Other finite difference approximation methods:

A weighted average approximation method:

we can introduce a weighted factor θ to some finite difference scheme, such as Crank-Nicolson scheme to produce a more general finite difference approximation to $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$.

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}=\frac{1}{(\Delta x)^2}\left[\theta(u_{i+1,j}^{n+1}-2u_{i,j}^{n+1}+u_{i-1,j}^{n+1})+(1-\theta)(u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n})\right] \quad ..(29)$$

If $\theta=0$ then explicit scheme obtain (equ.(10))

$\theta=1$ then fully implicit scheme obtain(equ.(12))

$\theta=\frac{1}{2}$ then Crank-Nicolson scheme obtain(equ.(14))

In practice $0\le\theta\le 1$.

Alternating direction implicit method (ADIM):

we consider the parabolic P.D.E.

$$\frac{\partial u}{\partial t}=\alpha(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) \quad \ldots\ldots\ldots\ldots..(30)$$

Define over a rectangle region $0<x<a \; and \; 0<y<b$,where α, a, b are constants, $u(x,y,t)$ is known on the boundary of the rectangle.define the coordinates (x,y,t) of mesh points as; $x=i\Delta x$, $y=j\Delta y$, $t=n\Delta t$,where $i, j, n \; are \; +ve$. The implicit Crank-Nicolson finite difference of above parabolic P.D.E. is

$$\frac{u_{i,j}^{n+q}-u_{i,j}^{n}}{\Delta t}=\frac{\alpha}{2}\left[(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\Big|_{i,j}^{n}+(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\Big|_{i,j}^{n+1}\right]\ldots\ldots\ldots(31)$$

This scheme produces $(M-1)(N-1)$ simultaneous equations for each time step.

However, the best method to solve these simultaneous equations is replaced $\frac{\partial^2 u}{\partial x^2}$ by implicit difference approximation and $\frac{\partial^2 u}{\partial y^2}$ by explicit difference approximation in $(n+1/2)$ time level, as

$$\frac{u_{i,j}^{n+1/2}-u_{i,j}^{n}}{\Delta t/2}=\frac{\alpha}{(\Delta x)^2}(\delta_x^2)u_{i,j}^{n+1/2}+\frac{\alpha}{(\Delta y)^2}(\delta_y^2)u_{i,j}^{n} \quad \ldots\ldots\ldots\ldots..(32)$$

These produce $(N-1)$ equations. For next time step, replace $\frac{\partial^2 u}{\partial x^2}$ by an explicit difference approximation and $\frac{\partial^2 u}{\partial y^2}$ by implicit difference approximation. These produce $(M-1)$ equations in $(M-1)$ unknowns. To obtain the solution at time step $(n+1)$

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n+1/2}}{\Delta t/2}=\frac{\alpha}{(\Delta x)^2}(\delta_x^2)u_{i,j}^{n+1/2}+\frac{\alpha}{(\Delta y)^2}(\delta_y^2)u_{i,j}^{n+1} \qquad \ldots\ldots\ldots\ldots(33)$$

Elliptic P.D.E.

Let the problem of determining the steady –state heat distribution in a thin square metal plate with dimensions 0.5m by 0.5m .two adjacent boundaries are held at $0^o C$,and the heat on the other boundaries increases linearly from $0^o C$ at one corner to $100^o C$ where the sides meet. If we place the sides with the zero boundary conditions along the $x-$ and $y-$axes, the problem is expressed as

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \ldots\ldots\ldots\ldots(34)$$

for $(x,y)\in R=\{(x,y)|\,0<x<0.5, 0<y<0.5\}$, with the boundary conditions

$u(0,y)=0, u(x,0)=0, u(x,0.5)=200x, u(0.5,y)=200y$.

If $M=N=4$, the problem has 5×5 grid and the finite difference equation

$$4u_{i,j}-\left(u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\right)=0 \qquad \ldots\ldots\ldots\ldots\ldots(35)$$

For $i=1,2,3$ *and* $j=1,2,3$.

