varepsilon_{i-1}^n)$$

Apply von Neumann analysis (equation (51)) for each term in above equation, we obtain

$$e^{\sqrt{-1}\beta(i\Delta x)}\zeta^{(n+1)} = (1-2r)e^{\sqrt{-1}\beta(i\Delta x)}\zeta^{(n)} + r(e^{\sqrt{-1}\beta(i+1)\Delta x} + e^{\sqrt{-1}\beta(i-1)\Delta x})\zeta^{(n)}$$

Divided by $e^{\sqrt{-1}\beta(i\Delta x)}\zeta^{(n)}$, we have

$$\zeta = (1-2r) + r(e^{\sqrt{-1}\beta\Delta x} + e^{-\sqrt{-1}\beta\Delta x})$$

$$= (1-2r) + 2r\cos\beta\Delta x$$

$$\zeta = 1 - 4r\sin^2\frac{\beta\Delta x}{2}$$

From the stability condition (50), we have $|\zeta| = \left|1 - 4r\sin^2\frac{\beta\Delta x}{2}\right| \le 1$,and this implies to $0 \le r \le \frac{1}{2}$(give the details to illustrate that)

<u>Exercise8:</u> Find the stability condition for an explicit finite difference formula that is used to approximation $u_t = u_{xx} + u_{yy}$.

<u>Exercise</u>9: Show that an explicit finite difference formula for approximation is stable for $0 \le r \le \frac{1}{6}$.

<u>Exercise10:</u> Consider the finite difference equation

$$u_i^{n+1} - 2u_i^n + u_i^{n-1} = \frac{r^2}{2}\left\{(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + (u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1})\right\}$$

(a) Find P.D.E. that is consistent with FDE
(b) Find the stability condition.

Exercise11: (1) Approximate $u_t + vu_x - \alpha u_{xx} = 0$ by ;

(a) Explicit method **(b)** Implicit method

(c) Crank-Nicolson method

(2) Find the truncation error and stability regions for all above finite difference methods.

(3) Approximating the first derivative in the P.D.E (part 1), by using the weight θ at two time levels, then, find the truncation error and stability condition.

Matrix stability analysis:

Assuming periodic initial data and neglecting the boundary conditions, we have used the von-Neumann method to determine the stability of the difference schemes. We now apply the matrix method, which automatically takes into account the boundary conditions of the problem, to difference schemes for the stability analysis. The two level difference scheme may be written as,

$$A_0 \vec{u}^{(n+1)} = A_1 \vec{u}^{(n)} + b^n, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52)$$

where b^n contains boundary conditions and $|A_0| \neq 0$. For $A_0 = \mathrm{I}$, the difference scheme(52) will be an explicit scheme otherwise an implicit scheme. We now assume that an error is introduced by round-off or some other source in to the solution $\vec{u}^n$ and call it $\vec{u}^{*n}$, then

$$A_0 \vec{u}^{*(n+1)} = A_1 \vec{u}^{*(n)} + b^n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53)$$

Subtracting equation(52) from equation(53), we get

$$A_0 \vec{\varepsilon}^{*(n+1)} = A_1 \vec{\varepsilon}^{*(n)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(54)$$

,where $\vec{\varepsilon}^{*(n)} = \vec{u}^{*(n)} - \vec{u}^{(n)}$ is the numerical vector error. In the stability analysis by the matrix method, we determine the condition under which the value of the numerical error vector $\left\|\vec{\varepsilon}^{*(n)}\right\| = \left\|\vec{u}^{*(n)} - \vec{u}^{(n)}\right\|$, where $\|\cdot\|$ denotes a suitable norm, remains bounded as n increases indefinitely, with k remaining fixed.

