Stability, Convergence and consistency:

After presented, how to approximate the derivatives that including in P.D.E. to generate the finite difference schemes for its numerical solution? Should be discussing the follow;

- Verity that these schemes are good approximation to the P.D.E. (consistent).
- Verify that the schemes are stable or no.
- Show that the numerical solution converges to the solution of P.D.E.

Let us to define

$$F_{h,k}(U) = b \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(43)$$

Is a finite scheme and,

$$F\,u = b \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(45)$$

is a partial differential equation. Now we need to light up some definition related to the property of finite difference schemes, as follows;

Definition: we say that a finite difference scheme (43) is consistent with P.D.E.(45) of order (k,h) , if for any smooth function

$$Fu - F_{k,h}U = O(k^r, h^s) \qquad \ldots\ldots\ldots\ldots\ldots(46)$$

To verify consistency expand u in Taylor series and make sure equation (46) holds.

Example: If $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + cu$ approximate by explicit finite difference method, then

(1) show that the finite difference equation given as

$$u_i^{n+1} = (1 + crh^2 + r^2\delta_x^2)u_i^n$$

$$,where \quad r = \frac{k}{h^2}, \; in \; which \; k = \Delta t \; and \; h = \Delta x$$

(2) Show that difference equation and P.D.E. are consistent with the truncation error

$$TE = \frac{k}{2}\frac{\partial^2 u}{\partial t^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. = O(k, h^2)$$

Solution: using forward finite difference approximation (15) for the first order time derivative, and central finite difference approximation(18)for the second order spatial derivative, finite difference equation so obtained is

$$\frac{u_{i+1}^{n+1} - u_i^n}{\Delta t} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + cu_i^n \qquad \ldots\ldots\ldots\ldots\ldots\ldots(43)$$

Rearrangement this equation, we have

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \Delta t c u_i^n \qquad \ldots\ldots\ldots\ldots\ldots\ldots(44)$$

If we put $r = \frac{\Delta t}{(\Delta x)^2}$ and using the definition of the central difference operator, then the finit difference equation becomes

$$u_i^{n+1} = (1 + cr(\Delta x)^2 + r^2\delta_x^2)u_i^n$$

Expand each term in equation(44) ,we obtain

$$u_i^n + \frac{\Delta t}{1!}\frac{\partial u_i^n}{\partial t} + \frac{(\Delta t)^2}{2!}\frac{\partial^2 u_i^n}{\partial t^2} + \frac{(\Delta t)^3}{3!}\frac{\partial^3 u_i^n}{\partial t^3} + \cdots\cdots =$$

$$u_i^n + r(u_i^n + \frac{\Delta x}{1!}\frac{\partial u_i^n}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u_i^n}{\partial x^2} + \frac{(\Delta x)^3}{3!}\frac{\partial^3 u_i^n}{\partial x^3} + \cdots\cdots$$

$$-2u_i^n + u_i^n - \frac{\Delta x}{1!}\frac{\partial u_i^n}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u_i^n}{\partial x^2} - \frac{(\Delta x)^3}{3!}\frac{\partial^3 u_i^n}{\partial x^3} + \cdots\cdots) + cr(\Delta x)^2 u_i^n$$

Rearrangement this equation to obtain

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - cu + \frac{(\Delta t)}{2}\frac{\partial^2 u_i^n}{\partial t^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u_i^n}{\partial x^4} = 0$$

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - cu + TE = 0 \quad ,where \quad TE = \frac{(\Delta t)}{2!}\frac{\partial^2 u_i^n}{\partial t^2} - \frac{(\Delta x)^2}{12}\frac{\partial^4 u_i^n}{\partial x^4} + \cdots\cdots\cdots\cdots$$

This equation of order $O(k,h^2)$.

<u>Exercise</u>7: Approximation P.D.E. in above example by implicit finite difference method ,then find its order of error

Definition: For a function $v = (\ldots\ldots\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\ldots\ldots\ldots)$ on the grid with step size Δx:

$$norm\, v = \|v\| = \left[\Delta x \sum_{m=-\infty}^{\infty} |v_m|^2\right]^{\frac{1}{2}}$$

And for a function f on the real time

$$norm\ f = \|f\| = \left[\int_{-\infty}^{+\infty} |f(x)|^2 dx\right]^{\frac{1}{2}}$$

Definition: a finite one-step difference scheme (43) for a first order P.D.E. is stable if there exist number $k_0 > 0$ *and* $h_0 > 0$ such that for any for any $T > 0$ there exist a constant C_T such that

$$\|v^n\| \le C_T \|v^0\| \quad , \text{For} \quad 0 < nk \le T, 0 < h \le h_0, 0 < k \le k_0$$

Definition: The initial value problem for the first order P.D.E. is well-posed, if for any time $T \ge 0$, there exist C_T such that any solution $u(x,t)$ satisfies

$$\|u(x,t)\| \le C_T \|u(x,0)\| \ \text{ for } \ 0 \le t \le T$$

Definition: A one-step finite difference scheme approximating a P.D.E. is convergent if for any solution to the P.D.E., $u(x,t)$ is approach to numerical solution $u(nh, mk)$ as $h,k \to 0$

Note: A consistent finite difference scheme for a P.D.E. for which the initial value problem well posed is convergent if it is stable.

