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Chapter One
Random Variables

Space Sample : The Sample denoted by S is the collection or totality of all possible outcomes of a conceptual experiment.

Random Variable : The random variable denoted by X or $X(.)$ is a real number which associated outcomes of the random experiment. (i.e. The random variable is a function with domain S (sample space) and codomain the real line R, that is a real function defined on the sample space). $P(\text{Random Variable}) : R \to [0,1]$

Discrete Random Variable: A r.v. X will be defined to be discrete if the range of X is countable , say $x_1, x_2, x_3, \ldots$

Continuous Random Variable : A r.v. X will be defined to the continuous if the range of X is uncountable, say $a \le x_i \le b \quad i = 1,2,3,\ldots$

Note: The space of a r.v. X is the real numbers $= \{x : X(w) = x, w \in S\}$

Ex.1: Consider the experiment of tossing a single coin; then the sample space $S = \{head, tail\}$. Let the r.v. X denote the number of heads , hence

$$X(w) = \begin{cases} 0 & , \text{if } w = tail \\ 1 & , \text{if } w = head \end{cases} \qquad X(w) = \begin{cases} 0, & \text{if } w = head \\ 1, & \text{if } w = tail \end{cases} \leftarrow \text{number of tails}$$

So, the r.v. X associates a real number with each outcomes of the experiment (i.e. $X = \{0, 1\}$)

Ex.2: Consider experiment of tossing two dice ,then the sample space $S = \{(i, j) : i, j = 1, 2, \ldots, 6\}$. Let the r.v. X denote the sum of the upturned faces , so $X(w) = i + j, \; w = (i, j) \in S$; so $X = \{2, 3, 4, \ldots, 12\}$.

Probability Density Function : Any function $f(x)$ with domain the real line R and codomain $[0, \infty)$ (i.e. $f : R \to [0, \infty)$) is called probability density function (p.d.f.) of a r.v. X if the following conditions are satisfies ;

1. $f(x) \ge 0 \quad , \forall x$

2. a) $\sum_x f(x) = 1 \quad$, if X a discrete r.v. $\qquad$ b) $\int_x f(x)dx = 1 \quad$, if X a continuous r.v

Remark :The probability $\Pr\{x \in A\}$ is completely determined by the probability density function $f(x)$ of the random variable X as following :

a) $\sum_{x \in A} f(x) = \Pr\{x \in A\}$, if X is discrete r.v.

b) $\int_A f(x)dx = \Pr\{x \in A\}$, if X is continuous r.v

②

Ex.3: Let the r.v. X have the p.d.f.

$$f(x)=\begin{cases}2c/(x-1)! & ,x=1,2,3,\dots\\ 0 & \text{o.w.}\end{cases}$$

Find 1) The constant c 2) The $Pr(A)$ **where** $A=\{x:2\le x\le 5\}$

Solution:

1. by properties of the p.d.f. $f(x)$ **of a r.v.** X **we get :**

① متسلسلة هندسية غير منتهية مثل $c+c^2+c^3+\dots$ مجموعها $= \boxed{\frac{c}{1-c}}$

$$\sum_{\forall x} f(x)=1 \Rightarrow \sum_{x=1}^{\infty}\frac{2c}{(x-1)!}=1$$

$$\Rightarrow 2c\left(\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\dots\right)=1$$

② متسلسلة هندسية منتهية مثل $\frac{1}{2}+\left(\frac{1}{2}\right)^2+\dots+\left(\frac{1}{2}\right)^{n}$ مجموعها $= \boxed{\frac{c(1-c^n)}{1-c}}$

$$\Rightarrow 2c(e^1)=1 \quad \left(\text{since } e^x=\sum_{i=1}^{\infty}\frac{x^i}{i!}=\frac{x^0}{0!}+\frac{x^1}{1!}+\dots\right)$$

$$\Rightarrow c=\frac{1}{2e} \Rightarrow c=0.367 \qquad \boxed{e^1=2.7182}$$

2) $Pr(A)=Pr(2\le X\le 5)=\sum_{x=2}^{5} f(x)=\frac{1}{e}\sum_{x=2}^{5}\frac{1}{(x-1)!}=\frac{1}{e}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}\right)=0.63$

③ $e^x=\sum_{i=0}^{\infty}\frac{x^i}{i!}=\frac{x^0}{0!}+\frac{x}{1!}+\frac{x^2}{2!}+\dots$

Ex.4: Let the r.v. X have the p.d.f. $f(x)=\begin{cases}kxe^{-x} & ,x\ge 0\\ 0 & \text{o.w.}\end{cases}$

,find the $Pr(A_1\cup A_2)$, $Pr(A_1\cap A_2)$ **where** $A_1=\{x:1\le x\le 3\}$ **and** $A_2=\{x:2\le x\le 5\}$

⓪ متسلسلة عددية $\sum_{i=1}^{x} i = 1+2+3+\dots+x$, $\frac{x}{2}(x+1)$ مجموعها $=\frac{x(x+1)}{2}$

Solution:

Firstly we find the constant k. by using the properties of p.d.f. $f(x)$

$$\int_{\forall x} f(x)\,dx=\int_0^{\infty} kxe^{-x}\,dx=1$$

By part integration method $\int u\,dv = uv-\int v\,du \Rightarrow \int_0^{\infty} xe^{-x}dx=-xe^{-x}+\int_0^{\infty} e^{-x}dx$

$u=x \Rightarrow du=dx$ $\qquad = \left[-xe^{-x}-e^{-x}\right]_0^{\infty}=[0+1]=1$

