MSc. course

Numerical Methods for Partial Differential Equations

Syllabus

* Introduction and preliminaries, operators,
* Finite difference formula,
- finite difference formula for linear partial differential equations (parabolic, elliptic, hyperbolic),high order formula for finite difference methods,
- finite difference formula for nonlinear partial differential equations, analysis of finite difference formula (consistency, convergence , stability).

* Finite elements method (FEM),
- properties,
- Lagrange polynomial
- relation between interpolation polynomial and shape function ,
- varitional methods, Rayleigh-Ritz method(RRM) ,relation between RRM and FEM.
- weighted residual method, collocation method ,least square method and Galerkin method.

* Differential quadrature method (DQM),
- analysis of DQM constructor ,
- applications.

* Other methods(Adomian decomposition method, variaitional iteration method, homotopy method,…)with applications.

References:

1- simth G.D."Numerical Solution of Partial Differential Equations,Finite Difference Methods" London.1978
2- Noye B.J. " Numerical Solution of Partial Differential Equations " North-Australia,Holand,1981
3- 3-RaoS.S. " The finite Element in Engineering" U.S.A 1982
4- Michel A.R." Computational Methods in Partial Differential Equations " London 1976
5- Grossmann C. and Roos H-G." Numerical Treatment of Partial Differential Equations" Springer-Verlag Berlin Heidelberg 2007
6- Strikwerda J.C." finite difference schemes and Partial Differential Equations" U.S.A 2004

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Numerical solution of partial differential equations

Numerical analysis is a branch of applied mathematics; the subject can be standard with a good skill in basic concepts of mathematics. This subject has many applications and wide uses in the area of applied sciences such as, physics, engineering, Biological, …ect. So, when any body wants to study this subject, should be to get answers, which do not agree with experiment or observation data. This is because there always has to be careful choice of the mathematical model that is to be used to describe a particular phenomenon. The problems of the real subject of P.D.Es are possible great complexity involving many physical effects (or other sciences) and a considerable set of non-linear equations. These problems can not be solved either by advanced techniques or by putting then on the computer. The techniques do not exist and the machines are neither powerful enough nor sophisticated enough (to reject spurious solutions). the problem only be omitting, after much careful thought, perhaps and special case can be dealt with analytically , and this will show what sort of calculation the machines must be programmed for more general case. After determine the mathematical model for the problems, should be try to solve it. For this situation, we need good mathematical procedure to simplified or linearized problems, which are non-linear or involving complex geometries, or both. Here the numerical techniques such as finite difference, finite elements, differential quadrature ,….ect;are play important role to computational of problems are described by a set of linear and/or non-linear equations.

Important examples of the three type equations are the

$$u_{xx} + u_{yy} = 0 \quad Laplace \quad equation$$

$$u_t = u_{xx} \quad Heat \quad equation$$

$$u_{tt} = u_{xx} \quad Wave \quad equation$$

Before derivation of finite difference formulas, which are using to approximation partial differential equations, we wanted to introduce classification of second order linear partial differential equations

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where $A, B, C, D, .E, F, and\ G$ are functions $x\ and\ y$

Now, If

$$B^2 - 4AC \begin{cases} < 0 \Rightarrow & Elliptic \quad P..D..E. \\ = 0 \Rightarrow & Parabolic \quad P..D..E. \quad examples \\ > 0 \Rightarrow & Hyperbolic \quad P..D..E. \end{cases} \qquad \begin{matrix} u_{xx} + u_{yy} = 0 & Laplace \quad equation \\ u_t = u_{xx} & Heat \quad equation \\ u_{tt} = u_{xx} & Wave \quad equation \end{matrix}$$

And, also we need to give information about u on the boundary (C) of $\boldsymbol{R}$ (Fig(1)

- given u on C [Dirichlet problem]
- given $\frac{\partial u}{\partial n}$ on C, where n the norm [Neuman problem]
- $\alpha u + \beta \frac{\partial u}{\partial n}$, where α, β are given, [Mixed problem]

Example 1:

One end of a bar $2\,ft$ long .whose sides are insulated, is kept at the temperature $0^0 C$,while the other end is kept at $10^0 C$. If the initial temperature distribution is linear along the bar, write down the boundary value problem that governing the temperature in the bar.

