## *Upper (lower) bounds, Maximum (Minimum) elements,*
## *Least (Greatest) bounds:*

***Definition***

Let $S \subseteq \mathbb{R}$ be a subset of real numbers. If there is a real numbers $b$ such that $x \leq b$ $(x \geq b)$ for all $x \in S$, then b is called ***an upper (a lower) bound*** for $S$ and we will say that $S$ ***is bounded above (below) by b.***

***Remark:***

**1.** If $b$ is an upper bound for $S$, then every real number greatest than $b$ will also be an upper bound for $S$ , i.e. if b is an upper bound for $S$ and $c \in \mathbb{R}$ such that $b \leq c$ , then $c$ is also an upper bound for $S$.

**2.** If $b$ is a lower bound for $S$ and $c \in \mathbb{R}$ such that $c \leq b$ then $c$ is also a lower bound for $S$.

***Definition***

Let $S \subseteq \mathbb{R}$ be a bounded above subset of real numbers. A real number $b$ is called ***a least upper bound*** for $S$ if:

**i.** $b$ is an upper bound for $S$, and;

**ii.** if a real number $c$ is an upper bound for $S$, then $b \leq c$, (i.e. there is no real number less than $b$ can be an upper bound for $S$).

If $b$ is a least upper bound for , we shall denote it by $\boldsymbol{b = Sup\ S}$.

***Definition***

Let $S \subseteq \mathbb{R}$ be a bounded below subset of real numbers. A real number $b$ is called ***a greatest lower bound*** for $S$ if :

**i.** $b$ is a lower bound for $S$, and;

**ii.** if a real number $c$ is a lower bound for $S$, then $c \leq b$, (i.e. there is no real number greater than $b$ is a lower bound for $S$.

If $b$ is a greatest lower bound for , we shall denote it by $\boldsymbol{b = Inf\ S}$.

***Definition***

Let $S \subseteq \mathbb{R}$. If $b$ is an upper bound for $S$ and $b \in S$, then $b$ is called ***a maximal element*** of $S$, i.e. if $b = Sup\ S$ and $b \in S$, then $b$ is said to be ***a maximal element*** of $S$, and we shall write in this case $\boldsymbol{b = Max\ S}$.

***Definition***

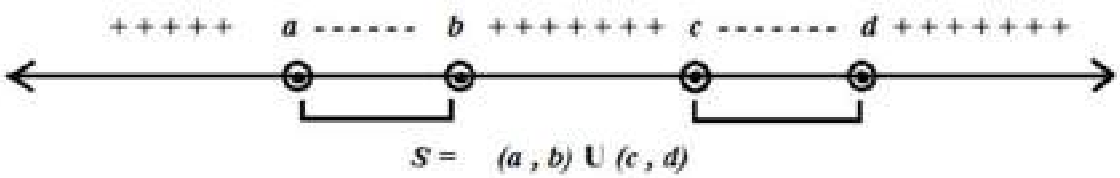
Let $\subseteq \mathbb{R}$ . If $b$ is a lower bound for $S$ and $b \in S$, then $b$ is called ***a minimal element*** of $S$, i.e. if $b = Inf\ S$ and $b \in S$, then $b$ is ***a minimal element*** of $S$, and we shall write in this case $\boldsymbol{b = Min\ S}$.

## *Completeness axiom:*

Every non-empty set of real numbers which is bounded above (bounded below) has a supremum (infimum), i.e. $\exists\ b \in \mathbb{R} \ni b = Sup\ S, (b = Inf\ S\ )$.

***Examples:***

1. The set $\mathbb{R}^+ = (0\ ,\infty)$ is unbounded above. It has no upper bounds, no maximal element and no supremum. The real numbers 0 is a lower bound of $\mathbb{R}^+$ and every real numbers less than 0 is also a lower bound of $\mathbb{R}^+$. $\mathbb{R}^+$ has no minimal element , and $Inf\ \mathbb{R}^+ = 0$.
2. $S = [0\ ,1]$ is bounded above by 1 (i.e. 1 is an upper bound for $S$ ) and is bounded below by 0 (i.e. 0 is lower bound for $S$). $Sup\ S = 1$ and $Inf\ S = 0$. Also $Max\ S = 1$ , and $Min\ S = 0$ .
3. $S = \{x: (x-a)(x-b)(x-c)(x-d) < 0\ ; a < b < c < d\} = (a\ ,b) \cup (c\ ,d)$




Note that, $a$ is a lower bound of S (hence any real number less than a is also a lower bound of $S$). $S$ is bounded below by $a$. $d$ is an upper bound for $S$ (hence any real number greater than a is also an upper bound of S). $S$ is bounded above by $d$ .$Inf\ S = a$ , and $S$ has no minimal element of $S$. Also $Sup\ S = d$, and $S$ has no maximal element of $S$.

***Remark:***

Supremum and Infimum of a subset of real numbers are uniquely determined whenever they exist.

***Explanation***:

Suppose $Sup\ S = b$ and $Sup\ S = c$ .

Since $Sup\ S = b$, then $b$ is an upper bound of $S$.

As $b$ is an upper bound of $S$ and $Sup\ S = c$, then $c \leq b$.

Also, as $Sup\ S = c$, then $c$ is an upper bound of $S$.

As $c$ an upper bound of $S$ and $Sup\ S = b$, that implies $b \leq c$.

Thus, $= b$ , and hence $Sup\ S$ is uniquely determined if it is exist .

Similarly, we can show that $Inf\ S$ is uniquely determined if it is exist.

## *Some properties of the Supremum:*

***Theorem (Approximation property):***

Let $S \subseteq$ be a non-empty subset of real numbers with an upper bound $b$. Then $Sup\ S = b$ if, and only if, for every $a \leq b$ there is some $a \in S$ such that $a < x \leq b$.

***Proof:***

Since $Sup\ S = b$, hence $x \leq b\ \forall\ x \in S$ ... $(*)$

**Wanted:** $\exists\ x \in S\ \ni a < x \leq b$ and from * above we need to show only:

$$\exists\ x \in S\ \ni a < x\ .$$

Suppose $x \leq a\ \ \forall\ x \in S$, then $a$ is an upper bound for $S$. But $Sup\ S = b$ is the least upper bound for $S$. Thus $b < a$ and this is a contradiction.

Therefore, $\exists\ x \in S\ \ni a < x$ and from $(*)$ above, we deduce that $a < x \leq b$ .

Conversely, suppose $\forall a < b, \exists\ x \in S\ \ni a < x \leq b$. **Wanted:** $Sup\ S = b$.

By contrary, assume that $Sup\, S \neq b$. That is, $\exists a < b$ such that $a$ is an upper bound of $S$, i.e. $x \leq a, \forall x \in S$ and this contradicts our assumption above. Thus, $Sup\, S = b$.

***Theorem (Additive property):***

Let $A, B \subseteq \mathbb{R}$, be non-empty subsets of real numbers and let $C = \{x + y \in \mathbb{R}: x \in A\,, y \in B\,\}$. If each of $A$ and $B$ has a supremum, then $C$ has a supremum and $Sup\, C = Sup\, A + Sup\, B$ .

***Proof:***

Let $Sup\, A = a\,, Sup\, B = b$. If $z \in C$ , then $\exists\, x \in A$ and $y \in B$ such that $z = x + y$. Since $Sup\, A = a\,, Sup\, B = b$, hence $x \leq a$ and $y \leq b$ and that implies $x + y \leq a + b \;\Rightarrow\; z \leq a + b$ .

Therefore $a + b$ is an upper bound of $C$ and the Supremum of $C$ exists, say $c = Sup\, C$. Therefore, $c \leq a + b$, i.e. $Sup\, C \leq Sup\, A + Sup\, B$ .

To show that $c = a + b$ (i.e. $Sup\, C = Sup\, A + Sup\, B$ .), we need to show that $a + b$ satisfied the approximation property for supremum.

So, assume $\epsilon > 0$. Thus, $a - \frac{\epsilon}{2} < a = Sup\, A$ and $b - \frac{\epsilon}{2} < b = Sup\, B$ .

From the approximation property for supremum, we imply that;

$\exists\, x \in A$ and $\exists\, y \in B \ni a - \frac{\epsilon}{2} < x \leq a$ and $-\frac{\epsilon}{2} < y \leq b$ .

Since $a - \frac{\epsilon}{2} < x$ and $b - \frac{\epsilon}{2} < y \;\Rightarrow\; a + b - \epsilon < x + y \leq a + b$.

But $+y = z \in C \ni a + b - \epsilon < z \leq a + b$. Therefore, $SupC = a + b$.

$$\Rightarrow Sup\, C = Sup\, A + Sup\, B$$

***Theorem (Comparison property):***

Let $A, B \subseteq \mathbb{R}$ be non-empty subsets of real numbers such that $x \leq y$ for every $x \in A$ and $y \in B$. If $B$ has a Supremum, then $A$ has Supremum and $up\, A \leq Sup\, B$ .

***Proof:***

Suppose that $B$ has a supremum, say $Sup\, B = b$, then $y \leq b \;\; \forall\, y \in B$.

But $x \leq y\ \forall\, x \in A$ and $y \in B$, so $x \leq b\ \forall\, x \in A$ and that implies $b$ is an upper bound for $A$. From completeness axiom $Sup\, A$ exists, say $a$=$Sup\, A$ . Since b is an upper bound for $A$ and $a = Sup\, A$, thus $a \leq b$, i.e. $Sup\, A \leq Sup\, B$.

## *As a home work prove the following properties of the infimum:*

***Theorem (Approximation property):***

Let $S \subseteq \mathbb{R}$ be a non-empty set of real numbers with a lower bound $b$. Then $b = Inf\ S$ if, and only if, for every $a > b$ there is some $x \in S$ such that $\leq x < a$ .

***Theorem (Additive property):***

Let $A, B \subseteq \mathbb{R}$ be non-empty subsets of real numbers and let $C = \{x + y : x \in A\, , y \in B\}$. If each of $A$ and $B$ has an infimum, then $C$ has an infimum and $Inf\ C = Inf\ A + Inf\ B$ .

***Theorem (Comparison property):***

Let $A, B \subseteq \mathbb{R}$ be non-empty subsets of real numbers such that $x \leq y$, for every $x \in A$ and $y \in B$ . If $A$ has a infimum, then $B$ has infimum and $Inf\ A \leq\ Inf\ B$.

***Theorem (Archimedean Property of the field of real numbers*** $\mathbb{R}$***):***

The set of real numbers $\mathbb{R}$ is unbounded above, i.e. if $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

***<u>Proof:</u>***

Let $x \in \mathbb{R}$. By contrary assume there is no $n \in \mathbb{N}$ such that $x < n$, i.e. $n \leq x, \forall n \in \mathbb{N}$ . Thus, $x$ is an upper bound of $\mathbb{N}$. Therefore, $\mathbb{N}$ has a supremum say $y = Sup\mathbb{N}$. Since $y - 1 < y$, hence there exists $m \in \mathbb{N} \ni y - 1 < m$, as an application of the approximation property of $y = Sup\mathbb{N}$. Then, $y < m + 1$, i.e. $\exists m+1 \in \mathbb{N} \ni y = Sup\mathbb{N} < m+1$ and this contradicts the assumption that $y$ is an upper bound of $\mathbb{N}$. Therefore, $\mathbb{R}$ is unbounded above.

***Exercises:***

1. Let $x, y \in \mathbb{R}$ be positive real numbers. Then:
   **a.** $\exists\, n \in \mathbb{N} \ni x < ny$.
   **b.** $\exists\, n \in \mathbb{N} \ni 0 < \frac{1}{n} < y$.
   **c.** $\exists\, n \in \mathbb{N} \ni n-1 \leq y < n$.
2. Let $x, y \in \mathbb{R}$. Then:
   **a.** $\exists\, r \in \mathbb{Q} \ni x < r < y$, (The Density theorem of the rational numbers).
   **b.** $\exists\, z \in \mathbb{Q}^c \ni x < z < y$, (The Density theorem of the irrational numbers).

## *Euclidean space* $\mathbb{R}^n$

When $n = 1$, a point in $\mathbb{R}$ is a real number.

When $= 2$ , a point in two dimensional space $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is an ordered pair of real numbers $(x_1, x_2)$.

When $n = 3$, a point in three-dimensional space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is a triple of real numbers $(x_1, x_2, x_3)$.

In general, a point in $n$- dimensional space $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ is an ordered **n-tuple** of real numbers $(x_1, x_2, \ldots, x_n)$. The real number $x_k$ is called the $k$-th coordinate of the point $(x_1, x_2, \ldots, x_n)$.

***Definition***

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbb{R}^n$ and $c \in \mathbb{R}$, We define:

i. **Equality:** $x = y \Leftrightarrow x_1 = y_1\ ,\ x_2 = y_2, \ldots, x_n = y_n$.

ii. **Sum:** $x + y = (x_1 + y_1\ ,\ x_2 + y_2, \ldots, x_n + y_n)$

iii. **Multiplication by real numbers (scalars):**

$$cx = c(x_1, x_2, \ldots, x_n) = (cx_1, c\,x_2, \ldots, cx_n)$$

iv. **Difference:** $x - y = (\,x_1 - y_1\ ,\ x_2 - y_2, \ldots, x_n - y_n)$

v. **Origin (zero vector):** $0 = (0\,,0\,,\ldots\,,\ 0)$

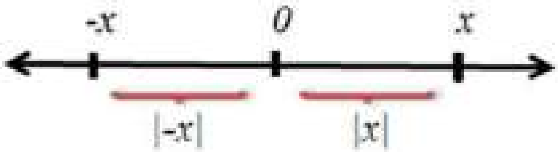
vi. **Inner product (dot product):**

$$x.y = x_1.y_1 + \ x_2.y_2 + \cdots + \ x_n.y_n$$

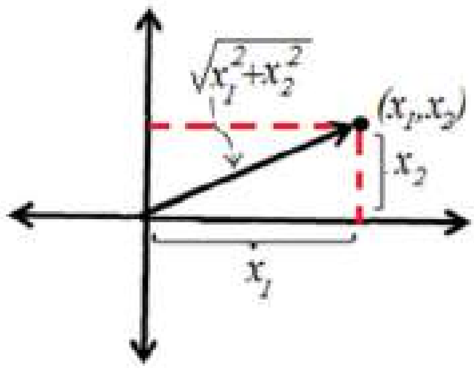
$$x.y = \sum_{i=1}^{n} x_k.y_k$$

vii. **Norm (length):** $\|x\| = \sqrt{x.x} = \sqrt{{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2}$ .

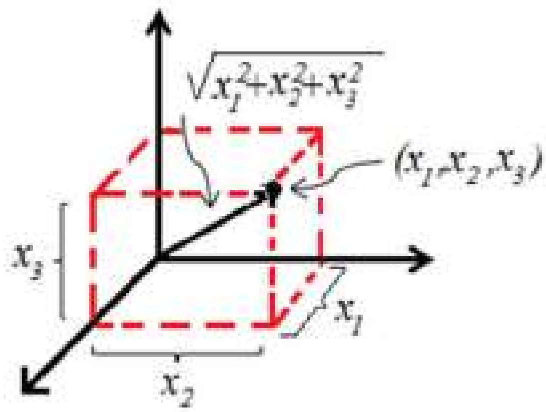
For $n = 1$;




For $n = 2$;

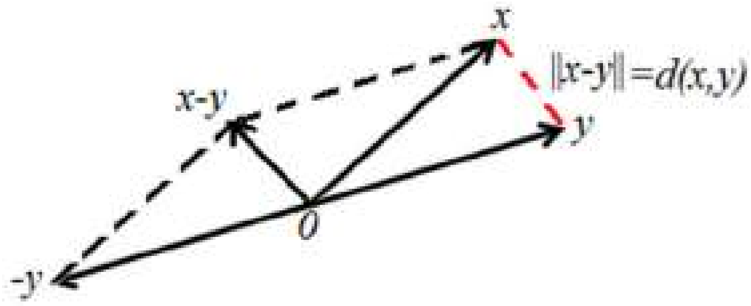



For $n = 3$;



**viii.** the norm $\|x-y\|$ is called **the distance** between $x=(x_1, x_2, \dots, x_n)$ and $y=(y_1, y_2, \dots, y_n)$;

$$\|x-y\| = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots+(x_n-y_n)^2}.$$




***Remark :***

$(\mathbb{R}^n, +, .)$ is a vector space over the filed $\mathbb{R}$ .

***Properties of the norm:***

Let $=(x_1, x_2, \dots, x_n)$ , $y=(y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. Then

**a)** $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x=0$ .

**b)** $\|cx\| = |c|\|x\|$ for any $c \in \mathbb{R}$, where $|c|$ denotes the absolute value of $c$.

**c)** $\|x-y\| = \|y-x\|$.

**d)** ***Cauchy - Schwartz inequality***: $|x-y| \leq \|x\|\|y\|$.

**e)** ***Triangle inequality***: $\|x-y\| \leq \|x\|+\|y\|$, sometimes the triangle inequality written in the form.

$$\|x-y\| \leq \|x-z\|+\|z-y\|.$$

**f)** $\|x-y\| \geq |\|x\|-\|y\||$.

## *Metric spaces:*

***Definition:***

A matric space is a pair $(M, d)$ consists of a non-empty set $M$ and a real valued function $d: M \times M \to \mathbb{R}$ called a ***metric function*** or ***distance function***, satisfying the following properties: for any $x, y, z \in M$.

$M_1: d(x, y) \geq 0$.

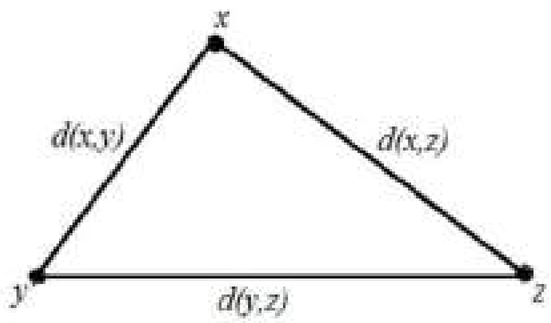
$M_2: d(x\,,y) = 0 \Leftrightarrow x = y.$

$M_3: d(x\,,y) = d(y\,,\ x).$

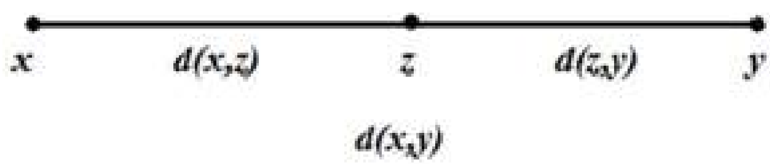
$M_4: d(x\,,z) \leq\ d(x\,,y) + d(y\,,z).$

***Remark:***

1. The real number $d(x\,,y)$ is called ***the distance*** from $x$ to $y$.
2. The properties $(M_1)$ and $(M_2)$ are state that the distance from any point to another is never negative, and that the distance from a point to itself is zero .
3. The property $(M_3)$ states that the distance from a point $x$ to a point $y$ is the same as the distance from $y$ to $x$.
4. The property $(M_4)$ is called ***the triangle inequality***, because if $x$, $y$ and $z$ are not collinear points in the plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as shown in the following figure




Then $M_4$ states that, the length $d(x\,,z)$ of one side of the triangle is less than to the sum $d(x\,,y) + d(y\,,z)$ of the lengths of the other two sides of the triangle. Moreover, if $x$, $y$ and $z$ are collinear points in the plane as shown in the following figure:




Then, $d(x\,,z) =\ d(x\,,y) + d(y\,,z)$

## *Example of metric spaces*:

***Example 1:***

let $M = \mathbb{R}^n , n \geq 1$ and let $d: M \times M \to \mathbb{R}$ be a function defined by;

$$d(x , y) = \|x - y\|, \ \forall\, x , y \in M;$$

$$\text{where } \|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}$$

$$= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Clearly, the function $d$ above is a metric on $M$ called ***the Euclidean metric*** and in fact the pair $(M , d) = (\mathbb{R}^n , \|.\| )$ is called ***the Euclidean space.***

***Remark***:

1) If $n = 1 \Rightarrow d(x , y) = |x - y| \ \forall x , y \in \mathbb{R}$.
2) If $n = 2 \Rightarrow d(x , y) = \|x - y\|$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2};$$

$\forall (x_1 , x_2), (y_1 , y_2) \in \mathbb{R}^2$ .