Expressing this in terms of relabeled interior grid points $p_1, p_2,\cdots\cdots p_9$ corresponding to $u_1,u_2,\cdots\cdots u_9$ implies that the equations at the points p_i are

$$\begin{aligned}&for\ j=3,\\ &i=1:\quad 4u_{1,3}-u_{2,3}-u_{1,2}=u_{0,3}+u_{1,4}\\ &i=2:\quad 4u_{2,3}-u_{3,3}-u_{1,3}-u_{2,2}=u_{2,4}\\ &i=3:\quad 4u_{3,3}-u_{1,2}-u_{0,3}=u_{4,3}+u_{3,4}\end{aligned} \qquad \ldots\ldots\ldots\ldots(36a)$$

$$\begin{aligned}&for\ j=2,\\ &i=1:\quad 4u_{1,2}-u_{2,2}-u_{1,3}-u_{1,1}=u_{0,2}\\ &i=2:\quad 4u_{2,2}-u_{3,2}-u_{1,2}-u_{2,3}-u_{2,1}=0\\ &i=3:\quad 4u_{32}-u_{2,3}-u_{3,3}-u_{3,1}=u_{4,2}\end{aligned} \qquad \ldots\ldots\ldots\ldots(36b)$$

$$\begin{aligned}
&for\ j=1,\\
&i=1:\quad 4u_{1,1}-u_{2,1}-u_{1,2}=u_{0,1}+u_{1,0}\\
&i=2:\quad 4u_{2,1}-u_{3,1}-u_{1,1}-u_{2,2}=u_{2,0}\\
&i=3:\quad 4u_{3,1}-u_{2,1}-u_{3,2}=u_{3,0}+u_{4,1}
\end{aligned} \qquad \ldots\ldots\ldots\ldots(36d)$$

Where the right sides of the equations are obtained from the boundary conditions. In fact, the boundary conditions imply that

$$u_{1,0}=u_{2,0}=u_{3,0}=u_{0,1}=u_{0,2}=u_{0,3}=0$$

$$u_{1,4}=u_{4,1}=25 \quad u_{2,4}=u_{4,2}=50 \quad u_{3,4}=u_{4,3}=75$$

The linear system associated with this problem has the matrix form

$$AU=B \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

where

$$A=\begin{bmatrix}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
-1 & 4 & -1 & 0 & -1 & & & & \\
0 & -1 & 4 & -1 & 0 & -1 & & & \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & & \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1\\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix},\ U=\begin{bmatrix}u_{1,3}\\u_{2,3}\\u_{3,3}\\u_{1,2}\\u_{2,2}\\u_{3,2}\\u_{1,1}\\u_{2,1}\\u_{3,1}\end{bmatrix},\ B=\begin{bmatrix}25\\50\\150\\0\\0\\50\\0\\0\\25\end{bmatrix}$$

Exercise 5: Write the linear system of algebraic equations associated with the problem of **exercise 3** in the matrix form.

Note: the linear system of algebraic equations that is resulting from approximation P.D.Es by finite difference method (finite difference equations), needed good procedure to solve it. There are two procedures to achieve this aim, the first is called direct methods(such as, Gauss elimination , LU factorization,……) and the second is called iterative method (such as;Jacobi iterative, Gauss-seidel iterative, successive over relaxation(SOR) iterative ,……..). For example, system (37) is solving by using Gauss-siedel method and the results are;

i,j	(1,3)	(2,3)	(3,3)	(1,2)	(2,2)	(3,2)	(1,1)	(2,1)	(3,1)
$u_{i,j}$	18.75	37.50	56.25	12.50	25.00	37.50	6.25	12.50	18.75
$u(x,y)=400xy$	18.75	37.50	56.25	12.50	25.00	37.50	6.25	12.50	18.75

This mean the truncation error is zero at each step.

Accuracy of the finite difference equation of the numerical scheme:

Accuracy of finite difference schemes can be determined by many ways such as , theoretically through the order of error of the finite difference equation and experimentally through the measurements of errors(L^2 norm, ….).

There is Richardson's approach to limit the order of accuracy of the finite difference equation. In this method, we need to know the truncation error of the difference equation.