Definition: Fourier transformation and inversion formula for u defined in region $\Re$ given as;

$$\hat{u}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} u(x)\, dx\, ;$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \hat{u}(p)\, dp$$

For a grid function $v = (\ldots\ldots\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots\ldots\ldots\ldots)$ with grid spacing Δx

$$\hat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} e^{-imh\xi} u_m \ \Delta x\, ;$$

$$u_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{-ipx} \hat{u}(\xi)\, d\xi$$

From the Parseval condition

$$\|u(x)\| = \|\hat{u}(p)\|, \qquad \text{where } \|\hat{u}(\xi)\|^2 = \int_{-\pi/h}^{\pi/h} |\hat{u}(\xi)|^2\, d\xi$$

$$\|\hat{u}(\xi)\| = \|u_m\|$$

Convergent:

The system of algebraic equations that is resulting from recurrence relation of finite difference schemes, written as

$$\vec{u}^{\,n+1} = M\vec{u}^{\,n} + \vec{c} \qquad \ldots\ldots\ldots\ldots\ldots\ldots.. (45)$$

Let $\hat{u}^n$ be the solution of the finite difference system (45) with a perturbed initial conditions;

$$\hat{\vec{u}}^{\,n+1} = M\hat{\vec{u}}^{\,n} + \vec{c} \qquad \ldots\ldots\ldots\ldots\ldots\ldots.. (46)$$

Let $\qquad \varepsilon = \vec{u}^{\,n} - \hat{\vec{u}}^{\,n}$

From (45) and(46) ,we have

$$\vec{u}^{\,(n+1)} - \hat{\vec{u}}^{\,(n+1)} = M(\vec{u}^{\,(n)} - \hat{\vec{u}}^{\,(n)})$$

$$\Rightarrow \vec{\varepsilon}^{\,(n+1)} = M\vec{\varepsilon}^{\,(n)} \Rightarrow \vec{\varepsilon}^{\,(n)} = M^{(n)}\vec{\varepsilon}^{\,(0)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(47)$$

If $M^n \to 0$, as $n \to \infty$,then $\vec{\varepsilon}^{\,(n)} \to 0$ (i.e. the system is convergent).

Definition: if $M^n \to 0$, as $n \to \infty$,then M is convergent.

Definition: Spectrum radius $\rho(M) = \max_i |\lambda_i|$, where λ_i are the eigenvalues of the matrix M .

Theorem: If M is the matrix coefficients and $\rho(M)$ is spectrum radius, then $\|M\| \geq \rho(M)$.

Proof : Suppose $\rho(M) = \max_i |\lambda_i| = \lambda_1$, then

$$M\vec{u} = \lambda_1 \vec{u} \Rightarrow \|M\vec{u}\| = \|\lambda_1 \vec{u}\| \ , \qquad \|\vec{u}\| \neq 0$$

$$\|M\bar{u}\| = |\lambda_1|\|\bar{u}\| \qquad \text{……………………(48)}$$

Also, $$\|M\bar{u}\| \le \|M\|\|\bar{u}\| \qquad \text{……………………(49)}$$

From equations (48)&(49), we get:

$$\|M\|\|\bar{u}\| \ge |\lambda_1|\|\bar{u}\|$$

Since $\|\bar{u}\| \ne 0 \;\; \Rightarrow \|\bar{u}\| > 0$, this implies that

$$\|M\| \ge |\lambda_1| = \rho(M)$$ ■

Theorem: If $\|M\| < 1$, then M is convergent.

Proof:

$$\|M^{(n)}\| = \|MM^{(n-1)}\|$$

$$\|MM^{(n-1)}\| \le \|M\|^{(n)} \quad ,$$ Prove that!

From these relations, we get

$$\|M^{(n)}\| \le \|M\|^{(n)}$$

If $\|M\| < 1$, then $\|M\|^{(n)} \to 0, \, as \; n \to \infty$, this implies that

$M \to 0$, as $n \to \infty$, from the previous definition ,we have
M is convergent ■

Corollary: If $\|M\| < 1$ for any norm then the iterative process for $\hat{\bar{u}}^{n+1} = M\hat{\bar{u}}^{n}$ will converge for every $u^{(0)}$.

Note: it is possible that for some norm that $\|M\| > 1$, but M is still convergent.

Theorem: If $\rho(M) \ge 1$, then M is not convergent.

Proof:

Suppose $\rho(M) = \max_i |\lambda_i| = \lambda_1$, then

$$M\bar{u} = \lambda_1\bar{u} \;\text{, where } \lambda_1 \ge 0 \,, \bar{u} \ne 0$$

From(47***), we have

$$\bar{\varepsilon}^{(n)} = M^{(n)}\bar{\varepsilon}^{(0)}$$

Let $\bar{u} = \bar{\varepsilon}^{(0)}$, then

$$\bar{\varepsilon}^{(1)} = M\bar{\varepsilon}^{(0)} = M\bar{u} = \lambda_1\bar{u}$$

$$\bar{\varepsilon}^{(n)} = M^{(n)}\bar{u} = \lambda_1^{(n)}\bar{u}$$

Since $\lambda_1 \ge 0$, $\bar{u} \ne 0$, then

$\|\bar{\varepsilon}^{(n)}\| = |\lambda_1|^{(n)}\|\bar{u}\|$,dose not approach to zero, as $n \to \infty$, thus
M is not convergent ■

Theorem: Necessary and sufficient condition for $M \ne 0$ be convergent iff $\rho(M) < 1$.