3) If $n = 3 \Rightarrow d(x , y) = \|x - y\|$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

$\forall (x_1 , x_2 , x_3), (y_1 , y_2 , y_3) \in \mathbb{R}^3$.

***Exercise:*** Prove the ***Murkowski's inequality:-*** For $p \geq 1$

$$\sqrt[p]{\sum_{i=1}^{n} |x_i + y_i|^p} \leq \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} + \sqrt[p]{\sum_{i=1}^{n} |y_i|^p}.$$

To show that, $(\mathbb{R}^n , \|.\| )$ is a metric space, let;

$$x = (x_1 , x_2 , \dots , x_n) , y = (y_1 , y_2 , \dots , y_n) \in \mathbb{R}^n.$$

$(\boldsymbol{M_1})$***:*** From the definition of $d(x , y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$, hence the rang of the function $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is equal to $[0 , \infty)$. Thus, $d(x , y) \geq 0. \forall\, x , y \in \mathbb{R}^n$.

$(\boldsymbol{M_2})$: $d(x , y) = 0 \Leftrightarrow \|x - y\| = 0 \Leftrightarrow \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} = 0;$

$\Leftrightarrow \sum_{i=1}^{n}(x_i - y_i)^2 \Leftrightarrow (x_i - y_i)^2 = 0 \Leftrightarrow x_i - y_i = 0 \Leftrightarrow x_i = y_i,\ \forall i = 1, \dots, n \Leftrightarrow x_1 = y_1\ , x_2 = y_2\ , \dots, x_n = y_n \Leftrightarrow x = y$.

$$(\boldsymbol{M_3}): d(x\,,y) = \|x - y\| = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = \sqrt{\sum_{i=1}^{n}\left(-(y_i - x_i)\right)^2}$$

$$= \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = \|y - x\| = d(y, x).$$

$$(\boldsymbol{M_4}): d(x\,,z) = \|x - z\| = \sqrt{\sum_{i=1}^{n}((x_i - y_i) + (y_i - z_i))^2}$$

$$\leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2};$$

$$= \|x - y\| + \|y - z\| = d(x\,,y) + d(y\,,z).$$

Therefore $(\mathbb{R}^n\,, \|.\|\,)$ is a metric space.

***Example (2):***

Let $M$ be a non-empty set and let $d: M \times M \to \mathbb{R}$ be a function defined by

$$d(x, y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}.$$

Then $d$ is a metric function on $M$ and hence $(M\,, d)$ is a metric space called **the discrete metric space**.

***<u>Sol. :</u>***

Let $x\,, y\,, z \in M$,

$(\boldsymbol{M_1})$: Since $d(x\,,y) = 0$ if $x = y$ and $d(x\,,y) = 1$ if $x \neq y$. Therefore, $d(x\,,y) \geq 0, \forall\, x\,, y \in M$.

$(\boldsymbol{M_2})$: $d(x\,,y) = 0 \Leftrightarrow x = y$

$$(\boldsymbol{M_3}): d(x, y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases} = \begin{cases} 0 & if\ y = x \\ 1 & if\ y \neq x \end{cases} = d(y\,,x)$$

$(\boldsymbol{M_4})$: We have the following cases:

**i.** $x = y\,, x \neq z\ (i.e. y \neq z)$

Since $1 \leq 0 + 1 \Rightarrow d(x\,,z) \leq d(x\,,y) + d(y\,,z)$.

**ii.** $x = z\,, y \neq z\ (i.e. y \neq x)$

since $0 \leq 1 + 1 \Rightarrow d(x\,,z) \leq d(x\,,y) + d(y\,,z)$.

**iii.** $z = y, x \neq y$ $(i.e. x \neq z)$

since $1 \leq 1 + 0 \Rightarrow d(x, z) \leq d(x, y) + d(y, z)$.

**iv.** $x = y = z$

since $0 \leq 0 + 0 \Rightarrow d(x, z) \leq d(x, y) + d(y, z)$.

**v.** $x \neq y \neq z$

since $1 \leq 1 + 1 \Rightarrow d(x, z) \leq d(x, y) + d(y, z)$.

Hence $(x, z) \leq d(x, y) + d(y, z), \forall\, x, y, z \in M$.

Therefore, $(M, d)$ is a metric space.

***Example (3):***

Let $(M, d)$ be a metric space. Define a function $e: M \times M \to \mathbb{R}$ by:

$$e(x, y) = Min\{1, d(x\ y)\};$$

for any $x, y \in M$. Therefore $(M, e)$ is a metric space.

***<u>Sol .:</u>***

Let $x, y, z \in M$.

$(\boldsymbol{M_1})$**:** Since, either $e(x, y) = 1$, (hence $e(x, y) > 0$) or $e(x, y) = d(x, y)$, (hence $e(x, y) \geq 0$). Therefore, $e(x, y) \geq 0$.

$(\boldsymbol{M_2})$: $e(x, y) = 0 \Leftrightarrow Min\{1, d(x, y)\} = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y$.

$(\boldsymbol{M_3})$: $e(x, y) = Min\{1, d(x, y)\} = Min\{1, d(y, x)\} = e(y, x)$.

$(\boldsymbol{M_4})$**:** Note that, in general, $e(x, y) = Min\{1, d(x, y)\} \leq 1, \forall x, y \in M$.

**Wanted:** $e(x, z) \leq e(x, y) + e(y, z)$. We have the following cases:

**i.** Suppose either, $e(x, y) = 1$ or $e(y, z) = 1$. To be definite, suppose $e(x, y) = 1$. We have that, in general $(x, z) \leq 1$ $\forall x, z \in M \Rightarrow e(x, z) \leq 1 \leq 1 + e(y, z) = e(x, y) + e(y, z)$. Similarly, if we suppose $e(y, z) = 1$, we can deduce that the triangle inequality is hold.

**ii.** Suppose both $e(x, y) < 1$ and $e(y, z) < 1 \Rightarrow e(x, y) = d(x, y)$ and $e(y, z) = d(y, z)$. Note that;

$$e(x, z) = Min\{1, d(x, z)\} \leq d(x, z) \leq d(x, y) + d(y, z)$$

$$= e(x\,,y) + e(y\,,z).$$

$$\Rightarrow\ e(x\,,z) \le e(x\,,y) + e(y\,,z).$$

Therefore, $(M\,,e)$ is a metric space.

***Example (4):***

Let $(M\,,d)$ be a metric space. Define a function $e: M \times M \to \mathbb{R}$ as:

$$e(x\,,y) = \frac{d(x\,,y)}{1+d(x\,,y)},\ \forall\, x\,,y \in M.$$

Then, $(M\,,e)$ is a metric space.

***<u>Sol.:</u>***

Let $x\,,y\,,z \in M$.

$(\boldsymbol{M_1})$: Since $d(x\,,y) \ge 0$, then clearly $e(x\,,y) \ge 0$.

$(\boldsymbol{M_2})$: $e(x\,,y) = 0 \Leftrightarrow \frac{d(x\,,y)}{1+d(x\,,y)} = 0 \Leftrightarrow d(x\,,y) = 0 \ \Leftrightarrow x = y$ .

$(\boldsymbol{M_3})$: $e(x\,,y) = \frac{d(x\,,y)}{1+d(x\,,y)} = \frac{d(y\,,x)}{1+d(y\,,x)} = e(y\,,x)$, since $(M\,,d)$ is a metric space.

$(\boldsymbol{M_4})$: **Wanted**: $e(x\,,z) \le e(x\,,y) + e(y\,,z)$ .

Note that, $\frac{d(x\,,y)}{1+d(x\,,y)+d(y\,,z)} \le \frac{d(x\,,y)}{1+d(x\,,y)} = e(x\,,y)$ and;

$$\frac{d(y\,,z)}{1+d(x\,,y)+d(y\,,z)} \le \frac{d(y\,,z)}{1+d(y\,,z)} = e(y\,,z)\ .$$

Since $(M\,,d)$ is a metric space, hence $d(x\,,z) \le d(x\,,y) + d(y\,,z)$ and we have the following;

$$e(x\,,z) = \frac{d(x\,,z)}{1+d(x\,,z)} \le \frac{d(x\,,y)+d(y\,,z)}{1+d(x\,,y)+d(y\,,z)}$$

$$= \frac{d(y\,,z)}{1+d(x\,,y)+d(y\,,z)} + \frac{d(x\,,y)+d(y\,,z)}{1+d(x\,,y)+d(y\,,z)} \le \frac{d(x\,,y)}{1+d(x\,,y)} + \frac{d(y\,,z)}{1+d(y\,,z)}$$

$$= e(x\,,y) + e(y\,,z) \Rightarrow\ e(x\,,z) \le e(x\,,y) + e(y\,,z)$$

Therefore, $(M\,,e)$ is a metric space.

***Definition (Metric subspace ):***

Let $(M\,,d)$ be a metric space and let $S$ be a non-empty subset of $M$. Then $(S\,,d)$ is also a metric space with the same metric $d$ or more precisely, with the restriction of $d$ on $S \times S$, $d = d_{S\times S} : S \times S \to \mathbb{R}$, as metric. We call $(S, d)$ a metric subspace of $(M, d)$.

***Examples:***

***Example 1:***

Let $(M, d)$ be a metric space, where $M = \mathbb{R}$ and $d(x, y) = |x - y|$, $\forall\, x, y \in M$. Let $S = \mathbb{Q}$, the set of rational numbers. Then $(S, d)$ is a matric subspace of $(M, d)$, i.e. $(\mathbb{Q}, |.|)$ is a metric subspace of $(\mathbb{R}, |.|)$.

***Example 2:***

Let $(\mathbb{R}^2, d)$ be the Euclidean space, where;

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

Define another metric $\acute{d}: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ on $\mathbb{R}^2$as;

$$\acute{d}(x, y) = \sqrt{(x_1 - y_1)^2 + 4(x_2 - y_2)^2}, \ \forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

Note that, $(\mathbb{R}^2, \acute{d})$ is not a metric subspace of $(\mathbb{R}^2, d)$, because the metric $\acute{d}$ is different from $d$.

## *Point-Set topology in metric spaces*

***Definition (Open ball):***

Let $(M, d)$ be a metric space and let $a \in M$. An open ball $B(a; r)$ with center $a$ and radius $r$ is defined by:

$$B_M(a; r) = \{x \in M |\ d(x, a) < r\}.$$

***Remark:***

If $(S, d)$ is a metric subspace of a metric space $(M, d)$ and $a \in S$, then the open ball $B_s(a; r)$ of $S$ is given by:

$$B_S(a; r) = S \cap B_M(a; r).$$

***Example 1:*** Consider the Euclidean metric space $(\mathbb{R}^n, d), n \geq 1$, where;

$$d(x,y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \forall (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$$

Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $r > 0$, therefore,

$$B(a;r) = \{x \in \mathbb{R}^n : d(x,y) < r\} = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

$$= \left\{x \in \mathbb{R}^n : \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} < r\right\}$$

$$= \{x \in \mathbb{R}^n : (x_1 - a_1)^2 + (x_2 - a_2)^2 + \dots + (x_n - a_n)^2 < r^2\}$$

Observe that;

**i.** When $n = 1$, $(\mathbb{R}, d)$ is the Euclidean metric space, where;

$$d(x,y) = |x - y|, \forall x, y \in \mathbb{R}.$$

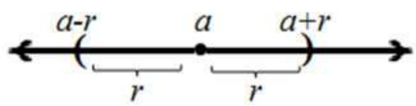
In this case;

$$B_M(a;r) = \left\{x \in \mathbb{R} : \sqrt{(x-a)^2} < r\right\}.$$

$$= \{x \in \mathbb{R} : |x - a| < r\} = \{x \in \mathbb{R} : -r < x - a < r\}$$

$$= \{x \in \mathbb{R} : a - r < x < a + r\} = (a - r, a + r).$$

Hence, in the Euclidean metric space $(\mathbb{R}, |.|)$, the open balls are open intervals.

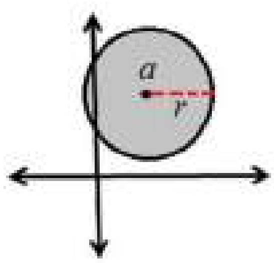



**ii.** When $n = 2$, $(\mathbb{R}^2, d)$ is the Euclidean metric space, where;

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \forall (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

In this case,

$$B_M(a;r) = \{x \in \mathbb{R}^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2\} = \text{Open circular disk.}$$

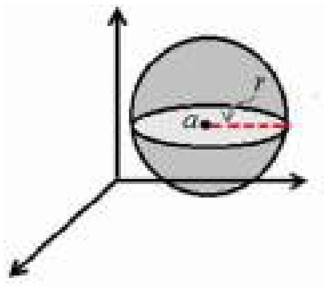


**iii.** When $n = 3$, $(\mathbb{R}^3, d)$ is the Euclidean metric space, where;

$$d(x\,,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}\,,$$
$$\forall\,(x_1\,,x_2,x_3\,)\,,(y_1\,,y_2,y\,) \in \mathbb{R}^3.$$

In this case ,

$$B_M(a;r) = \{x \in \mathbb{R}^2 : (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 < r^2\}$$
= Open solid sphere.




***Example 2:***

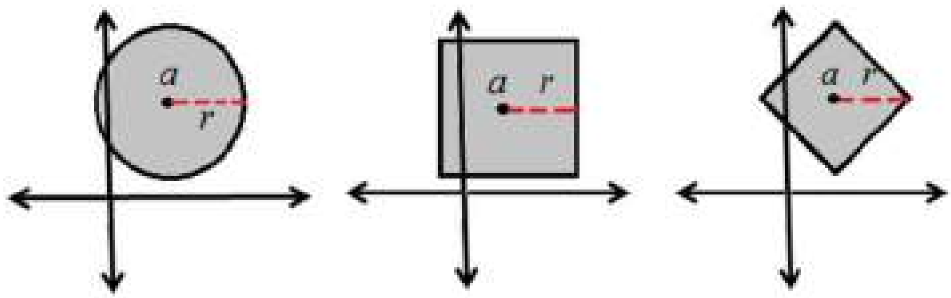
Let $M = \mathbb{R}^2$ with the following three metrics spaces on $M$ that given by:

**i.** $d(x\,,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\,, \forall\,(x_1\,,x_2\,)\,,(y_1\,,y_2\,) \in \mathbb{R}^2.$

**ii.** $d_1(x;y) = Max\{|x_1 - y_1|\,,|x_2 - y_2|\}, \forall\,(x_1\,,x_2\,)\,,(y_1\,,y_2\,) \in \mathbb{R}^2$ .

**iii.** $d_2(x;y) = |x_1 - y_1| + |x_2 - y_2|, \forall\,(x_1\,,x_2\,)\,,(y_1\,,y_2\,) \in \mathbb{R}^2.$

If $a \in \mathbb{R}^2$ and $r > 0$, we can draw the shape of the open ball $B(a;r)$ in $\mathbb{R}^2$ with respect to each of the above metrics as shown in the following figures:




***Definition (interior point):***

Let $(M\,,d)$ be a metric space and let $\emptyset \neq S \subseteq M$. A point $a \in S$ is called ***interior point*** of $S$ if, and only if, $\exists\, r > 0$ such that $B_M(a;r) \subseteq S$.

***Definition (open set):***

Let $(M\,,d)$ be a metric space. A non-empty subset $S$ of $M$ is said to be open in $M$ if, and only if, all points of $S$ are interior points of .

***Definition (interior of set):***

The set of all interior points of $S$ is called ***the interior of S*** and denoted by either $S^{\circ}$ or $nt(S)$ .

***Remark:*** In general, $S^{\circ} \subseteq S$.

***Example 1:*** Find the interior of the following sets:

1. In the Euclidean space $(\mathbb{R}, |.|)$ :

   $A = [-3,5]$ , $B = (1,4\}$ , $C = (5,8)$ $D = \{5\}$ $E = \mathbb{Z}$ .

   $A^{\circ} = (-3,5)$. Note that, for every $r > 0$, we have $B(-3;r) = (-3-r, -3+r) \nsubseteq A$. This shows that $-3$ is not an interior point of $A$. i.e. $-3 \notin A^{\circ}$. Similarly, $5 \notin A^{\circ}$. Deduce that, $B^{\circ} = (1,4)$, $C^{\circ} = (5,8)$, $D^{\circ} = \emptyset$ and $E^{\circ} = \emptyset$.

2. In the Euclidean space $(\mathbb{R}^2, \|.\|)$.

$$A = \{(x,y): x = y\} , B = \{(x,y) : x \geq 0, y \geq 0\} ,$$

$$C = \{(x,y): x^2 + y^2 = 1\}, D = \{(x,y): x^2 + y^2 \geq 1\}$$

$$E = \{(x,y): x^2 + y^2 < 1\}$$

$$A^{\circ} = \emptyset , B^{\circ} = \{(x,y) : x \geq 0, y \geq 0\} , C^{\circ} = \emptyset ,$$

$$D^{\circ} = \{(x,y): x^2 + y^2 \geq 1\} , F^{\circ} = F .$$

***Exercises:***

**1)** In a metric space $(M,d)$, show that both $\emptyset$ and $M$ are open sets in $M$ .

**2)** In a metric space $(M,d)$, show that every open ball $B_M(a;r)$ is an open set in $M$ .

**3)** In a discrete metric space $(M,d)$, show that every subset $S$ of $M$ is open set in $M$.

**4)** In a metric space $S = [0,1]$ of the Euclidean space $(\mathbb{R}, |\ |)$, show that every interval of the form $[0,x)$ or $(x,1]$ , where $0 < x < 1$, is an open set in $S$. Are these sets open in $\mathbb{R}$? explain that.

**Proof 2:**

**Wanted:** $B_M(a;r)$ open set in $M$.

Let $b \in B_M(a;r)$, we need to show $b$ is an interior point of $B_M(a;r)$, i.e. wanted: $\exists\ \delta > 0$ such that $B_M(b;\delta) \subseteq B_M(a;r)$.

Since $b \in B_M(a;r)$, hence $d(b\,,a) < r$.

Let $= Min\{\ d(b\,,a)\,, r - d(b\,,a)\}$ . Thus $\delta > 0$ and we will show that $B_M(b;\delta) \subseteq B_M(a;r)$. Let $x \in B_M(b;\delta)$, wanted: $x \in B_M(a;r)$, i.e. we need to show $d(x\,,a) < r$.

Since $x \in B_M(b;\delta)$, hence $d(x\,,b) < \delta$ and by using the triangle inequality we have; $d(x\,,a) \leq d(x\,,b) + d(b\,,a) \Rightarrow d(x\,,a) < \delta + d(b\,,a)$ ... $(*)$.

1. If $\delta = d(b\,,a) \Rightarrow \delta < r - d(b\,,a)$, then by recalling $(*)$ we have;

$$d(x\,,a) < \delta + d(b\,,a) < r - d(b\,,a) + d(b\,,a) = r$$

$$\Rightarrow d(x\,,a) < r$$

2. If $\delta = r - d(b\,,a)$, then $(*)$ implies that;

$$d(x\,,a) < \delta + d(b\,,a) < r - d(b\,,a) + d(b\,,a) = r$$

$$\Rightarrow d(x\,,a) < r$$

Therefore, $B_M(b;\delta) \subseteq B_M(a;r)$ and $B_M(a;r)$ is an open set in .

**Proof 3:**

**Wanted :** $S$ open in $M$. Let $x \in S$, we need to show that: $\exists r > 0$ such that $B_M(b;r) \subseteq S$.

Choose $r = \frac{1}{2} > 0$ , therefore;

$$B_M\left(x;\frac{1}{2}\right) = \left\{y \in M : d(y\,,x) < \frac{1}{2}\right\}$$

$$= \{y \in M : d(y\,,x) < 0\}$$

$$= \{y \in M : y = x\} = \{x\}.$$

$$\Rightarrow B_M\left(x;\frac{1}{2}\right) = \{x\}.$$

Since $x \in S \Rightarrow \{x\} \subseteq S \Rightarrow B_M\left(x;\frac{1}{2}\right) \subseteq S$.

Hence $S$ is an open set in $M$.

The important point to note here,

**i.** In the discrete metric space every singleton is an open ball and from exercise (2) above, we have every singleton is an open set.

