** **An explicit methods**
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**Definition:** An explicit method is one in which one unknown values in the $(n+1)^{th}$ level are specifying in terms of known values in the $(n)^{th}$ level.

**Definition:** An implicit method is one in which two or more unknown values in the $(n+1)^{th}$ level are specifying in terms of known values in the $(n)^{th}$ level.

Now, an explicit method can be reducing from the above general finite difference representation of the parabolic P.D.E., and may be writing as,

$$u_i^{n+1} = (1 + r\delta_x^2)u_i^n = u_i^n + r\delta_x^2 u_i^n = u_i^n + r(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$u_i^{n+1} = (1 - 2r)u_i^n + r(u_{i+1}^n + u_{i-1}^n) \qquad \text{...............(10)}$$

Unknown ($u_i^{n+1}$) known ($u_i^n$, $u_{i+1}^n$, $u_{i-1}^n$)

This expression is calling an explicit method involving four-point formula.
Other an explicit form can be obtaining as,

$$u_i^{n+1} = \frac{1}{2}(2 - 5r + 6r^2)u_i^n + \frac{3}{2}(2 - 2r)(u_{i+1}^n + u_{i-1}^n) - \frac{1}{12}(1 - 6r)(u_{i+2}^n + u_{i-2}^n) \text{ ......(11)}$$

This expression is calling an explicit method involving Six-point formula.
An implicit formula of the parabolic P.D.E. can be deriving as follows;
The finite difference formula $u_i^{n+1} = Exp(kD^2)u_i^n$

$$D^2 = \frac{1}{h^2}\left[\delta_x^2 - \frac{1}{12}\delta_x^4 + \frac{1}{90}\delta_x^6 + \cdots\cdots\cdots\right] \quad , \quad D^2 \approx \frac{\delta_x^2}{h^2}$$

$$(1 + 2r)u_i^{n+1} - r(u_{i+1}^{n+1} + u_{i-1}^{n+1}) = u_i^n \qquad \text{..............(12)}$$

This equation is call implicit formula of the parabolic P.D.E. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$. It is suggest by O'Brien et.el , which approximate $\frac{\partial^2 u}{\partial x^2}$ in the $(n+1)^{th}$ time level instead of the $(n)^{th}$ level.
Crank and Nicolson(1947) proposed a method is valid for all values of $r$. They replaced $\frac{\partial^2 u}{\partial x^2}$ by means of its finite difference representations on the $(n+1)^{th}$ time level and $(n)^{th}$ time level.

$$u_i^{n+1} = Exp(kD^2)u_i^n = Exp(\frac{kD^2}{2})Exp(\frac{kD^2}{2})u_i^n \qquad \ldots\ldots\ldots\ldots\ldots(13)$$

$$Exp(-\frac{kD^2}{2})u_i^{n+1} = Exp(\frac{kD^2}{2})u_i^n$$

$$Exp(-\frac{r}{2}\delta_x^2)u_i^{n+1} = Exp(\frac{r}{2}\delta_x^2)u_i^n$$

$$\left[1-\frac{r}{2}\delta_x^2 + \frac{r^2}{(4)\cdot(2!)}\delta_x^4 + \cdots\cdots\cdots\right]u_i^{n+1} = \left[1+\frac{r}{2}\delta_x^2 + \frac{r^2}{(4)\cdot(2!)}\delta_x^4 + \cdots\cdots\cdots\right]u_i^n$$

$$\left(1-\frac{r}{2}\delta_x^2\right)u_i^{n+1} \cong \left(1-\frac{r}{2}\delta_x^2\right)u_i^n$$

$$u_i^{n+1} - \frac{r}{2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) = u_i^n + \frac{r}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \quad \ldots\ldots(14)$$

This equation is call Crank-Nicolson implicit formula of the parabolic P.D.E. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

**<u>Exercise2</u>**: write down the following parabolic P.D.E. with variable coefficients

(1) $\frac{\partial u}{\partial t} = a(x)\frac{\partial^2 u}{\partial x^2}$ (Hint: put $L = a(x)D^2$)

(2) $\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a(x)\frac{\partial u}{\partial x})$