$dv=e^{-x}dx \Rightarrow v=-e^{-x}$ $\qquad uv+\int v\,du$

$0-(-1),\ 0+1=1$

$$\int_{\forall x} f(x)\,dx=k\int_0^{\infty} xe^{-x}\,dx=1 \Rightarrow k\{[-xe^{-x}]_{x=0}^{\infty}+\int_0^{\infty} e^{-x}\,dx\}=1$$

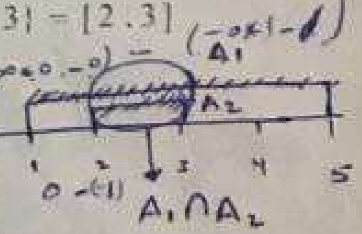
$(-\infty e^{-\infty}-e^{-\infty})-(-0e^{-0}-e^{-0})$

$$\Rightarrow -k[e^{-x}]_{x=0}^{\infty}=1 \Rightarrow k=1$$

$(-0\times 1-1)$

$=(-\infty\times 0-0)$

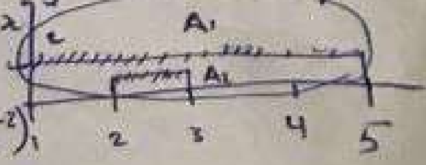
Now $A_1\cup A_2=\{x:1\le x\le 5\}=[1,5]$ $A_1\cap A_2=\{x:2\le x\le 3\}=[2,3]$



$$Pr(A_1\cup A_2)=\int_1^5 f(x)dx=\int_1^5 xe^{-x}dx=-xe^{-x}]_{x=1}^{5}+\int_1^5 e^{-x}dx$$

$$=-5e^{-5}+e^{-1}+[-e^{-x}]_{x=1}^{5}=2e^{-1}-6e^{-5}=0.74-0.04=0.69$$

$$Pr(A_1A_2)=\int_2^3 f(x)dx=\int_2^3 xe^{-x}dx$$



$$=-xe^{-x}\Big]_2^3+\int_2^3 e^{-x}dx=-xe^{-x}\Big]_2^3+\left[-e^{-x}\right]_2^3$$

$$=-3e^{-3}+2e^{-2}+(-e^{-3}+e^{-2})$$

$$=2e^{-2}-4e^{-3}$$

$$=0.27-0.199=0.07$$

③

دوال التوزيع التراكمي

Cumulative Distribution Function : **The cumulative distribution function of a r.v. X denoted by $F_X(x)$ or $F(x)$, is defined to be that function with domain the real line R and codomain the closed interval [0 , 1] (i.e. $f : R \to [0,1]$) which satisfies**

$$F(x) = \Pr(X \le x) = \begin{cases} \sum_{\forall u \le x} f(u) & , \text{ if X a discrete r.v.} \\ \int_{-\infty}^{x} f(u)\,du & , \text{ if X a continuous r.v} \end{cases}$$

s s s

$\lim_{x\to-\infty}$ [F(x)] $\lim_{x\to+\infty}$

0 1

Properties of a cumulative distribution function :

1) $0 \le F(x) \le 1$

2) $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$, $F(+\infty) = \lim_{x\to+\infty} F(x) = 1$

رتيبة (رتيبه) دالة غير تناقصية

3) $F(x)$ **is monotone, non-decreasing function; that is if** $a < b \Rightarrow F(a) \le F(b)$

4) $F(x)$ **is right continuous ; that is** $\lim_{0<h\to 0} F(x+h) = F(x)$ مستمرة من جهة اليمين

5) $\Pr(a < X \le b) = F(b) - F(a)$, $\forall a < b$

6) $\Pr(X = x) = F(x) - F(x^-)$

where $F(x^-)$ **are the left hand limit of** $F(x)$ **.**

⑦ $f(x) = \dfrac{dF(x)}{dx}$

دالة كثافة الاحتمال تمثل مشتقة دالة التوزيع

the p.d.f function the partial of c.d.v. for the random variable x

Ex.5: **Let** $f(x) = \begin{cases} x/15 & x = 1,2,3,\dots \\ 0 & o.w. \end{cases}$ **be the p.d.f. of a r.v. X**

ارسم

1) **Sketch graph of c.d.f.** $F(x)$ **.**

2) **Compute** $\Pr(2 \le X \le 5)$ **by using two methods.** → probability ? → Distribution function?

3) **Compute** $\Pr(X = 2)$ **by using two methods.** Same ?

Solution:

1)

$\sum_{i=1}^{x} i$ ~ مجموعها $\left[\frac{x}{2}(x+1)\right]$

$$F(x) = \Pr(X \le x) = \sum_{u=1}^{x} f(u) = \sum_{u=1}^{x} \frac{u}{15} = \frac{1}{15}(1 + 2 + \dots + x)$$