**ii.** There are many metric spaces satisfied the property; "every singleton is an open set". As a home work prove that: If $M = \{x_1, x_2, \dots, x_n\}$ is a finite set and $d: M \times M \to \mathbb{R}$ be any metric function can be defined on $M$, then the metric space $(M, d)$ satisfied the property "every singleton is an open set".

***Proof 4:***

We know that, if $B_M(a; r)$ is an open ball in a metric space $(M, d)$, then $B_S(a; r) = S \cap B_M(a; r)$ is an open ball in the metric subspace $(S, d)$. Note that, $B_{\mathbb{R}}(0; x) = (-x, x)$ is an open ball in $\mathbb{R}$, $\forall\ 0 < x < 1$.

$$\Longrightarrow B_S(0; x) = S \cap B_{\mathbb{R}}(0; x) = [0,1] \cap (-x, x) \ \ (\forall\ 0 < x < 1)$$
$$= [0, x) \ (\forall\ 0 < x < 1).$$

$\Longrightarrow B_S(0; x) = [0, x)$ is an open ball in the metric subspace $S$, and since each open ball is an open set, therefore $[0, x)$ is open set in the metric subspace $S$ for all $0 < x < 1$.

Similarly, $B_{\mathbb{R}}(1; x) = (1 - x, 1 + x)$ is an open ball in $\mathbb{R}$ $(\forall\ 0 < x < 1)$.

$$\Longrightarrow B_S(1; x) = [0,1] \cap B_{\mathbb{R}}(1; x) = [0,1] \cap (1 - x, 1 + x) \ (\forall\ 0 < x < 1)$$
$$= (1 - x, 1]$$

Note that , as $0 < x < 1 \Longrightarrow -1 < -x < 0 \ \Longrightarrow 0 < 1 - x < 1$

$$\Longrightarrow B_S(1; x) = (\acute{x}, 1], \qquad \forall\ 0 < \acute{x} < 1$$

$\Longrightarrow (\acute{x}, 1]$, ( $\forall\ 0 < \acute{x} < 1$) is an open set in the metric subspace $S$.

***Remark:***

Form the above we deduce that, if $(S, d)$ is a metric subspace of a metric space $(M, d)$, then the open sets in $(S, d)$ need not be open sets in $(M, d)$. For example recall exercise (4) above, we know that $[0, \frac{1}{2})$ is open set in the metric subspace = $[0,1]$ , while $[0,\frac{1}{2})$ is not open set in $\mathbb{R}$, since the point $0 \in [0,\frac{1}{2})$ is not an interior point of $[0,\frac{1}{2})$ w.r.t. the Euclidean space $(\mathbb{R},|\,|)$.

***Exercise:***

Let $(M,d)$ be a metric space and $x \in M$. If $r_2 > r_1 > 0$, prove that;

$$B(x;r_1) \subseteq B(x;r_2).$$

***Theorem:***

Let $(M,d)$ be a metric space. Then:

1. The intersection of a finite collection of open sets in $M$ is an open set in $M$.
2. The union of any collection of open sets in $M$ is an open set in $M$.

***Proof:***

***For 1:*** Suppose $G_1,\dots,G_n$ be open sets in $M$. **Wanted:** $\bigcap_{i=1}^{n} G_i$ is an open set in $M$, i.e. wanted: $\forall x \in \bigcap_{i=1}^{n} G_i\ \exists r > 0 \ni B(x;r) \subseteq \bigcap_{i=1}^{n} G_i$.

Let $x \in \bigcap_{i=1}^{n} G_i$. Then, $x \in G_i\ \forall i = 1,\dots,n$. But, $G_i$ is an open set in $M$, thus, $\exists r_i > 0 \ni B(x;r_i) \subseteq G_i \forall i = 1,\dots,n$. Put, $r = Min\{r_1,\dots,r_n\} > 0$. Since $r < r_i$, hence, $B(x;r) \subseteq B(x;r_i) \subseteq G_i\ \forall i = 1,\dots,n$. Thus, $B(x;r) \subseteq \bigcap_{i=1}^{n} G_i$. So, $x$ is an interior point in $\bigcap_{i=1}^{n} G_i$. Therefore, $\bigcap_{i=1}^{n} G_i$ is an open set.

***For 2:*** Assume, $G_\alpha$ be an open set in $M$ for all $\alpha \in I$. **Wanted:** $\bigcup_{\alpha\in I} G_\alpha$ is an open set, i.e. wanted: $\forall x \in \bigcup_{\alpha\in I} G_\alpha\ \exists r > 0 \ni B(x;r) \subseteq \bigcup_{\alpha\in I} G_\alpha$.

Let $x \in \bigcup_{\alpha\in I} G_\alpha$. Then, $x \in G_\beta$ for some $\beta \in I$. But, $G_\beta$ is an open set in $M$, therefore, $\exists r > 0 \ni B(x;r) \subseteq G_\beta \subseteq \bigcup_{\alpha\in I} G_\alpha$. Thus, $B(x;r) \subseteq \bigcup_{\alpha\in I} G_\alpha$. So, $x$ is an interior point in $\bigcup_{\alpha\in I} G_\alpha$. Therefore, $\bigcup_{\alpha\in I} G_\alpha$ is an open set.

***Remark:***

In general, the intersection of any collection of open sets in a metric space $(M,d)$ need not to be open set in $M$. As a counter example, the collection

$\left\{\left(\frac{-1}{n},\frac{1}{n}\right) \middle| n \in \mathbb{Z}^+\right\}$ is an infinite collection of open sets (open intervals) in the Euclidean space $\mathbb{R}$, but $\bigcap_{n\in\mathbb{Z}^+}\left(\frac{-1}{n},\frac{1}{n}\right) = \{0\}$ is not open in $\mathbb{R}$.

***Theorem:***

Let $(S\,,d)$ be a metric subspace of a metric space $(M\,,d)$ and let $X \subseteq S$. Then $X$ is open in $S$ if, and only if, $X = S \cap A$ for some set $A$ which is open in $M$.

***Proof:***

Suppose $X$ is an open set in $S$. Wanted: $\exists$ an open set $A$ in $\ni X = S \cap A$.

Since $X$ is an open set in $S$, hence, $\forall\, x \in S$ , $\exists\, r_x > 0 \ni B_s(x;\ r_x) \subseteq X$. It is clear that, $X = \cup_{x\in X} B_S(x;\ r_x)$. But $B_s(x;\ r_x) = S \cap B_M(x;\ r_x)$. So, if we let $A = \cup_{x\in X} B_M(x;\ r_x)$, then $A$ is a union of open sets in M, so it is an open set in $M$. To complete the proof, we need only to show that $X = S \cap A$.

$$\begin{aligned} X &= \cup_{x\in X} B_M(x;\ r_x) \\ &= \cup_{x\in X} (S \cap B_M(x;\ r_x) \\ &= S \cap (\cup_{x\in X} B_M(x;\ r_x)) \\ &= S \cap A \end{aligned}$$

Conversely, suppose $\exists$ an open set $A$ in $M$ such that $X = S \cap A$. Wanted: $X$ is open in $S$. Let $x \in X$, wanted: $x$ is an interior point of $X$ in $S$, i.e. $\exists\, r > 0 \ni B_s(x;\ r_x) \subseteq X$.

Since $x \in X = S \cap A \implies x \in A$. But $A$ is an open set in $M$, then $\exists\, r > 0 \ni B_M(x;\ r_x) \subseteq A \implies S \cap B_M(x;\ r_x) \subseteq S \cap A = X$.

But $B_s(x;\ r_x) = S \cap B_M(x;\ r_x)$ is an open ball in $S$, hence

$$B_s(x;\ r_x) \subseteq S \cap A = X$$
$$\implies B_s(x;\ r_x) \subseteq X$$

Hence, $x$ is an interior point of $X$ in $S$ and $X$ is open in $S$.

***Definition (closed set):***

Let $(M\,,d\,)$ be a metric space. A subset $S \subseteq M$ is called closed set in $M$ if, and only if, $S^c = M - S$ is open set in .

***Examples:***

In the Euclidean metric space $(\mathbb{R}^2, \|.\|)$ , the sets ,

$$A = \{(x, y): x = y\} \ , B = \{(x, y): x^2 + y^2 \leq 1\} ;$$

$$C = \{(x, y): x^2 + y^2 \geq 1\} \text{ and;}$$

$$D = \{(x, y): x^2 + y^2 = 1\} ;$$

are closed set in $\mathbb{R}^2$, while the set $E = \{(x, y): x^2 + y^2 < 1\}$ is not closed set in $\mathbb{R}^2$ .

***Exercises:***

Let $(M, d)$ be a metric space. Prove the following statements:

1. The union of a finite collection of closed sets in $M$ is closed set in $M$.
2. The intersection of any collection of closed sets in $M$ is closed set in $M$.
3. If $A$ is open set in $M$ and $B$ is closed set in $M$, show that $A - B$ is open set in $M$ and $B - A$ is closed set in $M$.

***Proof (1):***

Let $\mathcal{M} = \{G_i | i = 1, 2, \dots, n\}$ be a finite collection of closed sets in $M$. Wanted: $\cup_{i=1}^{n} G_i$ is closed set in $M$, i.e. wanted: $M - (\cup_{i=1}^{n} G_i)$ is open set in $M$.

Note that, $M - (\cup_{i=1}^{n} G_i) = \cap_{i=1}^{n} (M - G_i)$.

Since $G_i$ is closed set in $M \Longrightarrow M - (G_i)$ is open set in $\forall \ i = 1, 2, \dots, n$.

$\Longrightarrow \cap_{i=1}^{n} (M - (G_i))$ is open set in $M \ \forall \ i = 1, 2, \dots, n$.

$\Longrightarrow M - (\cup_{i=1}^{n} G_i)$ is open set in $M \ \forall \ i = 1, 2, \dots, n$.

$\Longrightarrow \cup_{i=1}^{n} G_i$ is closed set in $M$.

***Proof (3):***

**i.** Firstly wanted: $A - B$ is open set in $M$.

Note that, $A - B = A \cap B^c = A \cap (M - B)$. Since $B$ is closed set in $M$, then $M - B$ is open in $M$. But, $A$ is also open in $M$, then $A \cap (M - B)$ is open set in $M$ and hence $A - B$ is open set in $M$.

**ii.** Secondly wanted: $B - A$ is closed set in $M$, i.e. $M - (B - A)$ is open in $M$.

Note that,

$$M - (B - A) = M \cap (B \cap A^c)^c = M \cap (B^c \cup A)$$
$$= (M \cap B^c) \cup A = (M - B) \cup A.$$

Since $B$ is closed in $M$, then $M - B$ is open in $M$. But $A$ is open in $M$, thus $(M - B) \cup A$ is open in $M$. Hence $M - (B - A)$ is open in $M$. Therefore $(B - A)$ is closed set in $M$.

***Theorem:***

Let $(S, d)$ be a metric subspace of a metric space $(M, d)$ and let $Y \subseteq S$. Then $Y$ is closed in $S$ if, and only if, $Y = S \cap B$ for some closed set $B$ in $M$.

***Proof :***

Suppose that $Y$ is closed in $S$. Wanted: $\exists$ a closed set $B$ in $\ni Y = S \cap B$. Since $Y$ is closed in $S$, hence $S - Y$ is open in $S$. Thus, $\exists$ an open set $A$ in $M$ such that $S - Y = S \cap A$ (according to a previous result).

$\Longrightarrow Y = S - (S \cap A) = S \cap (S \cap A)^c$

$= S \cap (S^c \cup A^c) = (S \cap S^c) \cup (S \cap A^c)$

$= \emptyset \cup (S \cap A^c) = S \cap A^c = S \cap (M - A)$

$\Longrightarrow Y = S \cap (M - A)$.

Since $A$ is open in $M$, hence $M - A$ is closed in $M$. So, if we put $M - A = B$, then $B$ is closed set in $M$ such that $Y = S \cap B$ and our claim is hold.

Conversely, suppose $\exists$ a closed set $B$ in $M \ni Y = S \cap B$. Wanted: $Y$ is closed in $S$, i.e. $S - Y$ is open in $S$.

Note that,

$S - Y = S - (S \cap B) = S \cap (S \cap B)^c$

$= S \cap (S^c \cup B^c) = S \cap B^c = S \cap (M - B)$

Since $B$ is closed in $M$, then $A = M - B$ is open in $M$. Therefore, $S - Y = S \cap A$ is an open set in $S$, (according to a previous result) $\Longrightarrow Y$ is closed in $S$.

***Theorem (Axioms of an interior):***

Let $(M, d)$ be a metric space and $S, T \subseteq M$. Then:

1. $\emptyset^\circ = \emptyset$ and $M^\circ = M$.
2. If $S \subseteq T$, then $S^\circ \subseteq T^\circ$.
3. $S^\circ$ is the largest open set in $M$ that contained in $S$.
4. $S$ is open if, and only if, $S = S^\circ$.
5. $S^{\circ\circ} = S^\circ$.
6. $(S \cap T)^\circ = S^\circ \cap T^\circ$.
7. In general, $S^\circ \cup T^\circ \subseteq (S \cup T)^\circ$, but $(S \cup T)^\circ \neq S^\circ \cup T^\circ$.

***Proof 3:***

Let $\Omega = \{G \subseteq M |\ G \text{ is open in } M \text{ and } G \subseteq S\}$ be the collection of all open sets in $M$ that contained in $S$.

Firstly, we shall prove that $S^\circ = \bigcup_{G \in \Omega} G$.

**For $S^\circ \subseteq \bigcup_{G\in\Omega} G$:** Let $x \in S^\circ$, then $\exists r > 0 \ni B(x; r) \subseteq S$.

Wanted: $x \in \bigcup_{G\in\Omega} G$.

According to a previous result, $B(x; r)$ is an open set with $B(x; r) \subseteq S$. Thus, $B(x; r) \in \Omega$, so $\exists G' \in \Omega \ni B(x; r) = G'$. But $G' \subseteq \bigcup_{G\in\Omega} G$, then $B(x; r) \subseteq \bigcup_{G\in\Omega} G \Longrightarrow x \in \bigcup_{G\in\Omega} G \Longrightarrow S^\circ \subseteq \bigcup_{G\in\Omega} G$.

**For $\bigcup_{G\in\Omega} G \subseteq S^\circ$:** Let $x \in \bigcup_{G\in\Omega} G$. Wanted: $x \in S^\circ$.

Since $x \in \bigcup_{G\in\Omega} G$, hence $\exists G' \in \Omega \ni x \in G'$. That is, $G'$ is an open set in $M$ and $G \subseteq S$. Therefore, $x$ is an interior point of $G'$ and there exists $r > 0$ such that $B(x; r) \subseteq G' \subseteq S \Longrightarrow B(x; r) \subseteq S$. Thus, $x \in S^\circ$and $\bigcup_{G\in\Omega} G \subseteq S^\circ$.

Now, since $S^\circ = \bigcup_{G\in\Omega} G$ is a union of open sets in $M$, hence $S^\circ$is open in $M$ and it contained in $S$, since $S^\circ \subseteq S$. Thus, $S^\circ \in \Omega$. In fact, if $G$ is open and $G \subseteq$

$S$, then $G \subseteq \bigcup_{G\in\Omega} G = S^\circ$. Therefore, $S^\circ$ is the largest open set that contained in $S$.

***Proof 6:***

Wanted: $(S\cap T)^\circ = S^\circ \cap T^\circ$.

**i.** **For** $(S\cap T)^\circ \subseteq S^\circ \cap T^\circ$**:** Since $S\cap T \subseteq S$ and $S\cap T \subseteq T$, hence $(S\cap T)^\circ \subseteq S^\circ$ and $(S\cap T)^\circ \subseteq T^\circ$ as an application of axiom 2 above. Therefore, $(S\cap T)^\circ \subseteq S^\circ \cap T^\circ$.

**ii.** **For** $S^\circ \cap T^\circ \subseteq (S\cap T)^\circ$**:** Let $x \in S^\circ \cap T^\circ$. Wanted: $x \in (S\cap T)^\circ$, i.e. wanted: $\exists r > 0 \ni B(x;r) \subseteq S\cap T$.

Since $x \in S^\circ \cap T^\circ \Longrightarrow x \in S^\circ$ and $x \in T^\circ$;

$\Longrightarrow \exists r_1 > 0 \ni B(x;r_1) \subseteq S$ and $\exists r_2 > 0 \ni B(x;r_2) \subseteq T$;

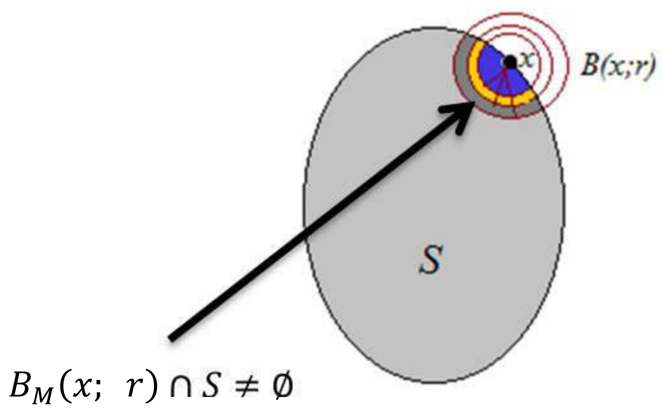
Put $r = Min\{r_1, r_2\}$. According to a previous result, $B(x;r) \subseteq B(x;r_i)$ for $i = 1,2$ $\Longrightarrow B(x;r) \subseteq S$ and $B(x;r) \subseteq T$ $\Longrightarrow B(x;r) \subseteq S\cap T \Longrightarrow$ $x \in (S\cap T)^\circ$.

From i and ii, $(S\cap T)^\circ = S^\circ \cap T^\circ$.

***Exercise:*** Prove the axioms 1,2,4,5 and 7 above.

***Definition (Adherent points):***

Let $(M, d)$ be a metric space and let $S \subseteq M$. A point $x \in M$ is called an ***adherent point*** of $S$ if, and only if, for every $r > 0$ the open ball $B_M(x;\ r)$ satisfied, $B_M(x;\ r) \cap S \neq \emptyset$.






Dr. Hana' M. Ali

***Definition (closure of a set):***

The set of all adherent points of a set $S$ is called ***the closure of a set*** $S$ which is denoted by $\bar{S}$.

***Remark:*** In general, $S \subseteq \bar{S}$. In fact, if $x \in S$, then $x \in B_M(x;\ r) \cap S, \forall r > 0$.

***Example 1:***

In the Euclidean metric space $(\mathbb{R}, |.|)$, let;

$$A = (-3, 4)\ ,\ B = [0, 1]\ ,\ C = [3, 7]\ ,\ D = \mathbb{Z}\ ,\ E = \mathbb{Q}.$$

Then, $\bar{A} = [-3, 4]\ ,\ \bar{B} = [0, 1]\ ,\ \bar{C} = [3, 7]\ ,\bar{D} = \mathbb{Z}\ ,\bar{E} = \mathbb{R}$.

***Example 2:***

In the Euclidean metric space $(\mathbb{R}^2, \|.\|)$, let;

$$A = \{(x, y): x^2 + y^2 < 1\}, B = \{(x, y): x^2 + y^2 > 1\},$$

$$C = \{(x, y): x^2 + y^2 = 1\},\ D = \{(x, y) : x \geq 0, y \geq 0\}.$$

$$\Longrightarrow \bar{A} = \{(x, y): x^2 + y^2 \leq 1\}\ ,\ \bar{B} = \{(x, y): x^2 + y^2 \geq 1\},$$

$$\bar{C} = \{(x, y): x^2 + y^2 = 1\}\ ,\bar{D} = \{(x, y) : x \geq 0, y \geq 0\}.$$

***Theorem (Axioms of a Closure):***

Let $(M, d)$ be a metric space and let $S, T \subseteq M$. Then

1. $\bar{\emptyset} = \emptyset$ and $\bar{M} = M$.
2. If $S \subseteq T$, then $\bar{S} \subseteq \bar{T}$.
3. $\bar{S}$ is the smallest closed set in $M$ such that $S \subseteq \bar{S}$.
4. $S$ is closed in $M \Leftrightarrow \bar{S} = S$.
5. $\bar{\bar{S}} = \bar{S}$.
6. $\overline{S \cup T} = \bar{S} \cup \bar{T}$.
7. In general, $(\overline{S \cap T}) \subseteq \bar{S} \cap \bar{T}$. But, $(\overline{S \cap T}) \neq \bar{S} \cap \bar{T}$.
8. $S^\circ = \overline{S^c}^c$

***Proof 3:***

Let $\Omega = \{F \subseteq M |\ F$ is closed in $M$ and $S \subseteq F\}$ be the collection of all closed sets in $M$ that contain $S$.