The equation (54) can be written in the form

$$\vec{\varepsilon}^{*(n+1)} = P\vec{\varepsilon}^{*(n)}$$

where $P = A_0^{-1} A_1$

It is simple to verify that $\vec{\varepsilon}^{*(n+1)} = P^{(n+1)}\vec{\varepsilon}^{*(0)}$

Thus the stability condition in the matrix method depends on the determination of a suitable estimate for $\|P\|$. When P is symmetric or similar to a symmetric matrix then $\|P\|_2$ is given by the spectral radius of P. Now, if the eigenvalues λ_i of P are distinct and the eigenvectors are $V^{(i)}$, we can expand the vector

$$\vec{\varepsilon}^{*(0)} = \sum_{i=1}^{M-1} C_i V^{(i)}$$

Then, we have

$$\vec{\varepsilon}^{*(n+1)} = \sum_{i=1}^{M-1} C_i \lambda_i^{(n+1)} V^{(i)}$$

Moreover, for the stability of difference scheme (52) we required each $|\lambda_i| \le 1$ for all i.

Hence, we get the result that error will not increase exponentially with n provided the eigenvalue with largest modulus has a modulus less than or equal one or

$$\|P\|_2 = \max_i |\lambda_i| \le 1$$

It is easy to see that the eigenvalues are the zeros of the characteristic equation

$$|A_1 - \lambda A_0| = 0$$

For the explicit method, we have

$$A_1 = \mathrm{I} + rC, \qquad A_0 = \mathrm{I}$$

The eigenvalues and eigenvectors of C are giving by

$$\lambda_i = -4\sin^2\frac{i\pi}{2M}, \quad 1 \le i \le M-1 \qquad\qquad \text{Prove that!}$$

$$V^{(i)} = \left[\sin\frac{i\pi}{M} \;\; \sin\frac{2i\pi}{M} \;\; \sin\frac{3i\pi}{M} \cdots\cdots\cdots\cdots\cdots \sin\frac{(M-1)i\pi}{M}\right]$$

It follows that the eigenvalues of $\mathrm{I} + rC$ are

$$\lambda_i = 1 - 4\sin^2\frac{i\pi}{2M}, \quad 1 \le i \le M-1$$

Therefore, the condition for the stability of the explicit method is

$$-1 \le 1 - 4r\sin^2\frac{\beta\Delta x}{2} \le 1$$

Hence, $0 \le r \le \frac{1}{2}$. The result obtain, which is identical with that obtained by application of the von-Neumann method.

Exercise12: Use this method to determine the stability of the difference equation that resulting in the previous exercise.

Gersschgorins theorem: The largest of the moduli of the eigenvalues of a square matrix A can not exceed the largest sum of the moduli of the elements along any row or any column.

$$|\lambda| \le |sum\ of\ any\ row\ or\ any\ column|$$

Brours theorem: Let P_i be the sum of the moduli of the elements along the i^{th} row excluding the diagonal elements a_{ii}. Then each eigenvalue of A lies inside or on the boundary of at least one of the circles

$$|\lambda - a_{ii}| \le P_i \text{ , where } P_i \equiv radius, \quad a_{ii} \equiv center.$$

For example ,from Crank-Nicolson formula ,we have

$$B\vec{u}^{(n+1)} = (4\mathrm{I} - B)\vec{u}^{(n)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(55)$$

$$\vec{u}^{(n+1)} = (4B^{-1} - \mathrm{I})\vec{u}^{(n)}$$

Where, $$B = \begin{bmatrix} 2+2r & -r & 0 & 0 & \cdot & \cdot & 0 \\ -r & 2+2r & -r & 0 & \cdot & \cdot & 0 \\ 0 & -r & 2+2r & -r & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & -r & 2+2r & -r \\ 0 & 0 & 0 & 0 & 0 & -r & 2+2r \end{bmatrix}$$

If the eigenvalue of matrix B is λ, then for the system to be stable $\left|\frac{4}{\lambda} - 1\right| \le 1$,for the matrix B: $\max_i P_i = |-r| + |-r| = 2r, \quad a_{ii} = 2 + 2r$ the Brours theorem leads to $2 \le \lambda \le 2 + 4r$, give more details about this application.

Exercise13: Then show that the equations (55) are unconditionally stable for $2 \le \lambda$.