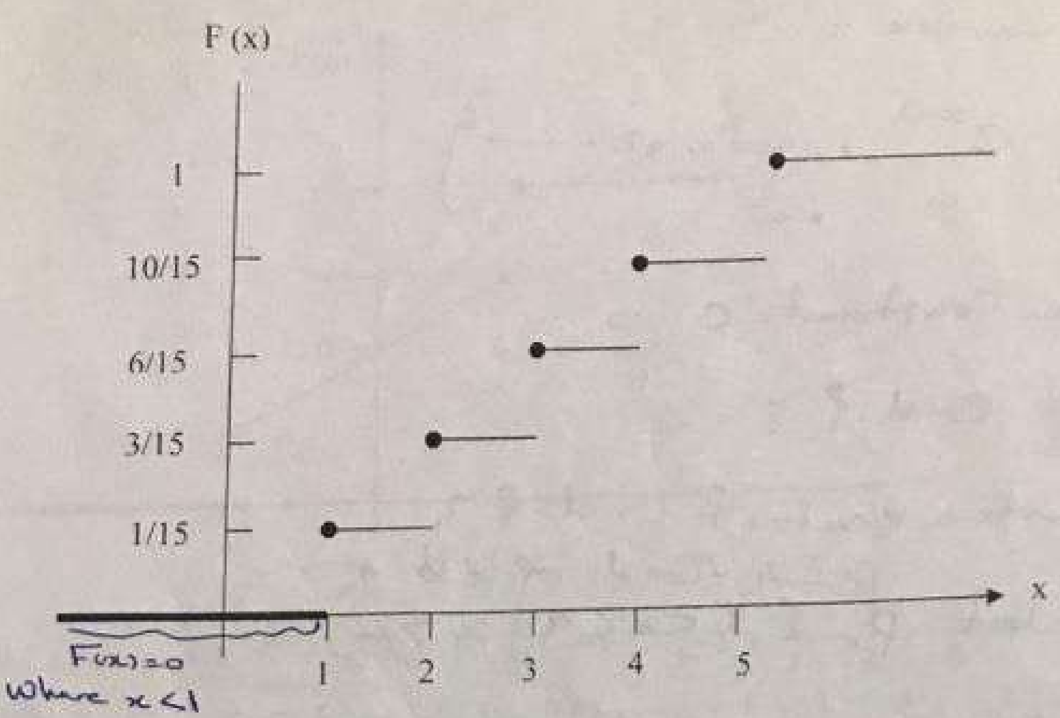
متسلسلة عددية

$$= \frac{1}{15}\left[\frac{x}{2}(x+1)\right] \text{ since } \sum_{i=1}^{x} i = 1 + 2 + \dots + x \text{ is a numerical series}$$

$$= \frac{1}{30}x(x+1) \quad x = 1,2,3,\dots$$

$$F(x) = \begin{cases} 0 & , x < 1 \\ 1/15 & , 1 \le x < 2 \quad = p(x=1) = \frac{1}{15} \\ 3/15 & , 2 \le x < 3 \quad p(x=2) = p(x=2) + p(x=1) = \frac{2}{15} + \frac{1}{15} = \frac{3}{15} \\ 6/15 & , 3 \le x < 4 \quad p(x=3) = p(x=3) + p(x=2) = \frac{3}{15} + \frac{3}{15} = \frac{6}{15} \\ 10/15 & , 4 \le x < 5 \quad p(x=4) = p(x=4) + p(x=3) = \frac{4}{15} + \frac{6}{15} = \frac{10}{15} \\ 1 & , 5 \le x \quad p(x=5) = p(x=5) + p(x=4) = \frac{5}{15} + \frac{10}{15} = \frac{15}{15} = 1 \end{cases}$$

4)



2) a. By depending on c.d.f. F(x)

$$\Pr(2 \le X \le 5) = \Pr(X \le 5) - \Pr(X \le 2) = F(5) - F(2^-) = 1 - \frac{1}{15} = \frac{14}{15}$$

b. By depending on p.d.f. f(x)

$$\Pr(2 \le X \le 5) = \sum_{x=2}^{5} f(x) = f(2) + f(3) + f(4) + f(5) = \frac{2}{15} + \frac{3}{15} + \frac{4}{15} + \frac{5}{15} = \frac{14}{15}$$

3) a. $\Pr(X = 2) = F(2) - F(2^-) = \frac{3}{15} - \frac{1}{15} = \frac{2}{15}$ **(by depending on c.d.f. F(x))**

b. $\Pr(X = 2) = \sum_{x=2}^{2} f(x) = f(2) = \frac{2}{15}$ **(by depending on p.d.f. f(x))**

Exercise: Let $f(x) = (c/3)^x \ , x = 1, 2, \ldots$ **be the p.d.f. of a r.v. X,find the value of** c **and sketch graph of c.d.f.** $F(x)$ **?**

Ex.6: Let $F(x) = \begin{cases} 0 & , \quad x < -1 \\ (x+2)/4 & , -1 \le x < 1 \\ 1 & , \quad 1 \le x \end{cases}$ **be the c.d.f. of a r.v. X ,**