Firstly, we shall prove that $\bar{S} = \bigcap_{F\in\Omega} F$.

**For $\bar{S} \subseteq \bigcap_{F\in\Omega} F$:** Let $x \in \bar{S}$, then $\forall r > 0 \ni B(x;r) \cap S \neq \emptyset$.

Wanted: $x \in \bigcap_{F\in\Omega} F$.

By contrary, assume that $x \notin \bigcap_{F\in\Omega} F$. So, $\exists F' \in \Omega \ni x \notin F' \Longrightarrow x \in F'^c$. But $F'^c$ is open set, since $F'$ is closed, that is $x$ is an interior point of $F'^c$, so $\exists\, r > 0 \ni B(x;r) \subseteq F'^c \Longrightarrow B(x;r) \cap F' = \emptyset$. But $F' \in \Omega$, i.e. it satisfied $S \subseteq F' \Longrightarrow B(x;r) \cap S \subseteq B(x;r) \cap F' = \emptyset$.

Thus, $\exists\, r > 0 \ni B(x;r) \cap S = \emptyset \Longrightarrow x \notin \bar{S}$ and that contradicts our assumption that $x \in \bar{S}$. Therefore, $x \in \bigcap_{F\in\Omega} F$.

**For $\bigcap_{F\in\Omega} F \subseteq \bar{S}$:** Let $x \in \bigcap_{F\in\Omega} F$. Wanted: $x \in \bar{S}$:

By contrary, suppose $x \notin \bar{S}$. That is, $\exists\, r > 0 \ni B(x;r) \cap S = \emptyset$. Thus, $S \subseteq (B(x;r))^c$. But $(B(x;r))^c$ is a closed set in $M$ and it contains $S$, so $(B(x;r))^c \in \Omega$. That is, $\exists F' \in \Omega \ni F' = (B(x;r))^c \Longrightarrow \bigcap_{F\in\Omega} F \subseteq F'$. But, $x \notin (B(x;r))^c \supseteq \bigcap_{F\in\Omega} F \Longrightarrow x \notin \bigcap_{F\in\Omega} F$ and that contradict our assumption that $x \in \bigcap_{F\in\Omega} F$. Therefore, $x \in \bar{S}$ and $\bigcap_{F\in\Omega} F \subseteq \bar{S}$.

Now, since $\bar{S} = \bigcap_{F\in\Omega} F$ is an intersection of closed sets in $M$, hence $\bar{S}$ is closed and it contains $S$, since $S \subseteq \bar{S}$. Thus, $\bar{S} \in \Omega$. In fact, if $F$ is closed and $S \subseteq F$, then $\bar{S} = \bigcap_{F\in\Omega} F \subseteq F$. Therefore, $\bar{S}$ is the smallest closed set that contain $S$.

***Proof 6:***

Wanted: $\overline{S \cup T} = \bar{S} \cup \bar{T}$.

i. **For $\bar{S} \cup \bar{T} \subseteq \overline{S \cup T}$:** Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, hence $\bar{S} \subseteq \overline{S \cup T}$ and $\bar{T} \subseteq \overline{S \cup T}$, as an application of axiom 2 above. Therefore, $\bar{S} \cup \bar{T} \subseteq \overline{S \cup T}$.

ii. **For $\overline{S \cup T} \subseteq \bar{S} \cup \bar{T}$:** Let $x \in \overline{S \cup T}$. Wanted: $x \in \bar{S} \cup \bar{T}$.

By contrary, assume $x \notin \bar{S} \cup \bar{T} \Longrightarrow x \notin \bar{S}$ and $x \notin \bar{T}$;

$\Longrightarrow \exists r_1 > 0 \ni B(x;r_1) \cap S = \emptyset$ and $\exists r_2 > 0 \ni B(x;r_2) \cap T = \emptyset$;

Put $r = Min\{r_1, r_2\}$. According to a previous result, $B(x;r) \subseteq B(x;r_i)$ for $i = 1,2$. Then;

$B(x;r) \cap S \subseteq B(x;r_1) \cap S = \emptyset$ and $B(x;r) \cap T \subseteq B(x;r_2) \cap T = \emptyset$;

$\Longrightarrow B(x;r) \cap S = \emptyset$ and $B(x;r) \cap T = \emptyset \Longrightarrow B(x;r) \cap (S \cup T) = \emptyset$.

Therefore, $x \notin \overline{S \cup T}$ (a contradiction). Thus, $x \in \bar{S} \cup \bar{T}$ and $\overline{S \cup T} \subseteq \bar{S} \cup \bar{T}$.

From i and ii, $\overline{S \cup T} = \bar{S} \cup \bar{T}$.

***Proof 8:***

Wanted: $S^{\circ} = \overline{S^c}^c$.

**For** $S^{\circ} \subseteq \overline{S^c}^c$**:** Let $x \in S^{\circ}$. Wanted: $x \in \overline{S^c}^c$.

Since $x \in S^{\circ} \Longrightarrow \exists r > 0 \ni B(x;r) \subseteq S \Longrightarrow B(x;r) \cap S^c = \emptyset$

$\Longrightarrow x \notin \overline{S^c} \Longrightarrow x \in \overline{S^c}^c \Longrightarrow S^{\circ} \subseteq \overline{S^c}^c$.

**For** $\overline{S^c}^c \subseteq S^{\circ}$**:** Let $x \in \overline{S^c}^c$. Wanted: $x \in S^{\circ}$.

Since $x \in \overline{S^c}^c \Longrightarrow x \notin \overline{S^c} \Longrightarrow \exists r > 0 \ni B(x;r) \cap S^c = \emptyset \Longrightarrow B(x;r) \subseteq S \Longrightarrow$ $x \in S^{\circ} \Longrightarrow \overline{S^c}^c \subseteq S^{\circ}$.

Therefore, our goal is down.

***Exercise:*** Prove the axioms 1,2,4,5 and 7 above.

***Definition (Accumulation (cluster) points of a set):***

Let $(M, d)$ be a metric space and let $S \subseteq M$. A point $x \in M$ is said to be an ***Accumulation point*** of $S$ if, and only if, for every open ball $B_M(x;r)$;

$$B_M(x;r) \cap S - \{x\} \neq \emptyset.$$

The set of all Accumulation points of a set $S$ is called ***the derived set*** of $S$ which is denoted by $S'$ or $dS$. Note that, $S' \subseteq S$.

***Remark:***

Let $(M, d)$ be a metric space and let $S \subseteq M$. Then:

1. $x$ is an Accumulation point of $S$ if, and only if, every open ball $B_M(x;r)$ contains points of $S$ different from $x$.
2. $x$ is an Accumulation point of $S$ if, and only if, $x$ is an adherent point of $S - \{x\}$.

***Example:***

In the Euclidean metric space $(\mathbb{R}, |.|)$, let;

$$A = (-3, 4) \ , \ B = [0, 1] \ , \ C = [3, 7] \ , \ D = \mathbb{Z} \ , \ E = \mathbb{Q}.$$

$$\Longrightarrow A' = [-3, 4] \ , \ B' = [0, 1] \ , \ C' = [3, 7] \ , D' = \emptyset \ , E' = \mathbb{R}.$$

***Example:***

In the Euclidean metric space $(\mathbb{R}^2, \|.\|)$, let ;

$$A = \{(x, y): x^2 + y^2 < 1\}, \qquad B = \{(x, y): x^2 + y^2 > 1\} \ ,$$

$$C = \{(x, y): x^2 + y^2 = 1\} \ , \ D = \{(x, y) : x \geq 0, y \geq 0\}.$$

$$\Longrightarrow A' = \{(x, y): x^2 + y^2 \leq 1\}, \qquad B' = \{(x, y): x^2 + y^2 \geq 1\},$$

$$C' = \{(x, y): x^2 + y^2 = 1\} \ , \ D' = \{(x, y) : x \geq 0, y \geq 0\}.$$

***Theorem (Axioms of a Derived set):***

Let $(M, d)$ be a metric space and let $S, T \subseteq M$. Then

1. $S \subseteq T \Longrightarrow S' \subseteq T'$.
2. $(S \cup T)' = S' \cup T'$.
3. In general, $(S \cap T)' \subseteq S' \cap T'$, but $(S \cap T)' \neq S' \cap T'$.
4. $\bar{S} = S' \cup S$.

***Proof 4:***

To show that, $\bar{S} = S' \cup S$, we need to prove:

**i.** $S' \cup S \subseteq \bar{S}$.

**ii.** $\bar{S} \subseteq S' \cup S$.

**For i:** From the definitions of the closure and the derived set of $S$, we have $S \subseteq \bar{S}$ and $S' \subseteq \bar{S}$. Therefore, $S' \cup S \subseteq \bar{S}$.

**For ii:** let $x \in \bar{S}$. Wanted: $x \in S' \cup S$.

By contrary, assume that $x \notin S' \cup S \Longrightarrow x \notin S'$ and $x \notin S$;

$x \notin S' \Longrightarrow \exists r > 0, \ B(x; r) \cap S - \{x\} = \emptyset$.

$\Longrightarrow \exists r > 0, \ B(x; r) \cap S = \emptyset$, (since $x \notin S$ and $S - \{x\} = S$).

$\Longrightarrow x \notin \bar{S}$, (a contradiction).

$\Longrightarrow x \in S' \cup S$. Accordingly, $\bar{S} \subseteq S' \cup S$.

From i and ii we have $\bar{S} = S' \cup S$.

***Definition (Boundary of a set):***

Let $(M, d)$ be a metric space and let $S \subseteq M$. A point $x \in M$ is said to be ***boundary point*** of a set $S$ if, and only if, for every open ball $B_M(x;r)$ contain at least one point of $S$ and at least one point of $S^c$, i.e. $(B(x;r) \cap S \neq \emptyset$ and $B(x;r) \cap S^c \neq \emptyset)$, i.e. $(x \in \bar{S} \cap \overline{S^c})$.

The set of all boundary points is called ***boundary set*** of $S$ and it denoted by $\partial S$. In fact; $\partial S = \bar{S} \cap \overline{S^c}$.

***Example:***

In the Euclidean metric space $(\mathbb{R}, |\ |)$, let $A = (-3, 3)$, $B = \mathbb{Z}$, $C = \mathbb{Q}$.

**i.** $\partial A = \bar{A} \cap \overline{A^c} = [-3, 3] \cap \big((-\infty, -3] \cup [3, \infty)\big) = \{-3, 3\}$.

**ii.** $\partial B = \bar{B} \cap \overline{B^c} = \mathbb{Z} \cap (\cup_{n \in z} [n, n+1]) = \mathbb{Z}$.

**iii.** $\partial C = \bar{C} \cap \overline{C^c} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

***Example:***

In the Euclidean metric space $(\mathbb{R}^2, \|\ \|)$, let;

$$A = \{(x, y): x^2 + y^2 < 1\}, \qquad B = \{(x, y): x^2 + y^2 > 1\},$$

$$C = \{(x, y): x^2 + y^2 = 1\}.$$

**i.** $\partial A = \bar{A} \cap \overline{A^c} = \{(x, y): x^2 + y^2 \leq 1\} \cap \{(x, y): x^2 + y^2 \geq 1\}$
$= \{(x, y): x^2 + y^2 = 1\}$.

**ii.** $\partial B = \bar{B} \cap \overline{B^c} = \{(x, y): x^2 + y^2 \geq 1\} \cap \{(x, y): x^2 + y^2 \leq 1\}$
$= \{(x, y): x^2 + y^2 = 1\}$.

**iii.** $\partial C = \bar{C} \cap \overline{C^c} = \{(x, y): x^2 + y^2 = 1\} \cap \{(x, y): x^2 + y^2 \geq 1\} \cup$
$\{(x, y): x^2 + y^2 \leq 1\} = \{(x, y): x^2 + y^2 = 1\}$.

***Exercises:***

Let $(M, d)$ be a metric space and let $A, B \subseteq M$. Then:

1. $\partial A = \emptyset$ if, and only if, $A$ is both open and closed in $M$.
2. $\partial(A^c) = \partial A$.

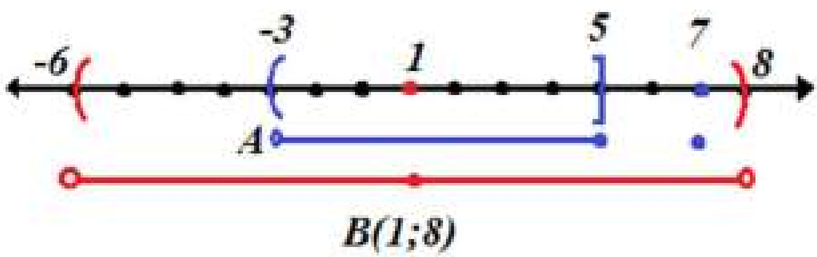
3. If $\bar{A} \cap \bar{B} = \emptyset$ , then $\partial(A \cup B) = \partial A \cup \partial B$.
4. If $A^{\circ} = B^{\circ} = \emptyset$ and if $A$ is closed in $M$, then $(A \cup B)^{\circ} = \emptyset$.

***Definition (Bounded set)***:

Let $(M, d)$ be a metric space. A subset $S$ if $M$ is called ***bounded*** if $S \subseteq B_M(x;\ r)$, for some $r > 0$ and some $a \in M$.

***Example:***

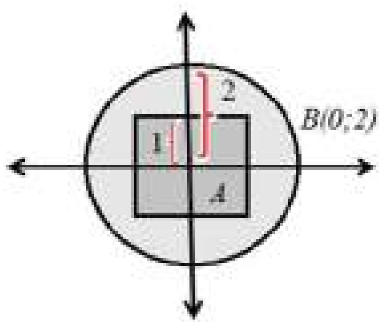
In the Euclidean metric space $(\mathbb{R}, |\,|)$, the set $A = (-3, 5] \cup \{7\}$ is bounded since we can find an open ball $B(1;7) = (-6, 8)$ such that $A \subseteq B(1;7)$, as shown in the following figure;




***Example:***

In the Euclidean metric space $(\mathbb{R}^2, \|\,\|)$,

the set $A = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$ is bounded set since we can find an open ball $B((0,0); 2) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\}$ such that $A \subseteq B((0,0); 2)$, as shown in the following figure;




***Theorem (Bolzano-Weierstrass):-***

Let $S$ be a bounded subsets of the Euclidean metric space $(\mathbb{R}^n, \|.\|)$ and it has infinitely many points. Then there is at least one point in $\mathbb{R}^n$ which is an accumulation point of $S$.

***Remark:*** To simplify the idea of the proof, we shall give it in the Euclidean space $\mathbb{R}$, (i.e. when $n = 1$).

***Proof:***

Since $S$ is bounded in $\mathbb{R}$, then we can find an open interval $(-a\,,a)$ such that $S \subseteq B(0;a) = (-a\,,a) \Rightarrow S \subseteq [-a\,,a]$.

**1.** Subdivide $[-a\,,a]$ into $[-a\,,0]$ and $[0\,,a]$. At least one of the subintervals $[-a\,,0]$ *or* $[0\,,a]$ contains an infinite subset of $S$. Denote such subinterval by $[a_1\,,b_1]$.

**2.** Bisect $[a_1\,,b_1]$ and obtain a subinterval $[a_2\,,b_2]$ containing an infinite subset of $S$ and continue this process.

**3.** In this way, a countable collection of closed subintervals $[a_1\,,b_1]$, $[a_2\,,b_2]\,, \ldots\,, [a_n\,,b_n], \ldots$ was obtained. The $n^{\text{th}}$ closed interval $[a_n\,,b_n]$ being of length $b_n - a_n = a/2^{n-1}$. Therefore, the length of $[a_n\,,b_n]$ is approach to zero as $n \longrightarrow \infty$.

**4.** Let $A = \{a_1\,,a_2\,,\ldots,a_n\,,\ldots\}$ and $B = \{b_1\,,b_2\,,\ldots,b_n\,,\ldots\}$. Since $a_i < b_1$, $\forall i = 1,2,\ldots$, hence $A$ is bounded above and $Sup(A)$ is exist. Moreover, $B$ is bounded below and $Inf(B)$ is exist, since $b_i > a_1\,, \forall i = 1,2,\ldots$ . In fact, we have;

$$a_1 < a_2 < \cdots < a_n < \cdots < b_n < \cdots < b_2 < b_1$$

Therefore, $Sup\{A\} = Inf\{B\} = x$ say, (**as an exercise prove that**). Notice that, $x$ may or may not belong to $S$.

**Now,** we shall prove that $x$ is an accumulation point of $S$, i.e. we need to show that $\forall r > 0,\ B(x;r) \cap S - \{x\} \neq \emptyset$.

Let $> 0 \Longrightarrow \frac{r}{4a} > 0$. By using a previous result;

$$\exists n \in \mathbb{Z}^+ \ni \frac{1}{2^n} < \frac{r}{4a} \Longrightarrow \frac{a}{2^{n-1}} < \frac{r}{2} \Longrightarrow b_n - a_n = \frac{a}{2^{n-1}} < \frac{r}{2}.$$

Thus, there exists a closed interval $[a_n\,,b_n]$ has length less than $\frac{r}{2}$. According, $x = Sup\{A\} = Inf\{B\}$, so $a_n < x < b_n$ and;

$$[a_n , b_n] \subseteq B\left(x;\frac{r}{2}\right) = \left(x-\frac{r}{2}, x+\frac{r}{2}\right) \subseteq B(x\,;r) = (x-r, x+r).$$

But $[a_n , b_n]$ contains an infinite subset of $S$. Therefore, $B(x\,;r)$ contains an infinite subset of $S \Rightarrow B(x\,;r) \cap S \neq \emptyset \Rightarrow B(x\,;r) \cap S - \{x\} \neq \emptyset$. Thus, for all open 1-ball $B(x\,;r) = (x-r, x+r)$ we have, $B(x\,;r) \cap S - \{x\} \neq \emptyset$. Hence $x$ is an accumulation point of $S$.

***Theorem:***

If $x$ is an accumulation point of a subset $S$ in the Euclidean space $\mathbb{R}^n$, then every open $n$-ball $B(x\,;r)$ contains infinitely many points of $S$ .

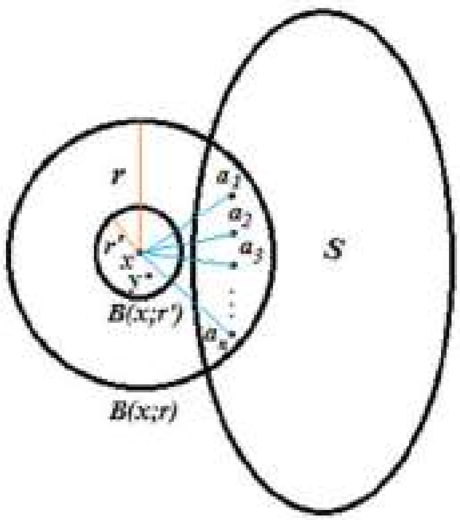
***Proof :*** By contrary, suppose there is an open $n$-ball $B(x\,;r)$ such that;

$$B(x\,;r) \cap S - \{x\} = \{a_1 , a_2 , \dots, a_n\}.$$

Since $a_1 , a_2 , \dots, a_n \in B(x\,;r)$, hence;

$$\|x-a_1\| < r , \|x-a_2\| < r , \dots, \|x-a_n\| < r.$$

Put $r' = \frac{1}{2} Min\{\|x-a_1\| , \|x-a_2\| , \dots, \|x-a_n\|\} > 0$. We need to show that, $B(x\,;r') \cap S - \{x\} = \emptyset$.




Suppose that $B(x\,;r') \cap S - \{x\} \neq \emptyset$

$\Rightarrow \exists$ at least $y \in B(x\,;r') \cap S - \{x\}$.

$\Rightarrow y \in B(x\,;r')$ and $y \in S - \{x\}$.

$\Rightarrow \|x-y\| < r'$ and $y \in S - \{x\}$.