The bar has the length $2\,ft$ (i.e. $\Omega = [0,1]$), then by conservation law of energy ,we have

$$s\rho A u_t(x_1, t)\Delta x = KA\left[u_x(x+\Delta x) - u_x(x,t)\right]$$

Where the constant s is the specific heat of the material, ρ is the mass per unit volume, and x_1 is between $x\ and\ x+\Delta x$, K is the thermal conductivity (positive constant), and A is the area of a cross section . Dividing through in this equation by $s\rho A\Delta x$ and then letting Δx approach to zero, we obtain the equation

$$u_t(x,t)\Delta x = \alpha\, u_{xx}(x,t) \quad 0 < x < 2,\ t > 0$$

Where $\alpha = K / s\rho$ is the thermal diffusivity of material.
One end kept at the temperature $0^0 C$ and the other end is kept at $10^0 C$

$$\Rightarrow u(0,t) = 0 \quad and \quad u(2,t) = 10 \quad , t \ge 0$$

The initial temperature distribution is linear along the bar

$$\Rightarrow u(x,0) = 5x \quad , 0 \le x \le 2$$

Therefore, the mathematical model for this problem is

$$u_t = \alpha\, u_{xx}$$
$$u(0,t) = 0 \quad and \quad u(2,t) = 10 \quad , t \ge 0$$
$$u(x,0) = 5x \quad , 0 \le x \le 2$$

Example 2:

A string is stretched between the fixed points $(0,0)\ and\ (1,0)$ and released at rest from the position $u = A\sin(\pi x)$,where A is a constant. Write down the mathematical model that governing the transverse displacement of a string.

The mathematical model for this problem is

$u_{tt} = \alpha^2 u_{xx}$ (**For** derivation you can see Churchill R. 'Fourier sires and Boundary Value Problems' page 5)

$u(0,t) = 0 \, and \, u(1,t) = 0 \, , t \geq 0$ (A string infixed at points $(0,0)$ *and* $(1,0)$ $\Rightarrow$ there is **no** displacement)

$u(x,0) = A\sin(\pi x) \quad , 0 \leq x \leq 1$ (Initial displacement, at $t = 0$)

Depending on the above information**,** the following is a rough summary of well-posed problems for second-order partial differential equations:

elliptic equation	plus boundary conditions
parabolic equation	plus boundary conditions with respect to space
	plus initial condition with respect to time
hyperbolic equation	plus boundary conditions with respect to space

Finite difference methods

One of the greatest needs in applied mathematics is a general and reasonably short method of solving partial differential equations by numerical methods. Several methods have been proposed for meeting this need, but none can be called entirely satisfactory. They are all long and laborious. Certain types of boundary value problems can be solved by replacing the differential equation by the corresponding difference equation and then solving the latter by a process of iteration. This method of solving partial differential equations was devise and first used by Richardson (1910). It was later improved by Liebmann(1918) and further improved more recently by Shortley &Weller (1938).the process is slow, but gives good results on boundary value problems which satisfy Laplace , Poisson, and several other partial differential equations. A strong point in its favor is that an automatic sequence-controlled calculating machine can do the computation.

A somewhat similar method is the relaxation method devised by Southwell. This method is shorter and more flexible than the iteration method, but is not adapted to automatic machine computation. In both of these methods the approximate solution of a partial differential equations with given boundary values, is found by finding the solution of the corresponding partial differential equation.

Operators: it is a mathematical operation on an operated function.

- Shifts (translation) operator $Ef(x) = f(x+h)$
- Difference operator $\Delta f(x) = f(x+h) - f(x)$
- Inverse difference operator $\nabla f(x) = f(x) - f(x-h)$
- Intermediate operator $\delta f(x) = f(x+h/2) - f(x-h/2)$

Properties of operators:

- Linearity of operator $E(f+g)=E(f)+E(g)$
- Product of operator $E\cdot E\cdot E\ f=E^3 f$
- Sum and difference operator $(E\mp D)f(x)=Ef(x)\mp Df(x)$
- Equality of operator $E_1=E_2 \Leftrightarrow E_1 f(x)=E_2 f(x)$
- Identity(unit) operator $I\ f(x)=f(x)$
- Null(zero) operator $0\ f(x)=0$

Exercise1: Prove that (a) $\Delta=E-1$
(b) $E^n D=DE^n$
(c) $E=e^{hD}$

Inverse operator: it is a mathematical operator that inverse the original operation.