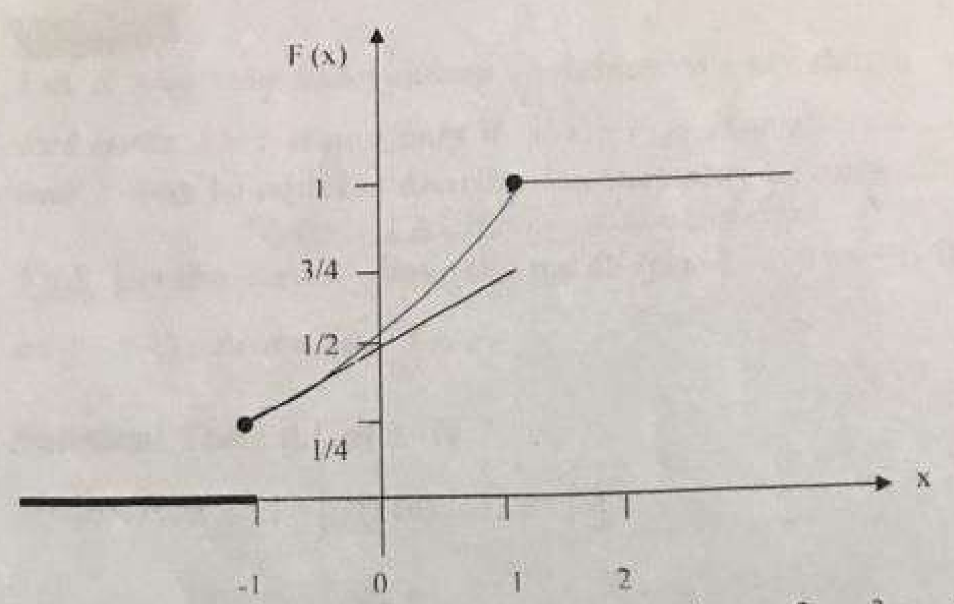
1) Sketch graph of c.d.f. $F(x)$ **.**

2) Compute

1) $\Pr(-\frac{1}{2} \le X \le \frac{1}{2})$ 2) $\Pr(2 < X \le 3)$ 3) $\Pr(X = 0)$ 4) $\Pr(X = 1)$

Solution:

5)



1) $Pr(-\frac{1}{2} \leq X \leq \frac{1}{2}) = F(\frac{1}{2}) - F(-\frac{1}{2}) = \frac{5}{8} - \frac{3}{8} = \frac{1}{4}$

2) $Pr(2 < X \leq 3) = F(3) - F(2) = 1 - 1 = 0$

3) $Pr(X = 0) = F(0) - F(0^-) = \frac{2}{4} - \frac{2}{4} = 0$

4) $Pr(X = 1) = F(1) - F(1^-) = 1 - \frac{3}{4} = \frac{1}{4}$

Remarks:

1) c.d.f. F(x) in above example(Ex.6) is a mixture of the continuous and discrete types.

2) If X is continuous r.v. , then p.d.f. $f(x)$ **can be obtained by derivative of c.d** $F(x)$ **, that is** $f(x) = \frac{dF(x)}{dx}$

Ex.7:Let $F(x) = \begin{cases} 0 & , \quad x < 0 \\ \sqrt{x}/2 & , \quad 0 \leq x < 4 \\ 1 & , \quad 4 \leq x \end{cases}$ **be the c.d.f. of a r.v. X,find the p.d.f.of X**

Solution:

Since X **is continuous r.v.**

Hence

$$f(x) = \frac{dF(x)}{dx} = \frac{1}{4\sqrt{x}} \quad , \quad 0 < x < 4$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{4\sqrt{x}} & , \quad 0 < x < 4 \\ 0 & , \quad o.w. \end{cases}$$

Remark:

Let X and Y be two random variables. We say that X and Y are equal in distribution and write $X \overset{D}{=} Y$ if and only if $F(x)=F(y)$, for all $x \in R$. It is important to note while X and Y may be equal in distribution they may be quite different. يعني الجديد بالملاحظة
ان x و y قد يكونان متساويين بالتوزيع لكنهما متغيرين مختلفين تماماً

Ex.8 Let the r.v. X have the p.d.f. $f(x)=1 \;,\; 0<x<1$. Define the random variable Y as $Y=1-X$. Prove that $X \overset{D}{=} Y$.

Solution: The c.d.f. of X is

$$F_X(x)=\Pr(X \le x)=\int_0^x f(u)\,du=\int_0^x du=[u]_0^x=x$$

$$\Rightarrow F_X(x)=\begin{cases} 0 & , \quad x<0 \\ x & , \quad 0 \le x<1 \\ 1 & , \quad 1 \le x \end{cases}$$

The space of Y is the interval $(0,1)$, the same as X (since $Y=1-X$). Further, the c.d.f of Y is

$$F_Y(y)=\Pr(Y \le y)=\Pr(1-X \le y)=\Pr(X \ge 1-y)=1-\Pr(X \le 1-y)=1-F_X(1-y)=1-(1-y)=y$$

$$\Rightarrow F_Y(y)=\begin{cases} 0 & , \quad y<0 \\ y & , \quad 0 \le y<1 \\ 1 & , \quad 1 \le y \end{cases}$$

Hence, Y has the same c.d.f. as X, i.e., $X \overset{D}{=} Y$, but $X \neq Y$.

التوقع الرياضي

Expectation: The expectation or expected value of the function $g(X)$ of the r.v. X, denoted by $E(g(X))$ or $\mu_{g(X)}$ is defined by ;

$$E(g(X))=\begin{cases} \sum_{x} g(x)\,f(x) & , \text{ if X a discrete r.v} \\ \int_{-\infty}^{\infty} g(x)f(x)dx & , \text{ if X a continuous r.v} \end{cases}$$

Where $f(x)$ is the p.d.f. of X.

Properties of Expected Value

1. $E(c)=c$ for a constant c.
2. $E(c\,g(X))=c\,E(g(X))$ for a constant c.
3. $E(\sum_{i} c_i g_i(X))=\sum_{i=1} c_i E(g_i(X))$
4. If $g_1(X) \le g_2(X)$ then $E(g_1(X)) \le E(g_2(X))$.