Since $a_i \in B(x\,;r) \Longrightarrow \|x-a_i\| < r , \forall\, 1 \leq i \leq n$.

But $r' < \|x - a_i\| < r$ , $\forall\, 1 \le i \le n$.

Therefore, $\|x - y\| < r' < r$ and $y \in S - \{x\}$.

$\Rightarrow \|x - y\| < r$ and $y \in S - \{x\}$.

$\Rightarrow y \in B(x\,; r)$ and $y \in S - \{x\}$.

$\Rightarrow y \in B(x\,; r) \cap S - \{x\}$.

$\Rightarrow y \in \{a_1 , a_2 , \dots , a_n\}$.

So, $\exists\; 1 \le i \le n \;\ni y = a_i$ and this contradicts the fact; $a_i \notin B(x\,; r')$, for all $1 \le i \le n$. Therefore, $B(x\,; r') \cap S - \{x\} = \emptyset \Longrightarrow x$ not an accumulation point of $S$ (a contradiction). Thus, every open ball $B(x\,; r)$ contains infinitely many points of $S$.

***Remark:***

The converse of the above theorem is not true in general. That is, if $S \subseteq \mathbb{R}^n$ is an infinite set of points, then $S$ need not has an accumulation point. For example, the set of integers $\mathbb{Z}$ is an infinite subset of $\mathbb{R}$, but it has no accumulation points, i.e. $\mathbb{Z}' = \emptyset$.

***Exercise:***

Prove that every finite set $S$ of $\mathbb{R}^n$ has no accumulation point.

***Cantor Intersection Theorem:***

Let $\{Q_1 , Q_2 , \dots , Q_n , \dots\}$ be a countable collection of non-empty sets in the Euclidean space $\mathbb{R}^n$ such that:

1. $Q_{k+1} \subseteq Q_k , \forall\, k = 1 ,2 , \dots$.
2. $Q_k$ is closed, $\forall\, k = 1 ,2 , \dots$. and;
3. $Q_1$ is bounded .

Then the intersection $\cap_{k=1}^{\infty} Q_k$ is closed and non-empty .

***Proof :*** Let $S = \cap_{k=1}^{\infty} Q_k$ . Since $Q_k$ is closed set in $\mathbb{R}^n$, $\forall\; k = 1 , 2, \dots$, hence $S$ is closed set in $\mathbb{R}^n$ (by applying a previous result that state: the intersection of any collection of closed sets is a closed set). We need only to show that, $S \ne \emptyset$.

**i.** If $Q_k$ is a finite set for some $k = 1, 2, \dots$, with $|Q_k| = n$, then from 1 above we have;

$$\dots \subseteq Q_{k+\ell+2} = \emptyset \subseteq Q_{k+\ell+1} = \emptyset \subseteq Q_{k+\ell} \subseteq \cdots \subseteq Q_{k+1} \subseteq Q_k \subseteq \cdots \subseteq Q_1,$$

for some $1 \le \ell \le n$. But, our assumption states $Q_k \ne \emptyset$, $\forall\, k = 1, 2, \dots$. That is the collection $\{Q_1, Q_2, \dots, Q_k, \dots\} = \{Q_1, Q_2, \dots, Q_{k+\ell}\}$ is finite and hence $S = \cap_{k=1}^{\infty} Q_k = Q_{k+\ell} \ne \emptyset$.

**ii.** Assume that each of $Q_k$ contains infinitely many points, $\forall\ k = 1, 2, \dots$. Let $A = \{x_1, x_2, \dots, x_k, \dots\}$, where $x_k \in Q_k$, , $\forall\ k = 1, 2, \dots$. Since $Q_k \subseteq Q_1$, $\forall\ k = 1, 2, \dots$, hence $A \subseteq Q_1$. But $Q_1$ is bounded and infinite in $\mathbb{R}^n$, so as an application of Bolzano-Weierstrass theorem, there exists an accumulation point say $x$ of $A$ in $\mathbb{R}^n$. We will show that, $x \in S$, i.e. $S \ne \emptyset$.

Since $x \in \mathbb{R}^n$ is an accumulation point of $A$, then;

$$\forall\, r > 0\ , B(x; r) \cap A - \{x\} \ne \emptyset$$

But $Q_k$ ( $\forall\ k = 1, 2, \dots$ ) contains all (except (possibly) a finite number) of the points of $A \Rightarrow B(x; r) \cap Q_k - \{x\} \ne \emptyset$, $\forall\ k = 1, 2, \dots$

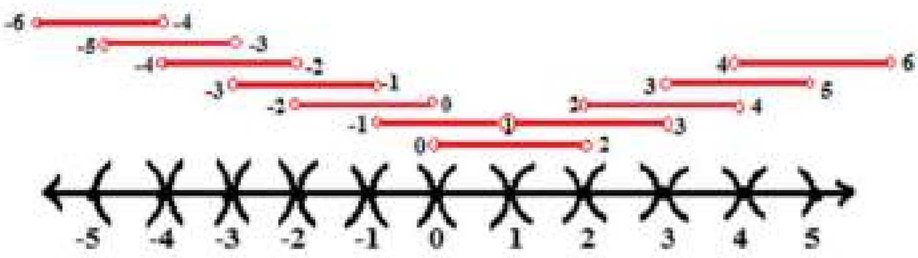
$$\Rightarrow x \in Q_k', \forall\ k = 1, 2, \dots$$

But $Q_k$ is closed in $\mathbb{R}^n$and $Q_k' \subseteq Q_k$, hence $x \in Q_k$, $\forall\ k = 1, 2, \dots$. Therefore, $x \in S = \cap_{k=1}^{\infty} Q_k \ne \emptyset$.

***Definition (covering):***

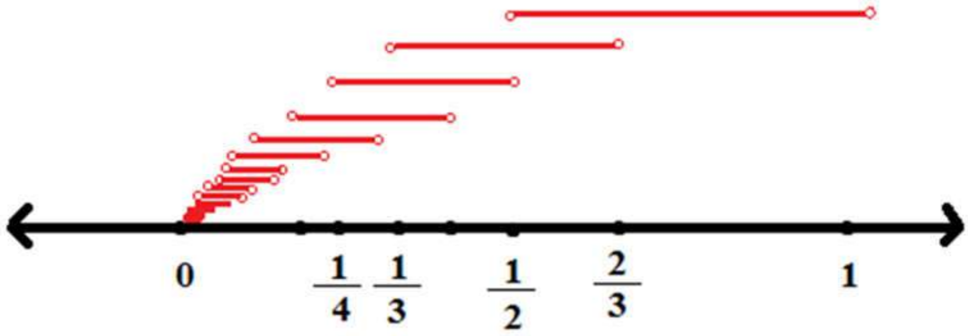
Let $(M, d)$ be a metric space and let $S \subseteq M$. A collection $\Omega = \{G_i |\ i \in I\}$ of a sets in $M$ is called a ***covering*** of $S$ if $S \subseteq \cup_{i \in I} G_i$. If $G_i$ is an open set in $M$ for all $i \in I$, then the collection $\Omega$ is called an ***open covering*** of $S$. If a finite subcollection of $\Omega$ is also a covering of $S$, then this finite subcollection of $\Omega$ is called a finite subcovering of $S$.

***Example 1:*** In the Euclidean space $\mathbb{R}$ , the collection $\Omega = \{(n, n+2) : n \in Z\}$ is a countable open covering of $\mathbb{R}$, as shown in the following figure:



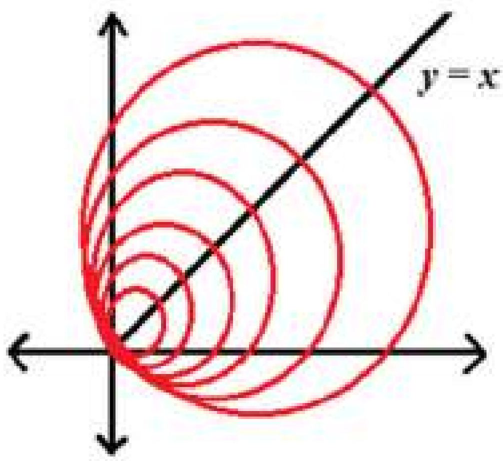

***Example 2:***

In the Euclidean space $\mathbb{R}$, the collection $\Omega = \{\left(\frac{1}{n}, \frac{2}{n}\right) : n = 2, 3, \dots\}$ is a countable open covering of the open interval (0.1), as shown in the following figure:




***Example 3:***

In the Euclidean space $\mathbb{R}^2$, the collection $\Omega = \{B((x, x)\,; x) | x > 0\}$ is an open covering of the set $S = \{(x\,, y) | x > 0\,,\ y > 0\}$. Note that, The collection $\Omega$ is not countable. In $\Omega = \{B(x\,; x) | x > 0 \text{ and } x \in \mathbb{Q}\}$, then $\Omega$ is a countable covering of $S$.



***Exercise:***

Let $\Psi = \{B_1 , B_2 , \dots \}$ denotes the countable collection of all n-balls having rational radii and centers at points with rational coordinates. Assume $x \in \mathbb{R}^n$ and $S$ be an open set in $\mathbb{R}^n$ such that $x \in S$. Prove that, there exists $B_k \in \Psi$ such that $x \in B_k \subseteq S$.

***Theorem (Lindelöf covering theorem):***

Let $A$ be a subset of the Euclidean space $\mathbb{R}^n$ and let $\Omega$ be an open covering of $A$. Then there is a countable subcollection of $\Omega$ which also covers $A$.

***Proof :***

Let $\Psi = \{B_1 , B_2 , \dots \}$ be the countable collection of all $n$-balls having centers with rational coordinates and rational radii. Since $\Omega$ is an open covering of $A \Longrightarrow A \subseteq \cup_{S \in \Omega} S \Longrightarrow \forall x \in A , \exists S_x \in \Omega \ni x \in S_x$. Since $S_x$ is an open set in $\mathbb{R}^n$ and $x \in S_x$, so by applying the above exercise we have;

$$\exists B_k \in \Psi \ni x \in B_k \subseteq S_x .$$

There are, of course infinitely many such $B_k$in $\Psi$ such that $x \in B_k \subseteq S_x$. So, we will choose only one of these open $n$-balls, for example the one of smallest index, say $m(x) = Min\{k : x \in B_k \subseteq S_x\} \Longrightarrow x \in B_{m(x)} \subseteq S_x$ ... (1)

From above we deduce the following, $\forall x \in A , \exists B_{m(x)} \in \Psi \ni x \in B_{m(x)}$.

$$\Rightarrow A \subseteq \cup_{x \in A} B_{m(x)} \quad \dots (2)$$

Therefore, $\{B_{m(x)} | x \in A\}$ is a countable subcollection of $\Psi$ which also covers $A$. From (1) and (2) above, we have;

$$A \subseteq \cup_{x \in A} B_{m(x)} \subseteq \cup_{x \in A} S_x .$$

Thus, $\{S_x \mid x \in A\}$ form a subcollection of $\Omega$ and an open covering of $A$. Since , $\forall x \in A , \exists S_x \in \Omega$ (and hence $\exists B_{m(x)} \in \Psi$ corresponding to the open set $S_x$ ) such that $x \in B_{m(x)} \subseteq S_x$. That is, there is 1-1 correspondence between $\{B_{m(x)} | x \in A\}$ and $\{S_x \mid x \in A\}$. Therefore, as $\{B_{m(x)} | x \in A\}$ is a countable covering of $A$, we deduce that $\{S_x \mid x \in A\}$ form a countable subcollection of $\Omega$ which also covers $A$.

***Remark:***

The Lindelöf covering theorem states that, from any open covering of a set $A$ in $\mathbb{R}^n$ we can extract a countable subcovering of $A$. The Hine-Borel theorem tells us that if, in addition, we know that $A$ is closed and bounded, we can reduce the countable subcovering of $A$ to a finite subcovering of $A$.

***Theorem (Hiene-Borel covering theorem ):***

Let $A$ be a closed and bounded set in the Euclidean space $\mathbb{R}^n$. If $\Omega$ is an open covering of $A$, then there is a finite subcollection of $\Omega$ which also covers $A$.

***Proof:***

Since $F$ is an open covering of $A$, hence by Lindelöf covering theorem, there exists a countable subcollection of $\Omega$ , say $\Psi = \{I_1 , I_2 , \dots\}$ also covers $A$, i.e. $A \subseteq \bigcup_{k\geq 1} I_k$. We shall show that $\exists m \geq 1 \ni A \subseteq \bigcup_{k=1}^{m} I_k$.

Now, consider for $m \geq 1$ the union $S_m = \bigcup_{k=1}^{m} I_k$. Clearly, $S_m$ is an open set of $\mathbb{R}^n$ since it is a union of open sets $I_1, I_2, \dots, I_m$, $\forall m \geq 1$. Therefore, $S_m^c = \mathbb{R}^n - S_m$ is closed $\forall\, m \geq 1$. Define a countable collection of sets $\{Q_1 , Q_2 , \dots\}$ as follows:

$$Q_1 = A \text{ and } Q_m = A \cap S_m^c,\ \forall\, m \geq 1.$$

We will show that $Q_m = \emptyset$ for some $m \geq 1$, which implies that, $A \cap S_m^c = \emptyset$, for some $m \geq 1$. This will give as $A \subseteq (S_m^c)^c = S_m = \bigcup_{k=1}^{m} I_k$ for some $m \geq 1$, i.e. $A \subseteq \bigcup_{k=1}^{m} I_k$ for some $m$ , and hence $\{I_1 , I_2 , \dots, I_m\}$ is a finite subcover of $A$ of $\Omega$, so, our aim is hold.

To do this, by contrary suppose that, $Q_m \neq \emptyset,\ \forall\, m \geq 1$. Observe that, the sets $Q_m,\ \forall\, m \geq 1$ have the following properties:

**i.** $Q_1 = A$ is closed and $Q_m$, is closed set (since $Q_m$ is the intersection of closed sets $A$ and $S_m^c$), $\forall\, m \geq 1$.

**ii.** $Q_m \supseteq Q_{m+1}$ $\forall\, m \geq 1$.(In fact: $S_m \subseteq S_{m+1}$ $m \geq 1 \Rightarrow S_m^c \supseteq S_{m+1}^c$ $\forall\, m > 1$ $\Rightarrow Q_m \supseteq Q_{m+1} \forall\, m > 1$. But $Q_m = A \cap S_m^c, \forall\, m > 1$.Therefore, $Q_m \subseteq Q_1$, $\forall\, m \geq 1$. Hence $Q_m \supseteq Q_{m+1}$, $m \geq 1$).

**iii.** $Q_1 = A$ is bounded.

From Cantor intersection theorem, we have $\bigcap_{m=1}^{\infty} Q_m \neq \emptyset$, i.e. $\exists x \in Q_1 \cap Q_2 \cap Q_3 \cap \dots \neq \emptyset$. But $A = Q_1$, thus$\exists x \in A \cap Q_2 \cap Q_3 \cap \dots \neq \emptyset$.

$\Longrightarrow \exists\ x \in A\ \ni x \in Q_m, \forall\ m \geq 1$, where $Q_m = A \cap S_m^c, \forall\ m \geq 1$.

$\Longrightarrow\ \exists\ x \in A\ \ni x \notin S_m = \bigcup_{k=1}^{m} I_k\ , \forall\ m \geq 1$.

$\Longrightarrow\ \exists\ x \in A\ \ni x \notin I_k, \forall\ k \geq 1 \Longrightarrow A \nsubseteq \bigcup_{k=1}^{m} I_k$, this is a contradiction. Hence, $Q_m = \emptyset$ for some $m \Longrightarrow A \subseteq S_m = \bigcup_{k=1}^{m} I_k$ for some $m \Longrightarrow \{I_1\ , I_2\ , \dots , I_m\}$ forms a finite open subcovering of $A$ contained of $\Omega$.

## *Compactness in metric spaces*

***Definition:***

Let $(M\ , d)$ be a metric space. A subset $S$ of $M$ is called ***compact*** if every open covering of $S$ contains a finite subcovering.

***Theorem:***

Let $S$ be a compact subset of a metric space $(M\ , d)$. Then:

1. $S$ is closed and bounded .
2. Every infinite subset of $S$ has an accumulation point in $S$.

***Proof (1):***

***Proof S bounded in M:***

Choose a point $p$ in $S$. The collection $\{B_M\,(p; k)|\ k = 1, 2\ ,3\ , \dots\}$ forms an open covering of $S$, i.e. $S = \bigcup_{m=1}^{\infty} B_M\,(p; k)$. But $S$ is compact, therefore there exists a finite subcovering of $S$, i.e. $S \subseteq \bigcup_{k=1}^{n} B_M\,(p; k)$. Since $\bigcup_{k=1}^{n} B_M\,(p\,; k) = B_M\,(p\,; n)$, hence $S \subseteq B_M\,(p\,; n)$ and $S$ is bounded in $M$.

***Proof S is closed set in M:***

We know that $S$ is closed in $M$ if and only if $S' \subseteq S$, i.e. if $S$ contains all its accumulation points. Consequently, $S$ is not closed in $M$ if, and only if, there exists an accumulation points of $S$ which is not belong to $S$, i.e. $\exists\ y \in S' \ni y \notin S$. We want to prove $S$ closed in $M$, so by contrary suppose $S$ is not closed in $M$, i.e. suppose that $\exists$ an accumulation point $y$ of $S$ such that $y \notin S$.

Now, for every $x \in S$, let $r_x = \frac{1}{2}d(x, y)$, where $r_x > 0 \ \forall\ x \in S$, since $y \notin S$. The collection $\{B_M(x; r_x) |\ x \in S\}$ forms an open covering of $S$, i.e. $S \subseteq \cup_{x\in S} B_M(x; r_x)$. But $S$ is compact $\Rightarrow$ $\exists$ a finite subcover say;

$$B_M(x_1; r_1), B_M(x_2; r_2), \dots, B_M(x_n; r_n), \text{ i.e. } S \subseteq \cup_{k=1}^{n} B_M(x_k; r_k).$$

Let $r = Min\{r_2, r_2, \dots, r_n\}$. We will show that, $B_M(y; r) \cap S - \{y\} = \emptyset$, i.e. $B_M(y; r) \cap S = \emptyset$ (since by our assumption $y \notin S$) and this will contradict the fact that $y$ is an accumulation point of $S$. To do this we need to show that $B_M(y; r) \cap B_M\left(x_k; r_{x_k}\right) = \emptyset$ for $k = 1, 2, 3, \dots, n$.

let $z \in B_M(y; r)$, we will show that $z \notin B_M\left(x_k; r_{x_k}\right)$ for all $k = 1, 2, 3, \dots, n$, i.e. $d(z, x_k) \geq r_{x_k}$. The triangle inequality gives as;

$$d(y, x_k) \leq d(y, z) + d(z, x_k)$$

$$\Rightarrow d(z, x_k) \geq d(y, x_k) - d(y, z) = 2r_{x_k} - d(y, z) > 2r_{x_k} - r$$

$$\geq 2r_{x_k} - r_{x_k} = r_{x_k}.$$

$$\Rightarrow d(z, x_k) > r_{x_k} \Rightarrow z \notin B_M\left(x_k; r_{x_k}\right)$$

$$\Rightarrow z \notin \cup_{k=1}^{n} B_M\left(x_k; r_{x_k}\right) \Rightarrow B_M(y; r) \cap \left(\cup_{k=1}^{n} B_M\left(x_k; r_{x_k}\right)\right) = \emptyset$$

But $S \subseteq \cup_{k=1}^{n} B_M\left(x_k; r_{x_k}\right) \ \Rightarrow\ B_M(y; r) \cap S = \emptyset \ \Rightarrow\ B_M(y; r) \cap S - \{y\} = \emptyset$. Therefore, $y$ is not accumulation point of $S$ (contradiction), Hence $S$ is closed in .