<u>Note:</u> we can approximate the derivatives (first-order &second-order) by finite difference depend on the following definition

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0}\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} \Rightarrow \frac{\partial u}{\partial x} \approx \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

Thus, this derivative at point $(i, j)$ can expression as follows,

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x} \qquad \textit{Forward difference} \qquad \ldots\ldots\ldots\ldots\ldots(15)$$

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} \approx \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \qquad \textit{Backward difference} \qquad \ldots\ldots\ldots\ldots\ldots(16)$$

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} \qquad \textit{Central difference} \qquad \ldots\ldots\ldots\ldots\ldots(17)$$

From these relations the second derivative can written as follows

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{\partial}{\partial x}(\left.\frac{\partial u}{\partial x}\right|_{i,j}) = (\left.\frac{\partial u}{\partial x}\right|_{i+1,j} - \left.\frac{\partial u}{\partial x}\right|_{i-1,j})/\Delta x$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2} \quad Central\,difference \qquad \ldots\ldots\ldots\ldots\ldots(18)$$

We can derivation these formulas by using Taylor's theorem. Apply Taylor's expand to the function $u$ at $x+\Delta x$ and $x-\Delta x$, we obtain

$$u(x+\Delta x,k) = u(x_i,t_n) + \frac{\Delta x}{1!}\frac{\partial u(x_i,t_n)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u(x_i,t_n)}{\partial x^2} + \cdots \qquad \ldots\ldots\ldots\ (19)$$

$$u(x-\Delta x,k) = u(x_i,t_n) - \frac{\Delta x}{1!}\frac{\partial u(x_i,t_n)}{\partial x} + \frac{(\Delta x)^2}{2!}\frac{\partial^2 u(x_i,t_n)}{\partial x^2} + \cdots \qquad \ldots\ldots\ (20)$$

Use these relations to prove above formulas.

**Discuss the truncation errors**

From equation( 19) , we get

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{u_{i+1}^n - u_i^n}{\Delta x} - \frac{\Delta x}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^2}{3!}\frac{\partial^3 u}{\partial x^3} + \cdots\cdots\cdots \\ &= \frac{u_{i+1}^n - u_i^n}{\Delta x} + O(\Delta x)\end{aligned} \qquad \ldots\ldots(21)$$

This is the Forward finite difference scheme .it is of the first-order of $\Delta x$.

Similarly, From equation(20) , we have

$$\frac{\partial u}{\partial x} = \frac{u_i^n - u_{i-1}^n}{\Delta x} + O(\Delta x) \qquad \ldots\ldots\ldots\ldots\ldots(22)$$

This is the Backward finite difference scheme .it is of the first-order of $\Delta x$

Subtracting (19) and (20), we obtain

$$\frac{\partial u}{\partial x} = \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + O((\Delta x)^2) \qquad \ldots\ldots\ldots\ldots(23)$$

This is the Central finite difference scheme .it is of the second-order of $(\Delta x)^2$

Adding (19) and (20), we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1}^n - 2u_i^2 + u_{i-1}^n}{(\Delta x)^2} + O((\Delta x)^2) \qquad \ldots\ldots\ldots(24)$$

This is the Central finite difference scheme .it is of the second-order of $(\Delta x)^2$

**Derivative boundary conditions**

For solving parabolic P.D.E $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ subject to the initial and boundary conditions

$u(x,0) = f(x) \quad for\ 0 \le x \le 1$ (where $f(x)$ is known function)

$$\frac{\partial u(0,t)}{\partial x} = u \quad for \;\; t > 0$$

$$\frac{\partial u(1,t)}{\partial x} = -u \quad for \;\; t > 0$$

by using explicit method, we obtain

$$u_i^{n+1} = (1-2r)u_i^n + r(u_{i+1}^n + u_{i-1}^n)$$

For $i = 0 \Rightarrow u_0^{n+1} = (1-2r)u_0^n + r(u_1^n + u_{-1}^n)$

Here appear problem in the computation of the recurrence relation exactly in the term $u_{-1}^n$. One of the treatments for this problem, one can used central difference scheme (equation (23)) for the boundary condition $\frac{\partial u(0,t)}{\partial x} = u$ gives