Proof :

1. Assume X is a continuous r.v.

$$E(c)=\int_{-\infty}^{\infty} c f(x)dx=c\int_{-\infty}^{\infty} f(x)dx=c \quad \left(\text{Since } f(x) \text{ is p.d.f. (i.e. } \int_{-\infty}^{\infty} f(x)dx=1)\right)$$

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3. **Assume X is continuous r.v.**

$$E(\sum_{i=1}^{n} c_i g_i(X)) = E(c_1 g_1(X) + c_2 g_2(X) + \cdots + c_n g_n(X))$$

$$= \int_{-\infty}^{\infty} (c_1 g_1(x) + c_2 g_2(x) + \cdots + c_n g_n(x)) f(x) dx$$

$$= c_1 \int_{-\infty}^{\infty} g_1(x) f(x) dx + c_2 \int_{-\infty}^{\infty} g_2(x) f(x) dx + \cdots + c_n \int_{-\infty}^{\infty} g_n(x) f(x) dx$$

$$= c_1 E(g_1(X)) + c_2 E(g_2(X)) + \cdots + c_n E(g_n(X)) = \sum_{i=1}^{n} c_i E(g_i(X)$$

Remarks:

1) If $E(X) = \mu_X \to \infty$, then we say that the mean of a r.v. X does not exist.

2) The r^{th} moments of the r.v. X, denoted by $E(X^r)$ or μ'_X is defined by :

$$E(X^r) = \begin{cases} \sum_{\forall x} x^r f(x) & , \text{ if X a discrete r.v.} \\ \int_{-\infty}^{\infty} x^r f(x) dx & , \text{ if X a continuous r.v} \end{cases}$$

where $f(x)$ is the p.d.f. of X .

3) The r^{th} central moment of the r.v. X about a is defined as $E((X-a)^r)$

Ex.9: Let $f(x) = 1/x^2$, $x \geq 1$ be the p.d.f. of a r.v. X, does $E(X)$?

Solution:

Since

$$E(X) = \int_{1}^{\infty} x f(x) dx = \int_{1}^{\infty} \frac{dx}{x} = [Ln(x)]_1^{\infty} = \lim_{x \to \infty} Ln(x) - Ln(1) = \infty$$

Hence $E(X)$ does not exist.

Ex.10: Let X be a r.v. with p.d.f. given by $f(x,\lambda) = \lambda e^{-\lambda x}$ $\lambda, x > 0$ (which is called the exponential distribution),find $E(X)$, $E(X^2)$

Solution :

$$E(X) = \int_{0}^{\infty} x f(x) dx = \int_{0}^{\infty} \lambda x e^{-\lambda x} dx$$

By part integration method

$u = x \Rightarrow du = dx$

$dv = \lambda e^{-\lambda x} dx \Rightarrow v = -e^{-\lambda x}$

$$E(X) = [-xe^{-\lambda x}]_{x=0}^{\infty} + \int_{0}^{\infty} e^{-\lambda x} dx = -\frac{1}{\lambda}[e^{-\lambda x}]_{x=0}^{\infty} = -\frac{1}{\lambda}(0-1) = \frac{1}{\lambda}$$

$$E(X^2) = \int_{0}^{\infty} x^2 f(x) dx = \int_{0}^{\infty} \lambda x^2 e^{-\lambda x} dx$$

By part integration method

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$u = x^2 \Rightarrow du = 2x\,dx$

$dv = \lambda e^{-\lambda x}\,dx \Rightarrow v = -e^{-\lambda x}$

$$E(X) = [-x^2 e^{-\lambda x}]_{x=0}^{\infty} + 2\int_0^{\infty} x e^{-\lambda x}\,dx = \frac{2}{\lambda}\int_0^{\infty} \lambda x e^{-\lambda x}\,dx = \frac{2}{\lambda}\left(\frac{1}{\lambda}\right) = \frac{2}{\lambda^2}$$

X **Ex.11: Let X be a r.v. with p.d.f.** $f(x,\theta) = \frac{\theta^x e^{-\theta x}}{x!}$ $x = 0,1,2,\dots$ $\theta > 0$ **(which is called the Poisson distribution), find** $E(X)$.

Solution :

$$E(X) = \sum_{x=0}^{\infty} x f(x) = \sum_{x=0}^{\infty} x \frac{\theta^x e^{-\theta}}{x!}$$

$$= 0 * \frac{\theta^0 e^{-\theta}}{0!} + 1 * \frac{\theta^1 e^{-\theta}}{1!} + 2 * \frac{\theta^2 e^{-\theta}}{2!} + 3 * \frac{\theta^3 e^{-\theta}}{3!} + \cdots$$

$$= \theta e^{-\theta}\left(\frac{\theta^0}{1!} + \frac{\theta^1}{2!} + \frac{\theta^2}{3!} + \cdots\right) = \theta e^{-\theta}(e^{\theta}) = \theta$$

Ex.12: Let X be a r.v. with p.d.f. given by

$f(x) = \frac{1}{x^2}$ $, x \ge 1$ **, Does E(X) exist ?**

Solution : $E(X) = \int_1^{\infty} x f(x)dx = \int_1^{\infty} \frac{1}{x}dx = [\text{Ln}\, x]_1^{\infty} = \text{does not exist}$

Remarks:

1) If X is a continuous r.v. having the c.d.f. $F(x)$ **then**

a. $E(X) = \int_0^{\infty} (1 - F(x))dx + \int_{-\infty}^{0} F(x)dx$

b. $E(X) = \int_0^{\infty} [1 - F(x) + F(-x)]dx$

Proof:

a. $E(X) = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} x F'(x)dx$

By part integration method

$u = x \Rightarrow du = dx$

$dv = F'(x)\,dx \Rightarrow v = F(x)$

Hence

$$E(X) = [xF(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} F(x)dx = \lim_{x\to\infty}[xF(x)] - \lim_{x\to-\infty}[xF(x)] - \int_{-\infty}^{\infty} F(x)dx$$

$$= \lim_{x\to\infty} x \,.\, \lim_{x\to\infty} F(x) - \lim_{x\to-\infty} x \,.\, \lim_{x\to-\infty} F(x) - \int_{-\infty}^{\infty} F(x)dx$$

$$= \int_0^{\infty} dx - [\int_{-\infty}^{0} F(x)dx + \int_0^{\infty} F(x)dx] = \int_0^{\infty} (1 - F(x)dx) - \int_{-\infty}^{0} F(x)dx$$

b. (H.W.)