***Proof (2):***

Let $T$ be an infinite subset of $S$. Want to show that: $\exists\ x \in S$ such that $x$ is an accumulation point of $T$. By contrary suppose that $x$ is not accumulation point of $T$ for all $x \in S \Rightarrow \forall\ x \in S\ \exists$ an open ball $B_M(x; r_x)$ such that;

$$B_M(x; r_x) \cap T - \{x\} = \emptyset$$

$$\Rightarrow B_M(x; r_x) \cap T = \emptyset \text{ (if } x \notin T) \text{ or } B_M(x; r_x) \cap T = \{x\} \text{ (if } x \in T)$$

$$\Rightarrow B_M(x; r_x) \text{ contains at most one point of } T\ \forall\ x \in S.$$

The collection $\{B_M(x; r_x)\ |x \in S\}$ forms an open covering of $S$ since $S \subseteq \bigcup_{x\in S} B_M(x; r_x)$. But $S$ is compact, then $\exists$ a finite subcovering say

$B_M(x_1;r_1), B_M(x_2;r_2), \ldots, B_M(x_n;r_n)$, i.e.$S \subseteq \cup_{k=1}^{n} B_M(x_k;r_k)$. Since $T \subseteq S \Rightarrow T \subseteq \cup_{k=1}^{n} B_M(x_k;r_k)$ ... $(*)$. But $B_M(x_k;r_k)\ \forall\ (k = 1, 2, \ldots, n)$ contains at most one point of $T$, therefore (from $(*)$ ) $T$ is finite set (contradiction). Hence, $\exists\ x \in S$ such that $x$ is an accumulation point of $T$.

***Remark:***

**i.** In the Euclidean space $\mathbb{R}^n$, each of properties (1) and (2) is equivalent to compactness, i.e. In the Euclidean space $\mathbb{R}^n$, the following three statements are equivalent: $S$ is compact in $\mathbb{R}^n \Leftrightarrow S$ is closed and bounded in $\mathbb{R}^n \Leftrightarrow$ every finite subset of $S$ has an accumulation point in $S$.

**ii.** In general, in any metric space $(M, d)$, we have

**a.** $S$ is compact in $M \Rightarrow S$ is closed and bounded in M.

**b.** $S$ is closed and bounded in $M \not\Rightarrow S$ is compact in $M$.

**c.** $S$ is compact in $M \Leftrightarrow$ every infinite subset of $S$ has an accumulation point in $S$ .

***Exercise:***

Consider the metric space $\mathbb{Q}$ (of rational numbers) of the Euclidean space $(\mathbb{R}, |.|)$ and let $S$ consists of the rational numbers in the open interval $(a, b)$, where $a$ and $b$ are irrational. Show that $S = (a, b) \cap \mathbb{Q}$ is closed and bounded in $\mathbb{Q}$ , but $S$ is not compact in $\mathbb{Q}$ .

***Theorem:***

Let $S$ be a closed subset of a compact metric space $M$. Then $S$ is compact in $M$.

***Proof :***

Let $\Omega = \{G_i \mid i \in I\}$ be an open covering of $S$, i.e. $S \subseteq \cup_{i \in I} G_i$. We show that a finite subcollection of $\Omega$ is also cover $S$. Since $S$ is closed in $M \Rightarrow S^c$ is open in $M \Rightarrow \Omega \cup \{S^c\}$ forms an open covering of $M$. But $M$ is compact, therefore $\exists$ a finite subcovering say $\{G_{i_1}, G_{i_2}, \ldots, G_{i_n}, S^c\}$ , i.e. $M = (\cup_{k=1}^{n} G_{i_k}) \cup S^c$. But

$\subseteq M \quad \Rightarrow S \subseteq (\cup_{k=1}^{n} G_{i_k}) \cup S^c$. But $S \cap S^c = \emptyset \Rightarrow S \subseteq (\cup_{k=1}^{n} G_{i_k}) \Rightarrow$ $\{G_{i_1}, G_{i_2}, \dots, G_{i_n}\}$ is a finite subcovering of $S \Rightarrow S$ is compact.

***Theorem:***

Let $(S, d)$ be a metric subspace of a metric space $(M, d)$ and let $X \subseteq S$. Then $X$ is compact in $S$ if and only if, $X$ is compact in $M$.

***Proof:***

Suppose $X$ is compact in $S$. Wanted: $X$ is compact in $M$, i.e. wanted: every open covering of $X$ in $M$ contains a finite subcovering. So, assume $\Omega = \{G_i | i \in I\}$ be an open covering of $X$ in $M$, i.e. $X \subseteq \cup_{i\in I} G_i$ and $G_i$ is an open set in $M$, $\forall i \in I$. Since, $X = X \cap S \subseteq (\cup_{i\in I} G_i) \cap S = \cup_{i\in I}(G_i \cap S)$, hence the collection $\Omega' = \{H_i = G_i \cap S | i \in I\}$ of open sets in $S$ forms an open covering of $X$ in $S$. But $X$ is compact in $S$, so $\Omega'$ contains a finite subcovering say $\{H_{i_1}, H_{i_2}, \dots, H_{i_n}\}$. That is, $X \subseteq \cup_{k=1}^{n} H_{i_k} = \cup_{k=1}^{n} (G_{i_k} \cap S) = (\cup_{k=1}^{n} G_{i_k}) \cap S$. Therefore, $X \subseteq \cup_{k=1}^{n} G_{i_k} \Rightarrow \{G_{i_1}, G_{i_2}, \dots, G_{i_n}\}$ is a finite subcovering of $\Omega$. Thus, $X$ is compact in $M$.

Conversely, assume $X$ is compact in $M$. Wanted: $X$ is compact in $S$, i.e. wanted: every open covering of $X$ in $S$ contains a finite subcovering. Let $\Omega' = \{H_i | i \in I\}$ be an open covering of $X$ in $S$, i.e. $X \subseteq \cup_{i\in I} H_i$ and $H_i$ is an open set in $S$, $\forall i \in I$. That is, for every $i \in I$, there exists an open set $G_i$in $M$ such that $H_i = G_i \cap S$. According to, $X \subseteq \cup_{i\in I} H_i = \cup_{i\in I}(G_i \cap S) = (\cup_{i\in I} G_i) \cap S$, we have $X \subseteq \cup_{i\in I} G_i$. That is, $\Omega = \{G_i | i \in I\}$ forms an open covering of $X$ in $M$. But $X$ is compact in $M$, so $\Omega$ contains a finite subcovering say $\{G_{i_1}, G_{i_2}, \dots, G_{i_n}\}$, i.e.;

$$X \subseteq \cup_{k=1}^{n} G_{i_k} \Rightarrow X = X \cap S \subseteq \left(\cup_{k=1}^{n} G_{i_k}\right) \cap S = \cup_{k=1}^{n} (G_{i_k} \cap S) = \cup_{k=1}^{n} H_{i_k}.$$

Therefore, $X \subseteq \cup_{k=1}^{n} H_{i_k} \Rightarrow \Omega'$ contains a finite subcovering $\{H_{i_1}, H_{i_2}, \dots, H_{i_n}\}$. Thus, $X$ is compact in $S$.

***Example:***

Let $((0,1),|\ |)$ be a subspace of the Euclidean space $(\mathbb{R},|\ |)$. The interval $(0,\frac{1}{2}]$ is closed and bounded subset of $(0,1)$ as a subspace of $\mathbb{R}$. On the other hand, $(0,\frac{1}{2}]$ is bounded, but not closed in $\mathbb{R}$, so it is not compact in $\mathbb{R}$ as an application of Hiene-Borel covering theorem and according to the above theorem $(0,\frac{1}{2}]$ is not compact in $(0,1)$ . This example is an illustration to the fact that, the closed and bounded subset of a metric space need not to be compact.

## *Sequences in metric spaces*

***Definition:***

Let $(M\ ,d)$ be a metric space and let $\mathbb{Z}^+ = \{1\ ,2\ ,3\ ,\dots\}$ be the set of positive integer numbers. Any mapping $f\colon \mathbb{Z}^+ \longrightarrow M$ is called a sequence in $M$.

***Remarks:***

**i.** A sequence in $M$ assigns to each $n \in \mathbb{Z}^+$ a uniquely determined point $x_n \in M$, i.e.;

$$1 \to f(1) = x_1 \in M$$
$$2 \to f(2) = x_2 \in M$$
$$\vdots$$
$$n \to f(n) = x_n \in M$$

The points $x_1\ ,x_2\ ,\dots,x_3\ ,\dots$ are called the terms (elements) of the sequence $f$ in $M$. The term $f(n) = x_n$ is called the $n_{th}$-term of $f$.

**ii.** We will denote the sequence $f\colon \mathbb{Z}^+ \longrightarrow M$ by any one of the following notations:

$$\langle x_n \rangle_{n\in\mathbb{Z}^+} = \langle x_1\ ,x_2\ \ ,\dots \rangle = \langle x_n |\ n \in \mathbb{Z}^+ \rangle = \langle x_n \rangle$$

**iii.** We have to distinguished between the sequence $\langle x_n \rangle = \langle x_n |\ n \in \mathbb{Z}^+ \rangle$ and its range, which is denoted by to be the set $= \{x_n | n \in \mathbb{Z}^+\} = \{x_1\ ,x_2\ \ ,\dots\}$ .

***Example:***

In the Euclidean space $\mathbb{R}$;

**i.** Consider the sequence $\langle x_n \rangle = \langle (-1)^n |\ n \in \mathbb{Z}^+ \rangle = \langle -1\ ,1\ ,-1\ ,1,\dots \rangle$. The range of the above sequence is $T = \{x_n | n \in \mathbb{Z}^+\} = \{-1\ ,1\}$.

**ii.** If $b \in \mathbb{R}$, the sequence $\langle x_n \rangle = \langle b\ ,b,\dots \rangle$, all of whose terms are equal to $b$, is called the constant sequence. The range of the above sequence is $T = \{b\}$.

***Example:***

In the Euclidean space $\mathbb{R}$, if $\langle x_n\rangle$ and $\langle y_n\rangle$ are sequences of real numbers then we can define:

a. **Sum:** $\langle x_n\rangle + \langle y_n\rangle = \langle x_n + y_n\rangle$
b. **Difference:** $\langle x_n\rangle - \langle y_n\rangle = \langle x_n - y_n\rangle$
c. **Multiplication:** $\langle x_n\rangle . \langle y_n\rangle = \langle x_n . y_n\rangle$
d. **Multiplication by a scalar:** if $c \in \mathbb{R}$ , $c\langle x_n\rangle = \langle cx_n\rangle$
e. **Quotient:** $\langle x_n\rangle / \langle y_n\rangle = \langle x_n / y_n\rangle$ provided that $y_n \neq 0$ for all $n \in \mathbb{Z}^+$.

For example, if $\langle x_n\rangle = \langle 2n\rangle = \langle 2 , 4 , 6, \dots\rangle$ and $\langle y_n\rangle = \langle\frac{1}{n}\rangle = \langle 1 , \frac{1}{2} , \frac{1}{3} , \dots\rangle$ be two sequences of real numbers, Then;

1. $\langle 2n\rangle + \langle\frac{1}{n}\rangle = \langle 2n + \frac{1}{n}\rangle = \langle\frac{2n^2+1}{n}\rangle = \langle 3, \frac{9}{2} , \frac{19}{3}, \dots\rangle$.
2. $\langle 2n\rangle - \langle\frac{1}{n}\rangle = \langle 2n - \frac{1}{n}\rangle = \langle 2n - \frac{1}{n}\rangle = \langle 3, \frac{7}{2} , \frac{17}{3} , \dots\rangle$.
3. $\langle 2n\rangle . \langle\frac{1}{n}\rangle = \langle 2n.\frac{1}{n}\rangle = \langle 2\rangle = \langle 2, 2 , 2 , \dots\rangle$.
4. $3\langle 2n\rangle = \langle 6n\rangle = \langle 6 , 12 , 18 , \dots\rangle$.
5. $\langle 2n\rangle / \langle\frac{1}{n}\rangle = \langle 2n / \frac{1}{n}\rangle = \langle\frac{2n}{1/n}\rangle = \langle 2n^2\rangle = \langle 2, 8 , 18 , \dots\rangle$.

Note that, if $\langle 1 + (-1)^n\rangle = \langle 0 , 2 , 0 , 2, \dots\rangle$, is a sequence of real numbers, therefore, $\langle 2n\rangle / \langle 1 + (-1)^n\rangle$ is not defined since some of the terms of the sequence $(1 + (-1)^n)$ are equal to 0.

***Definition:***

In the Euclidean space $\mathbb{R}$, a sequence $\langle x_n\rangle$ is called bounded above if $\exists\, M > 0$ such that $|x_n| \leq M,\ \forall\, n \in \mathbb{Z}^+$, while it is called bounded below if $\exists\, N > 0$ such that $N \leq |x_n|,\ \forall n \in \mathbb{Z}^+$.

***Example:***

The sequence of real numbers $\langle\frac{1}{n}\rangle = \langle 1 , \frac{1}{2} , \frac{1}{3} , \dots\rangle$ is bounded above since $\exists$a positive real number 2 such that $\left|\frac{1}{n}\right| \leq 2, \forall n \in \mathbb{Z}^+$. As well as, $\langle\frac{1}{n}\rangle$ is bounded below since $\exists$a positive real number 0 such that $0 \leq \left|\frac{1}{n}\right|,\ \forall n \in \mathbb{Z}^+$.

***Definition:***

In the Euclidean space $\mathbb{R}$ , a sequence $\langle x_n \rangle$ is called increasing if;

$$x_n \leq x_{n+1} \ \forall \, n \in \mathbb{Z}^+;$$

while it is called decreasing if, $x_n \geq x_{n+1} \ \forall \, n \in \mathbb{Z}^+$.

***Example :***

In the Euclidean space $\mathbb{R}$, a sequence $\langle \frac{1}{n} \rangle$ is decreasing since;

$$x_{n+1} = \frac{1}{n+1} < \frac{1}{n} = x_n, \ \forall \, n \in \mathbb{Z}^+.$$

The sequence $\langle n \rangle = \langle 1,2,3,\ldots \rangle$ is increasing since;

$$x_n = n < n+1 = x_{n+1}, \ \forall \, n \in \mathbb{Z}^+.$$

The sequence $\langle (-1)^n | n \in \mathbb{Z}^+ \rangle = \langle -1 \,, 1 \,, -1 \,, 1, \ldots \rangle$ is neither increasing nor decreasing.

***Definition (Convergent sequence in a metric space):***

A sequence $\langle x_n \rangle$ of points in a metric space $(M \,, d)$ is said to be converge if $\exists$ a point $p \in M$ with the following property:

$$\forall \, \epsilon > 0 \,, \ \exists \, N \in \mathbb{Z}^+ \ni \ d(x_n \,, p) < \epsilon, \forall n \geq N \ldots (*)$$

In this case, we say that $\langle x_n \rangle$ is converges to $p$ in $M$ and we write;

$$x_n \to p \ \text{ as } n \to \infty \text{ or } x_n \xrightarrow[n \to \infty]{} p.$$

If there is no such $p$ in $M$, the sequence $\langle x_n \rangle$ is said to be diverge.

***Remark:***

1. The above definition of convergence implies that;

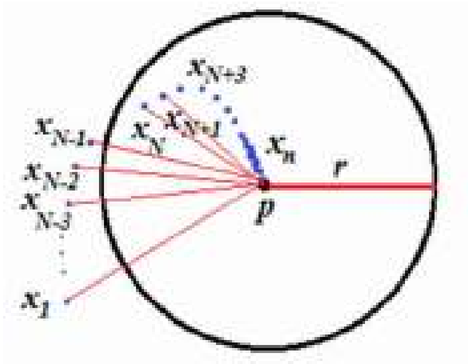
$$x_n \to p \text{ as } n \to \infty \iff d(x_n \,, p) \longrightarrow 0 \text{ as } n \to \infty.$$

   i.e. a sequence $\langle x_n \rangle$ converges to $p$ in $M$ if, and only if, the sequence $\langle d(x_n \,, p) \rangle$ of positive real numbers converges to 0 in $\mathbb{R}$.

2. The convergence condition $(*)$ can be written as;

$$\forall \, \epsilon > 0 \,, \ \exists \, N \in \mathbb{Z}^+ \ni \ x_n \in B(p \,;\, \epsilon), \forall n \geq N.$$

   i.e. the open ball $B(p \,;\, \epsilon)$ contains all the terms of the sequence $\langle x_n \rangle$ except a finite number of terms $x_1$, $x_2$,… and $x_{N-1}$ as shown in the following figure:




**3.** The greatest integer of $x$ denoted by $[x]$ is defined as follows:

$$[x] = \begin{cases} x & if\ x \in \mathbb{Z} \\ \text{the nearest integer no. to } x \text{ from the left} & if\ \ x \notin \mathbb{Z} \end{cases};$$

In fact, $[0] = 0,\ [0.79] = 0,\ [1] = 1\ ,\ [1.9] = 1$. In general, $[x] \le x\ ,\ \ \forall\ x \in \mathbb{R}$, also $[x] + 1 > x\ \forall\ \ x \in \mathbb{R}$

***Example :***

In the Euclidean metric space $\mathbb{R}$ , the sequence $\langle \frac{1}{n} \rangle = \langle 1, \frac{1}{2}, \dots \rangle$ converges to $0 \in \mathbb{R}$ .

***Solution:*** Let $\epsilon > 0$. Wanted: $\exists\ N \in Z^+ \ni\ n \in N\ \ \Rightarrow\ \left|\frac{1}{n} - 0\right| < \epsilon$.

For a moment assume that;

$$\left|\frac{1}{n} - 0\right| < \epsilon\ \Longrightarrow \left|\frac{1}{n}\right| < \epsilon\ \Longrightarrow \frac{1}{n} < \epsilon \Longrightarrow \frac{1}{\epsilon} < n\ \Longrightarrow n > \frac{1}{\epsilon}.$$

So, if we choose $N = \left[\frac{1}{\epsilon}\right] + 1 \in \mathbb{Z}^+$, then $\forall n \ge N\ \ \Rightarrow\ \ n \ge \left[\frac{1}{\epsilon}\right] + 1 \Rightarrow n > \frac{1}{\epsilon}$

$$\Rightarrow \frac{1}{n} < \epsilon \Rightarrow \left|\frac{1}{n}\right| < \epsilon\ \ \ \Rightarrow\ \left|\frac{1}{n} - 0\right| < \epsilon.$$

Therefore, $\langle \frac{1}{n} \rangle$ converges to 0 in $\mathbb{R}$.

***Theorem:***

A sequence in a metric space $(M\ , d)$ can converge to at most one point in $M$.

***Proof:***

Assume that $x_n \to p$ as $n \to \infty$ and $y_n \to q$ as $n \to \infty$ in $M$. We will prove that $p = q$. By contrary suppose $p \ne q$ and let $\epsilon = d(p, q) > 0$. As $x_n \to p \Rightarrow$

$\exists N_1 \in \mathbb{Z}^+$ such that $d(x_n, p) < \frac{\epsilon}{2}$, $\forall n \geq N_1$. Moreover as $y_n \to q \Rightarrow \exists N_2 \in \mathbb{Z}^+$ such that $d(y_n, q) < \frac{\epsilon}{2}$, $\forall n \geq N_2$. The triangle inequality gives us;

$$\epsilon = d(p, q) \leq d(p, x_n) + d(x_n, q) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \Longrightarrow \epsilon = d(p, q) < \epsilon;$$

and this is a contradiction. Therefore $p = q$.

***Remark :***

If a sequence $\langle x_n \rangle$ is converges in a metric space $M$, the unique point to which it converges, say $p$, is called the limit point of the sequence and it is denoted by, $p = \lim_{n\to\infty} x_n$.

***Remark :***

The convergence or divergence of a sequence depends on the underlying space as well as on the metric as we illustrate in the following:

***Example 1:***

From a previous example, we know that the sequence $\langle \frac{1}{n} \rangle$ is converge in the Euclidean space $\mathbb{R}$ to 0. The same sequence is diverge in the Euclidean subspace $= (0, 1]$ , since $0 \notin S$.

***Example 2:***

The sequence $\langle \frac{1}{n} \rangle$ is converge to 0 in the Euclidean metric space $(\mathbb{R}, |\,|)$. The same sequence does not converge to 0 in the discrete metric space $(\mathbb{R}, d)$. In fact, if we suppose that $\frac{1}{n} \to 0$ as $n \to \infty \Rightarrow d\left(\frac{1}{n}, 0\right) \to 0$ as $n \to \infty$. But $\frac{1}{n} \neq 0, \forall\ n = 1, 2, 3, \dots$ and $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the discrete metric, i.e.

$$d(x, y) = \begin{cases} 0 & if\ x = y \\ 1 & if\ x \neq y \end{cases}$$

Therefore, $d\left(\frac{1}{n}, 0\right) = 1\ \forall\ n = 1, 2, 3, \dots$. Hence $d(\frac{1}{n}, 0) = 1 \nrightarrow 0\ as\ n \to \infty$, this is a contradiction . Thus $\frac{1}{n} \nrightarrow 0$ as $n \to \infty$ in the discrete space $(\mathbb{R}, d)$.