For example; Shifts operator is $Ef(x)=f(x+h)$, the inverse of it is
$E^{-1}f(x)=f(x-h)\quad ((EE^{-1}=1))$
Difference operator $\Delta=E-1$, the inverse of it is
$\nabla=1-E^{-1}$

Show that
(1) $\delta_x=E^{1/2}-E^{-1/2}$
(2) $\delta_x^2=E+E^{-1}-2$
(3) $E^{-1/2}-\delta/2-\mu=0 \quad where \quad \mu=(E^{1/2}+E^{-1/2})/2$
(4) $(E^{-1/2}-\delta_x/2)^2-\delta_x^2/4=1$

Finite difference formulas:

Now, the area of integration $\boldsymbol{R}$ is covering by rectangular meshes $P_{ij}=P(i\Delta x, n\Delta t)$, are called mesh points. For a function u of a single variable, the familiar expression $\dfrac{u(x+\Delta x)-u(x)}{\Delta x}$, is called difference quotient, whose limiting value is the derivative of $u(x)$ with respect to x i.e.

$$u'(x)=\lim_{\Delta x\to 0}\frac{u(x+\Delta x)-u(x)}{\Delta x}\Rightarrow u'(x)\approx\frac{u(x+\Delta x)-u(x)}{\Delta x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

That is mean; a difference quotient approximates the derivative, the approximation becoming closer as Δx become small.

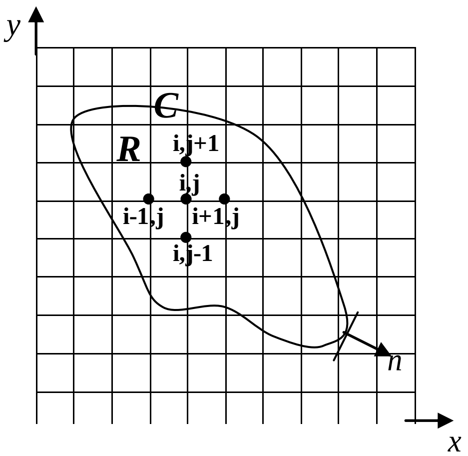

Figure (1)

Suppose we like to solve the parabolic P.D.E.

$$\frac{\partial u}{\partial t} = L(x,t,D,D^2)u$$

$$for\ example \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad L = D^2$$

Using Tayler's series expand $u(x.t+k)$ about (x,t)

$$u(x,t+k) = u(x_i,t_n) + \frac{k}{1!}\frac{\partial u(x_i,t_n)}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u(x_i,t_n)}{\partial t^2} + \cdots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$= (1 + \frac{k}{1!}\frac{\partial}{\partial t} + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2} + \cdots\cdots\cdots\cdots)u(x_i,t_n)$$

Using the finite difference code; $u_i^n = u(x_i,t_n)$

$$u_i^{n+1} = Exp(k\frac{\partial}{\partial t})u_i^n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

This is finite difference representation of the parabolic P.D.E.

As special case; let $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad L = D^2$

$$u_i^{n+1} = Exp(kD^2)u_i^n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$hD = \ell n(E) = \ell n(1+\Delta) = 2\ell n\left[\frac{\delta}{2} + (1+(\frac{\delta}{2})^2)^{1/2}\right] \qquad \text{Prove that!}$$

$$D = \frac{1}{h}\sinh^{-1}(\frac{\delta}{2}) \qquad \text{Prove that!}$$

$$D = \frac{1}{h}\left[\delta_x - \frac{1}{2^2 3!}\delta_x^3 + \frac{1}{2^4 5!}\delta_x^5 + \cdots\cdots\cdots\right] \qquad \ldots\ldots\ldots\ldots(6)$$

$$D^2 = \frac{1}{h^2}\left[\delta_x - \frac{1}{2^2 3!}\delta_x^3 + \frac{1}{2^4 5!}\delta_x^5 + \cdots\cdots\cdots\right]^2$$

$$= \frac{1}{h^2}\left[\delta_x^2 - \frac{1}{12}\delta_x^4 + \frac{1}{90}\delta_x^6 + \cdots\cdots\cdots\right] \qquad \ldots\ldots\ldots\ldots(7)$$

$$u_i^{n+1} = Exp(\frac{k}{h^2}\left[\delta_x^2 - \frac{1}{12}\delta_x^4 + \frac{1}{90}\delta_x^6 + \cdots\cdots\cdots\right])u_i^n$$

$$= Exp(r\delta_x^2)u_i^n \cdot Exp(-\frac{r}{12}\delta_x^4)u_i^n \cdot Exp(\frac{r}{90}\delta_x^6)u_i^n \cdots\cdots\cdots \qquad \ldots\ldots\ldots\ldots(8)$$

$$= ((1 + r\delta_x^2 + \frac{r}{2}(r - \frac{1}{6})\delta_x^4 + \frac{r}{6}(r^2 - \frac{r}{2} + \frac{1}{15})\delta_x^6 + \cdots\cdots\cdots)u_i^n$$

$$where \quad r = \frac{k}{h^2},\ in\ which\ k = \Delta t\ and\ h = \Delta x$$

This is the general finite difference representation of the parabolic P.D.E. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

Generally, there are two standard methods:-