9)

Ex.13: Let X be a r.v. with c.d.f. given by

$$F_X(x)=\begin{cases} 0 \ , & x<0 \\ x/2 \ , & 0\le x<2 \\ 1 \ , & 2\le x \end{cases}$$

, find E(X) .

Solution:

1) The first technique by depending on the definition
since X is continuous r.v.
hence $f(x)=F'(x)=x\ \frac{1}{2},\quad 0<x<2$

Now $E(X)=\int_0^2 x\,f(x)dx=\frac{1}{2}\int_0^2 x\,dx=\frac{1}{4}\left[x^2\right]_0^2=1$

$$E(X)=\int_{-\infty}^{0}F(x)dx+\int_0^{\infty}(1-F(x))dx=\int_0^2(1-x/2)\,dx=\left[\left(x-x^2/4\right)\right]_0^2=1$$

2) The second technique by depending on the c.df. (the above remark 1/a)

Now $E(X)=\int_{-\infty}^{0}F(x)dx+\int_0^{\infty}(1-F(x))dx=\int_0^2(1-x/2)\,dx=\left[\left(x-x^2/4\right)\right]_0^2=1$

Variance: The variance of the function $g(X)$ of the r.v. X ,denoted by $Var(g(X))$ or $\sigma^2_{g(X)}$ is defined by

$Var(g(X))=E(g(X)-\mu_{g(X)})^2=E[(g(X))^2]-[E(g(X))]^2$.

Properties of Expected Value

1. $Var(c)=0$ for a constant c.
2. $Var(c\,g(X))=c^2\,Var(g(X))$ for a constant c.
3. $Var(c\,g(X)\mp b)=c^2\,Var(g(X))$ where b , c are constants.

Proof :

1. $Var(c)=E(c^2)-(E(c))^2=c^2-c^2=0$

3.

$$\begin{aligned} Var(c\,g(X)\mp b) &= E[(c\,g(X)\mp b)^2]-[E(c\,g(X)\mp b)]^2 \\ &= E[c^2(g(X))^2\mp 2\,b\,c\,g(X)+b^2]-[c\,E(g(X))\mp b]^2 \\ &= c^2\,E[(g(X))^2]\mp 2\,b\,c\,E(g(X))+b^2 \\ &\quad -c^2[E(g(X))]^2\pm 2b\,c\,E(g(X))-b^2 \\ &= c^2\{E[(g(X))^2]-[E(g(X))]^2\}=c^2\,Var(g(X)) \end{aligned}$$

Remarks:

1. If $E(X)=\mu_X\to\infty$, then we say that the variance of a r.v. X does not exist.
2. If X is a continuous r.v. having the c.d.f. $F(x)$ then

   $Var(X)=\int_0^{\infty}2x[1-F(x)+F(-x)]dx-\mu_X^2$ (Prove that).

Let X denote the numbers ...
ariance, and moment generating function of X.

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Ex.14: Let $f(x) = 6x(1-x) \quad 0 < x < 1$ **be the p.d.f. of a r.v. X, find the variance** $Y = 1 - 2X$ **?**

Solution:

Firstly we find $E(X)$ **and** $E(X^2)$

$$E(X) = \int_0^1 x\, f(x)\,dx = 6\int_0^1 x(1-x)\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 0.5$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = 6\int_0^1 x^2(1-x)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 0.3$$

$$Var(X) = E(X^2) - (E(X))^2 = 0.3 - 0.25 = 0.05$$

$$Var(1-2X) = 4Var(X) = (4)(0.05) = 0.2$$

<u>Exercise</u> : Let X be a r.v. with c.p.d. $F(x) = 1 - e^{-\lambda x} \quad x, \lambda > 0$ **, find** $E(X)$

<u>Moment Generating Function</u> : The expected value of $e^{tg(X)}$ **is defined to be the moment generating function (m.g.f.) of** $g(X)$ **of the r.v.** X **if the expected value exists for every value of** t **some interval** $-h < t < h \;;\; h > o$ **. The moment generating function of** $g(X)$ **, denoted by** $M_{g(x)}(t)$ **,is**

$$M_{g(x)}(t) = E(e^{tg(X)}) = \begin{cases} \sum_{\forall x} e^{tg(X)} f(x) & , \text{ if X a discrete r.v.} \\ \int_{-\infty}^{\infty} e^{tg(X)} f(x)dx & , \text{ if X a continuous r.v} \end{cases}$$

where $f(x)$ **is the p.d.f. of** X **.**

<u>Properties of Moment Generating Function:</u>

1) $E(X) = M'_X(0)$

2) $Var(X) = M''_X(0) - (M'_X(0))^2$

Proof:

1. $M'_X(t) = \dfrac{dM_X(t)}{dt} = E(Xe^{tX}) \Rightarrow M'_X(0) = E(X)$

2.