***Exercises:***

***1.*** In the Euclidean space $\mathbb{R}$, let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two sequences such that $x_n \to p$ and $y_n \to q$ as $n \to \infty$. Prove that the following :

***a.*** **Sum:** $\langle x_n \rangle + \langle y_n \rangle$ converges to $p + q$.

***b.*** **Difference:** $\langle x_n \rangle - \langle y_n \rangle$ converges to $p - q$.

***c.*** **Multiplication:** $\langle x_n \rangle . \langle y_n \rangle$ converges to$pq$.

***d.*** **Multiplication by a scalar:** if $\in \mathbb{R}$ , $c\langle x_n \rangle$ converges to $cp$.

***2.*** In the Euclidean space $\mathbb{R}$, prove that the following :

***a.*** If $0 \leq y_n \leq x_n$ for all $n \in \mathbb{Z}^+$ and if $\langle x_n \rangle$ converge to 0, then $\langle y_n \rangle$ converge to 0.

***b.*** Let $\langle x_n \rangle$ be decreasing and bounded below. If $T = \{ x_n | n \in \mathbb{Z}^+\}$ is the range of $\langle x_n \rangle$, then $\langle x_n \rangle$ is converge to $Inf\ T$ (Give an example to explain that).

***c.*** Let $\langle x_n \rangle$ be increasing and bounded above. If $T = \{ x_n \mid n \in \mathbb{Z}^+\}$ is the range of $\langle x_n \rangle$, then $\langle x_n \rangle$ is converge to $Sup\ T$ (Give an example to explain that).

***Theorem:***

In the metric space $(M, d)$, assume that $\langle x_n \rangle$ is a convergent sequence such that $x_n \to p$ and let $T = \{x_1 , x_2 , \dots\}$ be the range of $\{x_n\}$. Then:

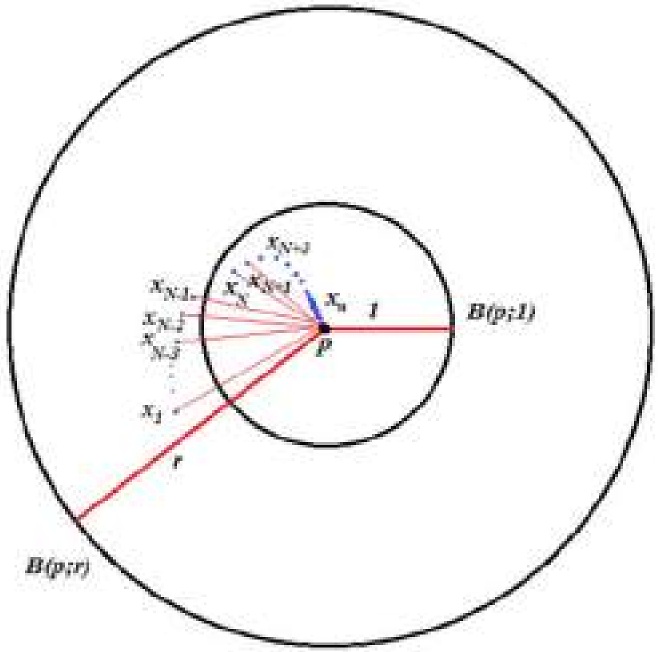
i. $T$ is bounded.

ii. $p$ is an adherent point of $T$.

***Proof (i):***

Wanted: $T$ is bounded, i.e. $\exists$ an open ball $B_M(p; r)$ such that $T \subseteq B_M(p; r)$. Let $\epsilon = 1$. Since $x_n \to p$ as $n \to \infty$, hence $\exists\ N \in \mathbb{Z}^+ \ni d(x_n , p) < 1, \forall\ n \geq N$ $\Rightarrow x_n \in B_M(p; 1)\ \forall\ n \geq N \Rightarrow\ x_n \in B_M(p; 1)\ \ \forall\ n \geq N$.

Let $r = 1 + Max\{d(p, x_1), d(p, x_2), \dots , d(p , x_{N-1})\}$.




In fact, if $\underline{n \geq N}$, $d(x_n, p) < 1 < r \Rightarrow x_n \in B_M(p;r)$ and if $\underline{n < N}$, $d(x_n, p) \leq Max\{d(p, x_1), d(p, x_2), \dots, d(p, x_{N-1})\} < r \Rightarrow x_n \in B_M(p;r)$ for all $n \geq 1 \Rightarrow T \subseteq B_M(p;r)$. Hence $T$ is bounded in $M$.

***Proof (ii):***

Wanted: $p \in \bar{T}$ (i.e. wanted: $\forall\, r > 0,\ B_M(p;r) \cap T \neq \emptyset$).

Let $r > 0$. Since $x_n \to p$ as $n \to \infty \Rightarrow \exists\, N \in \mathbb{Z}^+ \ni d(x_n, p) < r, \forall n \geq N$.

$\Rightarrow x_n \in B_M(p;r), \forall n \geq N$. But $x_n \in T\ \forall\ n \geq N \Rightarrow B_M(p;r) \cap T \neq \emptyset \Rightarrow p$ is an adherent point of $T$.

***Remark:***

**1.** If $\langle x_n \rangle$ is a convergent sequence in a metric space $M$ such that $x_n \to p$ and let $T = \{x_1, x_2, \dots\}$ be the range of $\langle x_n \rangle$, the point $p$ may not be an accumulation point of $T$. For example, in the Euclidean space $\mathbb{R}$, the sequence $\langle x_n \rangle = \langle 1,1,2,2,2, \dots \rangle$ is converge and converges to 2. The range of $\langle x_n \rangle$, $T = \{1,2\}$ is a finite subset of $\mathbb{R}$ which has no accumulation point in $\mathbb{R}$.Thus, 2 is not an accumulation point of $T$.

**2.** If $x_n \to p$ and T is infinite set, then $p$ is an accumulation point of $T$ since every open ball will contain infinitely points of T.

***Theorem:***

Given a metric space $(M, d)$ and a subset $S \subseteq M$. If a point $p \in M$ is an adherent point of , then there is a sequence $\langle x_n \rangle$ in $S$ which converge to $p$.

***Proof:***

Since $p \in M$ is an adherent point of $\Rightarrow \forall\, r > 0\;\; B_M(p; r) \cap S \neq \emptyset$.

Let $= \frac{1}{n}$, n=1 ,2 ,3 ,…⇒ $B_M(p; r) \cap S \neq \emptyset \quad \forall\, n \in \mathbb{Z}^+$. Thus, when:

$n = 1 \Rightarrow B_M(p; 1) \cap S \neq \emptyset \Rightarrow \exists\, x_1 \in B_M(p; 1) \cap S \Rightarrow x_1 \in S \text{and } d(x_1, p) < 1$

$n = 2 \Rightarrow B_M(p; 2) \cap S \neq \emptyset \Rightarrow \exists\, x_2 \in B_M(p; 2) \cap S \Rightarrow x_2 \in S \text{and } d(x_2, p) < \frac{1}{2}$

$n = 3 \Rightarrow B_M(p; 3) \cap S \neq \emptyset \Rightarrow \exists\, x_3 \in B_M(p; 3) \cap S \Rightarrow x_3 \in S \text{and } d(x_3, p) < \frac{1}{3}$

Therefore, $\forall\, n \in \mathbb{Z}^+ \;\exists$ a point $x_n \in S$ with $d(x_n, p) < \frac{1}{n}$. Thus, we have a sequence $\langle x_n \rangle$ in $S$ satisfied $d(x_n, p) \to 0$ as $n \to \infty$. Therefore, $x_n \to p$ as $n \to \infty$.

***Definition (Subsequence):***

Let $f: \mathbb{Z}^+ \to M$ be a sequence $\langle x_n \rangle$ in $M$, where $f(n) = x_n$, $\forall\, n \in \mathbb{Z}^+$ and let $k: \mathbb{Z}^+ \to \mathbb{Z}^+$ be an order preserving function, (i.e. $\forall m, n \in \mathbb{Z}^+$, if $m < n$, then $k(m) < k(n)$). Then the composition $f \circ k: \mathbb{Z}^+ \to M$ which is defined by, $f \circ k(n) = f\big(k(n)\big) = x_{k(n)}$ is called a subsequence $\langle x_{k(n)} \rangle$ of $\langle x_n \rangle$.

***Example:***

Consider the sequence $f = \langle \frac{1}{n} \rangle$ in $\mathbb{R}$ and let $k: \mathbb{Z}^+ \to \mathbb{Z}^+$be the order preserving function that defined as, $k(n) = 2^n$, $\forall n \in \mathbb{Z}^+$. Then $f \circ k = \langle \frac{1}{2^n} \rangle$ is a subsequence of $\langle \frac{1}{n} \rangle$. As well as each of the sequences $\langle \frac{1}{2n} \rangle$, $\langle \frac{1}{2n+1} \rangle$, $\langle \frac{1}{3^n} \rangle$ is a subsequence of $\langle \frac{1}{n} \rangle$. But the sequence $\langle \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \ldots \rangle$ is not a subsequence of $\langle \frac{1}{n} \rangle$.

***Exercise:*** In a metric space $(M\ d)$, prove that a sequence $\{x_n\}$ converges to $p$ if, and only if, every subsequence $\langle x_{k(n)} \rangle$ converges to $p$ .

## Cauchy sequences:

***Definition:***

A sequence $\langle x_n \rangle$ in a metric space $(M\ d)$ is called a Cauchy sequence, if it is satisfy the following condition:

$$\forall \epsilon > 0 \ , \exists\ N \in \mathbb{Z}^+ \ni \ d(x_m, x_n) < \epsilon, \forall m, n \ \geq N \ .$$

***Example:***

In the Euclidean space $\mathbb{R}$, the sequence $\langle x_n \rangle = \langle \frac{1}{n} \rangle$ is a Cauchy sequence.

***<u>Sol :</u>***

Let $\epsilon > 0$. Wanted: $\exists\ N \in \mathbb{Z}^+ \ni \ d(x_n, x_m) < \epsilon, \forall m, n \ \geq N$.

So, assume that there exists such $N$, satisfied;

$$|x_m - x_n| < \epsilon, \forall m, n \ \geq N.$$

$\Rightarrow |x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| = \left|\frac{1}{m} + (-\frac{1}{n})\right| \leq \left|\frac{1}{m}\right| + \left|-\frac{1}{n}\right| = \frac{1}{m} + \frac{1}{n}.$

$\Rightarrow |x_m - x_n| \leq \frac{1}{m} + \frac{1}{n}.$

Since, $n\ , m \ \geq N \Rightarrow \frac{1}{m} \leq \frac{1}{N}$ and $\frac{1}{n} \leq \frac{1}{N}$, hence$|x_m - x_n| \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N}$.

So, if we choose the positive integer $N = \left[\frac{2}{\epsilon}\right] + 1$, that satisfied;

$$N > \frac{2}{\epsilon} \Rightarrow \frac{1}{N} < \frac{\epsilon}{2} \Rightarrow \frac{2}{N} < \epsilon.$$

Therefore, $|x_m - x_n| \leq \frac{2}{N} < \epsilon, \forall m, n \ \geq N$ and $\langle x_n \rangle$ is a Cauchy sequence.

*Exercise:*

Let $(S, d)$ be a metric subspace of a metric space $(M, d)$. Prove that, a sequence $\langle x_n \rangle$ is a Cauchy sequence in $S$ if, and only if, $\langle x_n \rangle$ is a Cauchy sequence in $M$.

***Theorem:***

In a metric space $(M, d)$, every convergent sequence is Cauchy sequence.

***Proof:*** Let $\langle x_n \rangle$ be a convergent sequence in $M$ and $x_n \to p$ with $p \in M$. Wanted: $\langle x_n \rangle$ is a Cauchy sequence in $M$. Let $\epsilon > 0$. Wanted: $\exists\ N \in \mathbb{Z}^+ \ni$ $d(x_n, x_m) < \epsilon, \forall m, n \ \geq N$.

Since $\epsilon > 0$ and $x_n \to p \quad \Rightarrow \exists\, N \in \mathbb{Z}^+ \ni d(x_n, p) < \frac{\epsilon}{2}$, $\forall\, n \geq N$. So, if $m \geq N$, then $d(x_m, p) < \frac{\epsilon}{2}$. Now, if $n \geq N$ and $m \geq N$, by the triangle inequality we have:

$$d(x_n, x_m) \leq d(x_n, p) + d(x_m, p) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \Rightarrow \; d(x_n, x_m) < \epsilon$$

Thus, $\langle x_n \rangle$ is a Cauchy sequence in $M$.

***Example:***

The converse of the above theorem needs not to be true in general. For example, the metric subspace $(S = (0,1], |.|)$ of the Euclidean metric space $(\mathbb{R}, |.|)$. The sequence $\langle x_n \rangle = \langle \frac{1}{n} \rangle$ is a sequence of points in $S$. We know that, $\langle \frac{1}{n} \rangle$ is a Cauchy sequence in $\mathbb{R}$, and $\frac{1}{n} \to 0$. Thus, $\langle \frac{1}{n} \rangle$ is a Cauchy sequence in $S$, while it is diverge in $S$ since $0 \notin S$.

## *Complete metric space:*

***Definition:***

A metric space $(M, d)$ is called complete, if every Cauchy sequence in $M$ is converge in $M$. A subset $S$ of $M$ is called ***complete metric subspace*** of $(M, d)$, if $S$ is complete as a metric space.

***Example:***

The Euclidean space $\mathbb{R}^k$ is complete, $(k \geq 1)$.

***Proof:*** Let $\langle x_n \rangle$ be a Cauchy sequence in $\mathbb{R}^k$. Wanted, $\langle x_n \rangle$ is a convergent sequence in $\mathbb{R}^k$. Wanted: $\exists\, p \in \mathbb{R}^k \ni \; x_n \to p$.

Let $T = \{\, x_n : \; n \in \mathbb{Z}^+\}$ be the range of the sequence $\langle x_n \rangle$. There are two cases to be discussed:

**<u>The first one</u>**, if $T$ is finite, then all except a finite number of the terms of the sequence $\langle x_n \rangle$ are equal and hence $\langle x_n \rangle$ is converge to this common value. This show that $\mathbb{R}^k$ is complete in this case.

**The second one**, if $T$ is infinite. We will use the Bolzano-Weierstrass theorem to show that $T$ has an accumulation point $p \in \mathbb{R}^k$, and then we show that $x_n \to p$. To do this, we need first to show $T$ is bounded set in $\mathbb{R}^k$.

So, let $\epsilon = 1$. Since $\langle x_n \rangle$ is a Cauchy sequence in $\mathbb{R}^k$, hence;

$$\exists\, N \in \mathbb{Z}^+ \ni \ \|x_n - x_m\| < 1, \forall n, m \geq N.$$

Thus, if $n \geq N$ we have $\|x_n - x_N\| < 1$. Let;

$$r' = Max\{\|x_1\|, \|x_2\|, \dots, \|x_N\|\} \text{ and } r = 1 + r'.$$

However, if $1 \leq n \leq N$, we have $d(x_n, 0) = \|x_n\| \leq r' < r$. As well as, if $n > N$, we have $d(x_n, 0) = \|x_n\| \leq \|x_n - x_N\| + \|x_N\| < 1 + r' = r$. That is;

$$x_n \in B(0; r) \ \ \forall\, n \in \mathbb{Z}^+ \Rightarrow T \subseteq B(0; r).$$

Therefore, $T$ is bounded set in $\mathbb{R}^k$.

Now, in our second case $T$ is infinite and bounded, so from Bolzano-weierstrass theorem, $T$ has an accumulation point say, $p \in \mathbb{R}^k$. We need only to show that, $x_n \to p$.

Let $\epsilon > 0$. Wanted: $\exists\, N \in \mathbb{Z}^+ \ni \ \|x_n - p\| < \epsilon, \forall n \geq N$.

Since $\epsilon > 0$ and $\langle x_n \rangle$ is a Cauchy sequence in $\mathbb{R}^k$, hence;

$$\exists\, N \in \mathbb{Z}^+ \ni \|x_n - x_m\| < \frac{\epsilon}{2}, \forall n, m \geq N$$

Since $p$ is an accumulation point of $T$, hence $B(p; \frac{\epsilon}{2})$ contains infinitely many points of $T$ and there is at least a point $x_m$ with $m \geq N$ such that $x_m \in B(p; \frac{\epsilon}{2})$, i.e. $\|x_m - p\| < \frac{\epsilon}{2}$. By the triangle inequality, for $n \geq N$, we have;

$$\|x_n - p\| \leq \|x_n - x_m\| + \|x_m - p\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \Rightarrow \|x_n - p\| < \epsilon\,.$$

Therefore, $x_n \to p$ and $\mathbb{R}^k$ is complete.

***Example:***

For $n \geq 1$, The space $(\mathbb{R}^n, d)$ with the metric $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that defined as;

$$d(x, y) = Max\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\};$$

for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, is a complete metric space.

***Proof:*** Let $\langle x_m \rangle$ be a Cauchy sequence in $\mathbb{R}^n$ with respect to the metric $d$. Wanted: $\exists\, p \in \mathbb{R}^n \ni x_m \to p$ with respect to the metric $d$.

Let $\epsilon > 0$. Since $\langle x_m \rangle$ be a Cauchy sequence in $\mathbb{R}^n$ with respect to the metric $d \Rightarrow \exists\, N \in \mathbb{Z}^+ \ni d(x_m, x_r) < \epsilon, \forall m, r \geq N$, where $x_m = (x_m^1, x_m^2, \dots, x_m^n)$, $x_r = (x_r^1, x_r^2, \dots, x_r^n) \in \mathbb{R}^n$.

Since, for $m, r \geq N$, $d(x_m, x_r) < \epsilon$;

$\Rightarrow Max\{|x_m^1 - x_r^1|, |x_m^2 - x_r^2|, \dots, |x_m^n - x_r^n|\} < \epsilon$

$\Rightarrow |x_m^1 - x_r^1| < \epsilon$ , $|x_m^2 - x_r^2| < \epsilon, \dots, |x_m^n - x_r^n| < \epsilon$

$\Rightarrow \langle x_m^1 \rangle, \langle x_m^2 \rangle, \dots, \langle x_m^n \rangle$ are Cauchy sequences in $\mathbb{R}$ with respect to the Euclidean metric $|.|: \mathbb{R} \to \mathbb{R}$. But the Euclidean metric $(\mathbb{R}, |.|)$ is complete (see the above example). Thus, there are $p_1, p_2, \dots, p_n \in \mathbb{R}$ such that $x_m^1 \to p_1$, $x_m^2 \to p_2, \dots, x_m^n \to p_n$. Put $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$. **As an exercise,** show that

$$x_m = (x_m^1, x_m^2, \dots, x_m^n) \to (p_1, p_2, \dots, p_n) = p \Rightarrow x_m \to p \text{ in } (\mathbb{R}^n, d).$$

Hence, $(\mathbb{R}^n, d)$ is complete.

## ***Continuous functions:***

***Definition:***

Let $(S, d_s)$ and $(T, d_T)$ be metric spaces and $f: S \to T$ be a function. The function $f$ is said to be continuous at a point $p \in S$ if,

$\forall\, \epsilon > 0, \exists\, \delta > 0$ (depend on $\epsilon$ and $p$) $\ni$

$$d_S(x, p) < \delta \;\Rightarrow\; d_T\big(f(x), f(p)\big) < \epsilon.$$

Or equivalently: $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that $f(B_S(p; \delta)) \subseteq B_T(f(p); \epsilon)$.

We say that, $f$ is continuous on a set $A \subseteq S$ if, $f$ is continuous at every point of $A$.