$$u_{-1}^n = u_1^n - 2\Delta x\, u_0^n$$

Similarly, For $i = M \Rightarrow u_M^{n+1} = (1-2r)u_M^n + r(u_{M+1}^n + u_{M-1}^n)$, such that problem is appear in the term $u_{m+1}^n$,so by using the same previous way ,we have

$$u_{M+1}^n = u_{M-1}^n - 2\Delta x\, u_M^n$$

**Exercise3:**

A bar, with ends at $x = 0$ and $x = a$ ,with insulated ends, has an initial temperature distribution $u(x,0) = f(x)$

1) Write down the boundary value problem that corresponding to the physical problem.
2) Approximate the P.D.E. resulting in part(1) by:
   **a**- explicit method **b**-implicit method **c**-Crank-Nicolson method.

**Exercise4:**

Derive the following expressions:

1) $\left.\frac{\partial^2 u}{\partial x \partial y}\right|_{i,j} \approx \frac{u_{i+1,j+1} + u_{i-1,j-1} - u_{i-1,j+1} + u_{i+1,j-1}}{4(\Delta x \Delta y)}$ ,which is second order mixed central difference with respect to $x \; and \; y$.

2) $\left.\frac{\partial u}{\partial y}\right|_{i,j} \approx \frac{1}{6\Delta y}(-11u_{i,j} + 18u_{i,j+1} - 9u_{i,j+2} + 2u_{i,j+3})$ , which is third order difference with respect to $y$.

**Two-dimension P.D.E.**

(1) the 2D heat equation in the $(x, y, t)$ plane may be written as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \ldots\ldots\ldots\ldots\ldots(25)$$

Here we can define the differential operator as $L = D_x^2 + D_y^2$ , thus, equation(25) can be written as

$$\frac{\partial u}{\partial t} = Lu \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Using the finite difference code; $u_{i,j}^{n} = u(x_i, y_j, t_n)$, approximation of equation (26) is

$$u_{i,j}^{n+1} = Exp(KL)u_{i,j}^{n}$$
$$= \left(Exp(KD_x^2)\cdot Exp(KD_y^2)\right)u_{i,j}^{n}$$
$$= (Exp(r(\delta_x^2 - \frac{\delta_x^4}{12}\cdots\cdots))\cdot Exp(r(\delta_y^2 - \frac{\delta_y^4}{12}\cdots\cdots)))u_{i,j}^{n} \qquad \ldots\ldots..(27)$$
$$= (Exp(r\delta_x^2)\cdot Exp(r\delta_y^2))u_{i,j}^{n}$$
$$= (1 + r\delta_x^2)(1 + r\delta_y^2)u_{i,j}^{n}$$

This is explicit scheme for 2D parabolic P.D.E.

(2) Using an explicit method to approximation the2D parabolic P.D.E.

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}(a(x,y)\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(b(x,y)\frac{\partial u}{\partial y}) \qquad \ldots\ldots\ldots\ldots(28)$$

(3)Use an implicit scheme to approximation the 2D parabolic P.D.E.

**Explicit method:**

Advantage — Relatively simple to set up and program

Disadvantage — in terms of above example, for a given $\Delta x, \Delta t$ must be less than some limit imposed by stability constraints. In some cases, $\Delta t$ must be very small to maintain stability; this can result in long computer running times to make calculations over a given interval of $t$.

**Implicit method:**

Advantage — stability can be maintained overt much large values of $\Delta t$, hence using considerable fewer time steps to make calculations over a given interval of $t$. This result in less computer time.

Disadvantage — more complicated to set up and program

Disadvantage — Since massive matrix manipulations are usually required at each time step, the computer time per time step is much larger than in the explicit approach.

Disadvantage — Since large $\Delta t$ can be taken, the truncation error is large, and the use of implicit methods to follow the exact transient (time variation of the independent variable) may not be as accurate as an explicit approach. However, for a time- dependent solution in which the steady state is the desired result, this relative time wise inaccuracy is not important.