$$M''_X(t) = \frac{d^2M_X(t)}{dt^2} = E(X^2 e^{tX}) \Rightarrow M''_X(0) = E(X^2)$$

$$Var(X) = E(X^2) - (E(X))^2 = M''_X(0) - (M'_X(0))^2$$

$f(x) = \dfrac{dF(x)}{dx} = -\lambda e^{-\lambda x}$

$E(x) = \int_0^\infty x \cdot f(x)\, dx$

$= \int_0^\infty x \cdot (-\lambda e^{-\lambda x})\, dx$

$=$

Ex.15: **Let X be a r.v. with p.d.f. given by** $f(x,\lambda)=\lambda e^{-\lambda x}\ \lambda, x>0$ **, find the moment generating function (m.g.f.) of** X **.**

Solution:

$$M_X(t)=E(e^{tX})=\int_0^{\infty} e^{tx} f(x)dx=\int_0^{\infty} e^{tx}\lambda e^{-\lambda x}dx=\lambda\int_0^{\infty} e^{-(\lambda-t)x}dx$$

$$=\frac{-\lambda}{\lambda-t}[e^{-(\lambda-t)x}]_0^{\infty}=\frac{-\lambda}{\lambda-t}(0-1)$$

$$=\frac{\lambda}{\lambda-t}\ ,(\lambda-t>0\Rightarrow t<\lambda)$$

Ex.16: **Let X be a r.v. with p.d.f.** $f(x)=f(x,n,p)=\begin{cases}\binom{n}{x}p^x q^{1-x} & x=0,1,\dots,n\\ 0 & \text{o.w.}\end{cases}$ **where the two parameters** n, p **satisfy** $n=1,2,\dots$, $0\le p\le 1$ **and** $q=1-p$ **.(which is called the Bernoulli distribution), find** $M_X(t)$, $E(X)$ and $Var(X)$

Solution:

$$M_X(t)=E(e^{Xt})=\sum_{x=0}^{n} e^{xt} f(x)=\sum_{x=0}^{n}\binom{n}{x}e^{xt}p^x q^{n-x}$$

$$=\binom{n}{0}e^{0t}p^0q^n+\binom{n}{1}e^{t}p^1q^{n-1}+\binom{n}{2}e^{2t}p^2q^{n-2}+\dots+\binom{n}{n}e^{nt}p^nq^0$$

$$=\binom{n}{0}q^n+\binom{n}{1}(e^tp)^1q^{n-1}+\binom{n}{2}(e^tp)^2q^{n-2}+\dots+\binom{n}{n}(e^tp)^n$$

$$=(q+pe^t)^n$$

Now $M'_X(t)=n\,pe^t\left(pe^t+q\right)^{n-1}\Rightarrow E(X)=M'_X(0)=n\,p\left(p+q\right)^{n-1}=n\,p\ \ (\text{since } q=1-p)$

$$M''_X(t)=n(n-1)(pe^t)^2\left(pe^t+q\right)^{n-2}+n\,pe^t\left(pe^t+q\right)^{n-1}$$

$$\Rightarrow E(X^2)=M''_X(0)=n(n-1)p^2+n\,p$$

hence

$$Var(X)=E(X^2)-(E(X))^2=n(n-1)p^2+n\,p-n^2p^2=np(1-p)=npq$$

Characteristic Function : **The expected value of** $e^{itg(X)}$ **is defined to be the characteristic function of** $g(X)$ **of the r.v.** X **if the expected value exists for every value of** t **some interval** $-h<t<h\,;h>o\ ,i=\sqrt{-1}$ **. The characteristic function of** $g(X)$ **, denoted by** $\phi_{g(X)}(t)$ **,is**

$$\phi_{g(X)}(t)=E(e^{itg(X)})=\begin{cases}\sum_{\forall x} e^{itg(X)}f(x) & ,\ \text{if X a discrete r.v.}\\ \int_{-\infty}^{\infty} e^{itg(X)}f(x)dx & ,\ \text{if X a continuous r.v}\end{cases}$$

where $f(x)$ **is the p.d.f. of** X **.**

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Properties of Characteristic Function:

1) $\phi_{g(X)}(t) = E(e^{itg(X)}) = E(Cos(g(tX)) + i\,Sin(g(tX)))$

2) $-1 \le \phi_{g(X)}(t) \le 1 \quad \forall -h < t < h, h > 0$

3) $\phi'_X(0) = iE(X) = iM'_X(0)$, $\phi''_X(0) = i^2 E(X^2) = i^2 M''_X(0)$

4) $\phi''_X(0) - (\phi'_X(0))^2 = i^2\,Var(X)$

Proof:

2. $\phi'_X(t) = \dfrac{d\phi_X(t)}{dt} = E(iXe^{itX}) \Rightarrow \phi'_X(0) = iE(X) = iM'_X(0)$

$$\phi''_X(t) = \frac{d^2\phi_X(t)}{dt^2} = E(i^2X^2e^{itX}) \Rightarrow \phi''_X(0) = i^2E(X^2) = i^2M''_X(0)$$

Ex.17: Let X be a r.v. with p.d.f. $f(x) = f(x,n,p) = \binom{10}{x}(1/3)^x(2/3)^{10-x}$, $x = 0,1,\ldots,10$, **find** $\phi_X(t)$, $E(2X-1)$ **and** $Var(2-X/5)$.