Let u represents the solution of differential equation
And U represents the solution of finite difference equation
The discretization error is

$$u-U = Ak + Bh^2 + Ck^2 + Dh^4 + \cdots \qquad \ldots\ldots\ldots\ldots(38)$$

If we use k_1 , h_1 as mesh size to produce U_1, and k_2 , h_2 as mesh size to produce U_2 ,then

$$u-U_1 = Ak_1 + Bh_1^2 + Ck_1^2 + Dh_1^4 + \cdots \qquad \ldots\ldots\ldots(39)$$

$$u-U_2 = Ak_2 + Bh_2^2 + Ck_2^2 + Dh_2^4 + \cdots \qquad \ldots\ldots(40)$$

Subtracting these two equations, we obtain

$$u = \frac{1}{h_2^2 - h_1^2}(h_2^2 U_1 - h_1^2 U_2) + A\frac{k_1 h_2^2 - k_2 h_1^2}{k_1}$$

If we neglect the term involving A , then

$$u = \frac{1}{h_2^2 - h_1^2}(h_2^2 U_1 - h_1^2 U_2), \qquad \text{if } h_1 = h_2 \text{ and } k_1 = k_2, \text{ then}$$

$$u = \frac{1}{3}(4U_1 - U_2) \text{ the error is } O(h^4)$$

Definition: $u-U$ is the discitization error which be reduced by decreasing h *and* k .

Definition: Let $F_{i,j}(U)=0$ represent the finite difference equation at $(i,j)^{th}$ mesh point, then $F_{i,j}(U)$ is called **local truncation** error.

Definition: If N is a numerical solution of finite difference equation that is produce from each calculation is carried up to a finite number of decimal places. Thus, $U-N$ is the **global rounding** error.

Total error= discitization error+ global rounding error$= u_{i,j} - N_{i,j}$.

High accurate formula (high order) for elliptic P.D.E. can be derived by using more terms from the operators series. These formulas are useful when the boundary in highly irregular.

For example: An elliptic P.D.E.(equation(34)) can be approximation by central difference operators as;

$$\frac{1}{(\Delta x)^2}(\delta_x^2)u_{i,j}+\frac{1}{(\Delta y)^2}(\delta_y^2)u_{i,j}=0$$

For high accuracy**,** using the definition given by equation (7)

$$\frac{1}{h^2}\left[\delta_x^2-\frac{1}{12}\delta_x^4+\cdots\cdots\cdots\right]u_{i.j}+\frac{1}{h^2}\left[\delta_y^2-\frac{1}{12}\delta_y^4+\cdots\cdots\cdots\right]u_{i.j}=0$$

Now, for the first two terms, we have

$$\left[\delta_x^2-\frac{1}{12}\delta_x^4\right]u_{i,j}+\left[\delta_y^2-\frac{1}{12}\delta_y^4\right]u_{i,j}=0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(41)$$

The first term

$$\left[\delta_x^2-\frac{1}{12}\delta_x^4\right]u_{i,j}=\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)-\frac{1}{12}\delta_x^2\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)$$

$$=\frac{4}{3}u_{i+1,j}-\frac{5}{2}u_{i,j}+\frac{4}{3}u_{i-1,j}-\frac{1}{12}u_{i+2,j}-\frac{1}{12}u_{i-2,j}$$

Similarly for the second term

$$\left[\delta_y^2-\frac{1}{12}\delta_y^4\right]u_{i,j}=\frac{4}{3}u_{i,j+1}-\frac{5}{2}u_{i,j}+\frac{4}{3}u_{i,j-1}-\frac{1}{12}u_{i,j+2}-\frac{1}{12}u_{i,j-2}$$

Now, equation (41) becomes

$$u_{i,j}=\frac{1}{5}\left\{\frac{4}{3}\left(u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1}\right)-\frac{1}{12}\left(u_{i+2,j}+u_{i-2,j}+u_{i,j+2}+u_{i,j-2}\right)\right\} \qquad \ldots.(42)$$

This finite difference equation is called nine-point formula.

<u>Exercise 6</u>:

1) Determine the order of truncation error of equation(42) .
2) Show that the truncation error of the Laplace equation is $TE=\frac{h^2}{12}(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4})+\cdots\cdots$