Solution:

$$\phi_X(t) = E(e^{tX}) = \sum_{x=0}^{10} e^{txt} f(x) = \sum_{x=0}^{10}\binom{10}{x} e^{ixt}(1/3)^x\,(2/3)^{10-x}$$

$$= \binom{10}{0}e^{0it}(1/3)^0\,(2/3)^{10} + \binom{10}{1}e^{it}(1/3)^1\,(2/3)^9 + \cdots + \binom{10}{10}e^{10it}(1/3)^{10}\,(2/3)^0$$

$$= \binom{10}{0}(2/3)^{10} + \binom{10}{1}(e^{it}(1/3))^1\,(2/3)^9 + \cdots + \binom{10}{10}(e^{it}(1/3))^{10}$$

$$= \left(\frac{2}{3} + \frac{1}{3}e^{it}\right)^{10}$$

Now

$$\phi'_X(t) = \frac{10}{3}ie^{it}\left(\frac{2}{3} + \frac{1}{3}e^{it}\right)^9 \Rightarrow iE(X) = \phi'_X(0) = \frac{10}{3}i \Rightarrow E(X) = \frac{10}{3}$$

Hence

$$E(2X-1) = 2E(X) - 1 = \frac{20}{3} - 1 = \frac{17}{3}$$

Also

$$\phi''_X(t) = \frac{90}{9}i^2e^{2it}\left(\frac{2}{3} + \frac{1}{3}e^{it}\right)^9 + \frac{10}{3}i^2e^{it}\left(\frac{2}{3} + \frac{1}{3}e^{it}\right)^9$$

$$\Rightarrow i^2E(X^2) = \phi''_X(0) = 10i^2 + \frac{10}{3}i^2 \Rightarrow E(X^2) = \frac{40}{3}$$

Hence

$Var(X) = E(X^2) - (E(X))^2 = \dfrac{40}{3} - (\dfrac{10}{3})^2 = \dfrac{20}{3}$ **and** $Var(2 - X/5) = \dfrac{1}{25}Var(X) = (\dfrac{1}{25})(\dfrac{20}{3}) = \dfrac{4}{15}$

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Exercises(1)

1) Let $F(x) = \begin{cases} 0 & , \quad x < 0 \\ x/8 & , \quad 0 \le x < 2 \\ x^2/16 & , \quad 2 \le x < 4 \\ 1 & , \quad 4 \le x \end{cases}$ be the c.d.f. of X. Find

$\Pr(|X-\mu_x| < 2\sigma_x)$ and $\Pr(X^2 \ge 3X-2)$ by using p.d.f. and c.d.f..

2) Let $f(x) = 2^{-x}$, $x = 1, 2, \ldots$ be the p.d.f. of a r.v. X. Find :

a) Moment generating function for $Y = a + bX$, where a and b are constants.

b) Characteristic function for X.

c) Mean and variance of X .

3) Let $G_x(t) = \ln(M_x(t))$ where $M_x(t)$ be moment generating function of a r.v. X. Find the mean and variance of X .

4) Let the random variable X have mean μ, standard deviation σ, and m.g.f. $M_x(t)$, $-h < t < h$. Show that

$$E\left(\frac{X-\mu}{\sigma}\right)=0 \ , E\left(\left(\frac{X-\mu}{\sigma}\right)^2\right)=1, \text{ and } M_{\left(\frac{X-\mu}{\sigma}\right)}(t)=e^{-\mu t/\sigma}M_X(t) \ , \ -\sigma h<t<\sigma h$$

5) Let X be a random variable of the continuous type with p.d.f. $f(x)$, which is posative provided $0<x<b<\infty$. Show that $E(X)=\int_0^b (1-F(x))\,dx$, where $F(x)$ is the c.d.f. of X.

7) Let X be a random variable of the discrete type with p.d.f. $f(x)$, which is positive on the nonnegative integers. Show that $E(X)=\sum_{x=0}^{\infty}(1-F(x))$, where $F(x)$ is the c.d.f. of X.

7) Let X have the p.d.f. $f(x)=1/N$, $x=1,2,3,\ldots,N$. Show that the m.g.f.is

$$M_X(t)=\begin{cases} \dfrac{e^t(1-e^{Nt})}{N(1-e^t)} & t \ne 0 \\ 1 & t=0 \end{cases}$$

8) Find the moments of the distribution that has m.g.f. $M_x(t)=(1-t)^{-3}$, $t<1$.
(Hint: Find the MacLaurin's series for $M_x(t)$).

9) Let X be a random variable have p.d.f $f(x)=1/b$, $0 \le x \le b$ such that $\Pr(X < M_X''(0))=1/12$, copmpute the value b.

10) Let X denote the numbers obtained when one dice is throw. Compute the mean, variance, and moment generating function of X.

11) Let X have p.d.f. $f(x)=\frac{1}{\lambda}e^{-x/\lambda}$, $x>0$, $\lambda>0$. Find m.g.f, mean, and variance of X.

12) If X is a random variable has $f(x)=\frac{\lambda^{x}e^{-\lambda}}{x!}$, $x=0,1,2,3,\ldots$, $\lambda>0$ such that $\Pr(X=1)=\Pr(X=2)$, compute $\Pr(X\geq 1)$.

13) Let X have a probability density function $f(x)$ that is positive at $x=-1,0,1$ and is zero elsewhere. If $f(0)=1/2$, compute $E(X^{2})$.

14) Given $f(x)=\frac{4}{x}f(x-1)$, $x=1,2,\ldots$ Find the probability density function $f(x)$.