***Remark:***

If $p$ is an isolated point of $S$, i.e. $p \notin S' \cap S$, then every function $f: S \to T$ defined at $p$ will be continuous at $p$. To explain that: let $\epsilon > 0$. Since $p \notin S' \cap S$, hence $\exists\, \delta > 0 \ni B_S(p; \delta) \cap S - \{p\} = \emptyset \Rightarrow B_S(p; \delta) \cap S = \{p\}$. Thus, $B_S(p; \delta) = \{p\}$. In fact, $f(p) \in B_T(f(p); \epsilon)$, so;

$$f\big(B_S(p;\delta)\big) = f(\{p\}) = \{f(p)\} \subseteq B_T(f(p);\epsilon).$$

Therefore, $f$ is continuous at $p$.

***Theorem:***

Let $f: S \to T$ be a function from a metric space $(S, d_S)$ to another metric space $(T, d_T)$, and assume that $p \in S$. Then $f$ is continuous at $p \in S$ if, and only if, for every sequence $\langle x_n \rangle$ in $S$ converges to $p$, the sequence $\langle f(x_n) \rangle$ in $T$ converges to $f(p)$, i.e. $\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$.

***Proof :***

Suppose that $f$ is continuous at $p \in S$ and let $\langle x_n \rangle$ be a sequence in $S$ converges to $p$. Wanted: the sequence $\langle f(x_n) \rangle$ converges to $f(p)$.

Let $\epsilon > 0$. Wanted: $\exists\, N \in \mathbb{Z}^+ \ni d_T\big(f(x_n), f(p)\big) < \epsilon, \forall n \geq N$.

Since $f: S \to T$ is continuous at $p \in S \Rightarrow \exists\, \delta > 0$ such that if $x \in S$ with,

$$d_S(x,p) < \delta \;\; \Rightarrow d_T\big(f(x), f(p)\big) < \epsilon \quad \ldots\ldots(1)$$

Since $\delta > 0$ and $x_n \to p$ in $\Rightarrow \exists\, N \in \mathbb{Z}^+ \ni d_s(x_n, p) < \delta, \forall n \geq N$. From (1) above, $d_T\big(f(x_n), f(p)\big) < \epsilon, \forall n \geq N$. Therefore, $\langle f(x_n) \rangle$ in $T$ converges to $f(p)$.

Conversely, suppose that for every sequence $\langle x_n \rangle$ in $S$ converges to $p$, the sequence $\langle f(x_n) \rangle$ in $T$ converges to $f(p)$. Wanted: $f$ is continuous at $\in S$ .

By contrary, suppose that $f$ is not continuous at $p \in S \Rightarrow \exists\; \epsilon > 0$ such that $\forall\, \delta > 0, \exists\, x \in S$ such that;

$$d_s(x,p) < \delta \;\text{ and } d_T\big(f(x), f(p)\big) \geq \epsilon.$$

Let $\delta = \frac{1}{n}$ $, n \in \mathbb{Z}^+$. So;

$$\text{if } n = 1 \Rightarrow \delta = 1, \exists\, x_1 \in S \ni d_s(x_1, p) < 1 \text{ and } d_T\big(f(x_1), f(p)\big) \geq \epsilon;$$

$$\text{if } n = 2 \Rightarrow \delta = \tfrac{1}{2}, \exists\, x_2 \in S \ni d_s(x_2, p) < \tfrac{1}{2} \text{ and } d_T\big(f(x_2), f(p)\big) \geq \epsilon;$$

$$\text{if } n \in \mathbb{Z}^+ \Rightarrow \delta = \tfrac{1}{n}, \exists\, x_n \in S \ni d_s(x_n, p) < \tfrac{1}{n} \text{ and } d_T\big(f(x_n), f(p)\big) \geq \epsilon.$$

Therefore, we will obtain a sequence $\langle x_n \rangle$ in $S$ such that;

$$d_s(x_n, p) < \frac{1}{n}, \text{ but } d_T\big(f(x_n), f(p)\big) \geq \epsilon.$$

That means, $\langle x_n \rangle$ is sequence in $S$ converges to $p \in S$, but the sequence $\langle f(x_n) \rangle$ in $T$ is not converges to $f(p)$and this is a contradiction. Thus, $f: S \to T$ is continuous at $p \in S$.

***Theorem:***

Let $(S, d_S)$, $(T, d_T)$ and $(U, d_U)$ be metric spaces. Let $f: S \to T$ and $g: T \to U$ be functions, and let $g \circ f: S \to U$ be the composite function defined on $S$ by;

$$g \circ f(x) = g\big(f(x)\big), \text{ for } x \in S.$$

If $f$ is continuous at $p \in S$ and $g$ is continuous at $f(p) \in T$, then $g \circ f$ is continuous at $p$.

***Proof:*** Let $\epsilon > 0$. Wanted: $g \circ f$ is continuous at $p \in S$, i.e. wanted, $\exists\, \delta > 0$ such that;

$$d_S(x, p) < \delta \;\; \Rightarrow d_U(g(f(x)), g(f(p))) < \epsilon$$

Since $\epsilon > 0$ and $g: T \to U$ is continuous at $f(p) \Rightarrow \exists \delta_1 > 0 \ni$

$$d_T\big(y, f(p)\big) < \delta_1 \Rightarrow d_U(g(y), g(f(p))) < \epsilon \;\; \ldots\ldots (1)$$

Since $\delta_1 > 0$ and $f: S \to T$ is continuous at $p \Rightarrow \exists \delta > 0 \ni$;

$$d_S(x, p) < \delta \;\; \Rightarrow d_T\big(f(x), f(p)\big) < \delta_1 \ldots\ldots (2)$$

Form (1) and (2) above we have;

$$d_S(x, p) < \delta \;\; \Rightarrow d_T\big(f(x), f(p)\big) < \delta_1 \Rightarrow d_U(g(f(x)), g(f(p))) < \epsilon.$$

Therefore, $g \circ f$ is continuous at $p \in S$.

***Remark:***

Let $f: X \to Y$ be a function from a set $X$ into a set $Y$ and let $A \subseteq X$, $B \subseteq Y$. Then:

1. $f(A) = \{y \in Y \mid y = f(x), x \in A\} = \{f(x) \in Y \mid x \in A\}$
2. $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$
3. $f^{-1}f(A) \supseteq A \; and \; f^{-1}f(A) = A \;\; \Leftrightarrow f$ is onto.
4. $ff^{-1}(B) \subseteq B \; and \; ff^{-1}(B) = B \;\; \Leftrightarrow f$ is one-to-one.

***Theorem:***

Let $(S, d_S)$ and $(T, d_T)$ be metric spaces and let $f: S \to T$ be a function. Then:

1. $f$ is continuous on $S$ if, and only if, $f^{-1}(B)$ is an open set in $S$ for every open set $B$ in $T$.
2. $f$ is continuous on in $S$ if, and only if, $f^{-1}(B)$ is a closed set in $S$ for every closed set $B$ on $T$.

***Proof:***

***<u>For (1):</u>*** Suppose that $f$ is continuous on $S$ and let $B$ be an open set in $T$. Wanted: $f^{-1}(B)$ is an open set in $S$, i.e. wanted: each point in $f^{-1}(B)$ is an interior point of $f^{-1}(B)$.

Let $p \in f^{-1}(B)$. Wanted: $\exists\, \delta > 0 \ni B_S(p; \delta) \subseteq f^{-1}(B)$.

Since $p \in f^{-1}(B) \Rightarrow f(p) \in B$. But $B$ is open set in $T \Rightarrow f(p)$ is an interior point of $B \Rightarrow \exists\, \epsilon > 0 \ni B_T(f(p); \epsilon) \subseteq B \ldots (*)$

Since $\epsilon > 0$ and $f: S \to T$ is continuous at $\in S \Rightarrow \exists\, \delta > 0$, such that;

$$f(B_S(p; \delta)) \subseteq B_T(f(p); \epsilon)$$

$$\Rightarrow f^{-1}f(B_S(p; \delta)) \subseteq f^{-1}(B_T(f(p); \epsilon))$$

But $B_S(p; \delta) \subseteq f^{-1}f(B_S(p; \delta)) \Rightarrow B_S(p; \delta) \subseteq f^{-1}\big(B_T(f(p); \epsilon)\big) \ldots (*\,2)$

From $(*)$ we have, $f^{-1}\big(B_T(f(p); \epsilon)\big) \subseteq f^{-1}(B) \ldots (*\,3)$

From $(*\,2)$ and $(*\,3)$, we have $B_S(p; \delta) \subseteq f^{-1}(B)$. Thus, $f^{-1}(B)$ is an open set in $S$.

Conversely, assume that $f^{-1}(B)$ is open in $S$, for every open set $B$ in $T$. Wanted: $f$ is continuous on $S$.

Let $p \in S$. Wanted: $f$ is continuous at $p \in S$. Let $\epsilon > 0$. Wanted:

$$\exists\, \delta > 0 \ni f(B_S(p; \delta)) \subseteq B_T(f(p); \epsilon).$$

Since $B_T(f(p); \epsilon)$ is open set in $T$ containing $f(p)$, hence $f^{-1}\big(B_T(f(p); \epsilon)\big)$ is open set in $S$ containing $p$, i.e. $p \in f^{-1}\big(B_T(f(p); \epsilon)\big) \Rightarrow p$ is an interior point of $f^{-1}\big(B_T(f(p); \epsilon)\big) \Rightarrow \exists\, \delta > 0 \ni B_S(p; \delta) \subseteq f^{-1}\big(B_T(f(p); \epsilon)\big)$;

$$\Rightarrow \exists\, \delta > 0 \ni f(B_S(p;\delta)) \subseteq ff^{-1}\big(B_T(f(p);\epsilon)\big).$$

But, $ff^{-1}\big(B_T(f(p);\epsilon)\big) \subseteq B_T(f(p);\epsilon) \Rightarrow f(B_S(p;\delta)) \subseteq B_T(f(p);\epsilon)$. Thus $f$ is continuous at $p$.

***For (2):*** Suppose $f$ is continuous on $S$ and let $B$ be a closed set in $T$. Wanted: $f^{-1}(B)$ is a closed set in $S$, i.e. wanted: $S - f^{-1}(B)$ is an open set in $S$.

Since $B$ is closed in $T \Rightarrow T - B$ is open in $T$. But $f$ is continuous on $S \Rightarrow$ from part (1) above , $f^{-1}(T-B)$ is an open set in $S$. Since;

$$f^{-1}(T-B) = f^{-1}(T) - f^{-1}(B) = S - f^{-1}(B).$$

$\Rightarrow S - f^{-1}(B)$ is an open set in $S \Rightarrow f^{-1}(B)$ is a closed set in $S$.

Conversely, assume $f^{-1}(B)$ is closed in $S$ for closed set $B$ in $T$. Wanted: $f$ is continuous on $S$.

Let $A$ be an open set in $T$. Wanted: $f^{-1}(A)$ is open in $S$, (i.e. we will use part (1) above to show our aim). Since $A$ is open in $T \Rightarrow T - A$ is closed in $T \Rightarrow$ $f^{-1}(T-A)$ is closed in $S$, (this implies from our assumption). Since;

$f^{-1}(T-A) = S - f^{-1}(A) \Rightarrow S - f^{-1}(A)$ is closed in $S$.

$$\Rightarrow S - (S - f^{-1}(A)) \text{ is open in } S.$$

But $S - \big(S - f^{-1}(A)\big) = f^{-1}(A) \Rightarrow f^{-1}(A)$ is open in $S$. Thus, $f$ is continuous on $S$.

***Theorem:***

Let $f: S \to T$ be a continuous function from a metric space $(S, d_S)$ into a metric space $(T, d_T)$. If $X$ is a compact subset of $S$, then $f(X)$ is compact subset of $T$, in particular $f(X)$ is closed and bounded .

***Proof:*** Let $\{G_i | i \in I\}$ be an open covering of $f(X)$, i.e. $f(X) \subseteq \cup_{i\in I} G_i$ , where $G_i$ is open in $T$, $\forall i \in I$. Wanted: $\{G_i | i \in I\}$ contains a finite subcover of $f(X)$.

According, $f(X) \subseteq \cup_{i\in I} G_i$, we have $f^{-1}(f(X)) \subseteq f^{-1}(\cup_{i\in I} G_i)$.

Since, $X \subseteq f^{-1}f(X)$ and $f^{-1}(\cup_{i\in I} G_i) = \cup_{i\in I} f^{-1}(G_i)$, hence $X \subseteq \cup_{i\in I} f^{-1}(G_i)$.

But $G_i$ is open in $T$ and $f$ is continuous on $S$, therefore $f^{-1}(G_i)$ is open in $S$, $\forall\, i \in I \Rightarrow \{f^{-1}(G_i) | i \in I\}$ froms an open covering of $X$. But $X$ is compact in $S$

$\Rightarrow\ \exists$ a finite subcover of $\{f^{-1}(G_i)|i \in I\}$ for $X$ say $\{f^{-1}(G_1), \dots, f^{-1}(G_n)\}$, i.e. $X \subseteq \cup_{i=1}^{n} f^{-1}(G_i) \Rightarrow f(X) \subseteq f\left(\cup_{i=1}^{n} f^{-1}(G_i)\right) = \cup_{i=1}^{n} ff^{-1}(G_i)$.

But $ff^{-1}(G_i) \subseteq G_i$, so $\cup_{i=1}^{n} ff^{-1}(G_i) \subseteq \cup_{i=1}^{n} G_i \Rightarrow f(X) \subseteq \cup_{i=1}^{n} G_i$.

$\Rightarrow \{G_i : i = 1, \dots, n\}$ forms a finite subcover of $\{G_i|i \in I\}$ for $f(X)$. Hence, $f(X)$ is compact in $T$ and from a previous result, we implies that $f(X)$ is closed and bounded in $T$.

## *Complex valued functions and vector valued functions:*

***Definition:***

Let $(S, d_S)$be a metric space and let $f: S \to \mathbb{C}$ and $g: S \to \mathbb{C}$ be complex valued functions. The sum $+g: S \to \mathbb{C}$ , the difference $f - g: S \to \mathbb{C}$, the product $f.g: S \to \mathbb{C}$ and the quotient $f/_g : S \to \mathbb{C}$ are defined respectively by:

**1.** $f \pm g(x) = f(x) \pm g(x), \forall x \in S$.

**2.** $f.g(x) = f(x).g(x), \forall x \in S$.

**3.** $f/_g\,(x) = {f(x)}/_{g(x)}, \forall x \in S$ such that $g(x) \neq 0$.

***Exercise:***

Let $(S, d_S)$be a metric space and let $f: S \to \mathbb{C}$ and $g: S \to \mathbb{C}$ be complex valued functions. If $f$ and $g$ are continuous at $p \in S$, prove that;

$f + g, f - g, f.g: S \to \mathbb{C}$ are continuous functions at $p$.

***Definition:***

Let $(S, d_S)$be a metric space and let $f: S \to \mathbb{R}^n$ and $g: S \to \mathbb{R}^n$ be vector valued functions. The sum $f + g: S \to \mathbb{R}^n$, the scalar product $\alpha.f: S \to \mathbb{R}^n$, where $\alpha \in \mathbb{R}$, the inner (or dot) product $f.g: S \to \mathbb{R}^n$ and the norm $||f||: S \to \mathbb{R}$ are defined respectively by:

**1.** $f + g(x) = f(x) + g(x), \forall x \in S$.

**2.** $\alpha.f(x) = \alpha.f(x), \forall x \in S$.

**3.** $f.g(x) = f(x).g(x), \forall x \in S$.

**4.** $\|f\|(x) = \|f(x)\|, \forall x \in S$.

***Exercises:***

**1.** Let $(S, d_S)$be a metric space and let $f: S \to \mathbb{R}^n$ and $g: S \to \mathbb{R}^n$ be vector valued functions. If $f$ and $g$ are continuous at $p \in S$ and $\alpha \in \mathbb{R}$, prove that;

$$f + g, \alpha.f, f.g, \|f\| : S \to \mathbb{R}^n \text{ are continuous functions at } p.$$

**2.** Let $(S, d_S)$be a metric space and let $f: S \to \mathbb{R}^n$ be a vector valued function defined by, $f(x) = (f_1(x), f_2(x), \dots, f_n(x))$, for $x \in S$. Prove that, $f$ is continuous at $p \in S$ if, and only if, $f_i: S \to \mathbb{R}$ is continuous at $p$, for all $i = 1,2, \dots, n$.

## ***Bounded functions:***

***Definition:***

A function $f: S \to \mathbb{R}^n$ from a metric space $(S, d_S)$ into the Euclidean space $(\mathbb{R}^n, \|.\|)$, is called bounded on $S$, if there exists a positive real number $M > 0$, such that;

$$\|f(x)\| \leq M, \forall x \in S.$$

Or equivalently: $f$ is bounded if, and only if, $f(S)$ is bounded subset of $\mathbb{R}^n$.

***Theorem:***

Let $f: S \to \mathbb{R}^n$ be a function from a metric space $(S, d_S)$ into the Euclidean space $(\mathbb{R}^n, \|.\|)$. If $f$ is continuous on a compact subset $X$ of $S$, then $f$ is bounded.

***Proof***: Since $f$ is continuous on $X$ and $X$ is compact, then $f(X)$ is compact as a metric subspace of $\mathbb{R}^n$. So, $f(X)$ is compact subset of $\mathbb{R}^n$ and as an application of a previous result $f(X)$ is closed and bounded. Therefore, $f$ is bounded.

***Remark:***

If $f: S \to \mathbb{R}$ is a real valued function which is bounded on $X \subseteq S$, then $f(X)$ is bounded of $\mathbb{R} \Rightarrow f(X)$ is b is bounded above bounded above and bounded below $\Rightarrow f(X)$ has $Sup(f(X))$ and $Inf(f(X)) \Rightarrow$

$$Sup(f(X)) \leq f(x) \leq Inf(f(X)), \forall x \in X.$$

***Exercise:***

Let $f: S \to \mathbb{R}$ be a real valued function from a metric space $(S, d_S)$ into the Euclidean space $(\mathbb{R}, |.|)$. Prove that, if $f$ is continuous on a compact subset of $S$, then there exist two points $p, q \in X$ such that;

$$f(p) = Inf(f(X)) \text{ and } f(q) = Sup(f(X)).$$

***Theorem:***

Let $f$ be defined on an interval $S$ of $\mathbb{R}$. Assume that, $f$ is continuous at a point $c$ in $S$ and that $f(c) \neq 0$. Then, there is an open ball $B(c;\delta)$ such that $f(x)$ has the same sign as $f(c)$ in $B(c;\delta) \cap S$.

***Proof:***

Suppose that $f(c) > 0$. Let $\epsilon = \frac{1}{2} f(c) \Rightarrow \epsilon > 0$.

Since $\epsilon > 0$ and $f$ is continuous at $c \in S \Rightarrow \exists\ \delta > 0$ such that if $x \in S$ and;

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c)| < \epsilon.$$

Therefore, if $x \in B(c;\delta) \Rightarrow -\epsilon < f(x) - f(c) < \epsilon$

$\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon;$

$\Rightarrow f(c) - \frac{1}{2} f(c) < f(x) < f(c) + \frac{1}{2} f(c);$

$\Rightarrow 0 < \frac{1}{2} f(c) < f(x) < \frac{3}{2} f(c)$, since $f(c) > 0$;

$\Rightarrow f(x) > 0.$

Therefore, $f(x)$ has the same sign as $f(c)$ in $B(c;\delta) \cap S$. The proof is similar if $(c) < 0$ , except that we take in this case $\epsilon = -\frac{1}{2} f(c)$.

***Theorem (Bolzano's theorem for continuous functions):***

Let $f$ be a real-valued and continuous function on a compact interval $[a, b]$ in $\mathbb{R}$, and suppose that $f(a)$ and $f(b)$ have opposite signs , i.e. $f(a)f(b) < 0$. Then, there is at least one point $c \in (a, b)$ such that $f(c) = 0$.

***Proof:***

For definiteness, assume that $f(a) > 0$ and $f(b) < 0$. Let;

$$A = \{x \mid x \in [a, b] \text{ and } f(x) \geq 0\}.$$

Since $a \in [a\,,b]$ and $f(a) > 0 \Rightarrow a \in A \Rightarrow A \neq \emptyset$. Since, $A \subseteq [a\,,b] \Rightarrow x \leq b$, $\forall x \in A \Rightarrow b$ is an upper bound of $A \Rightarrow Sup\ A$ exists. Let $c = Sup\ A$.

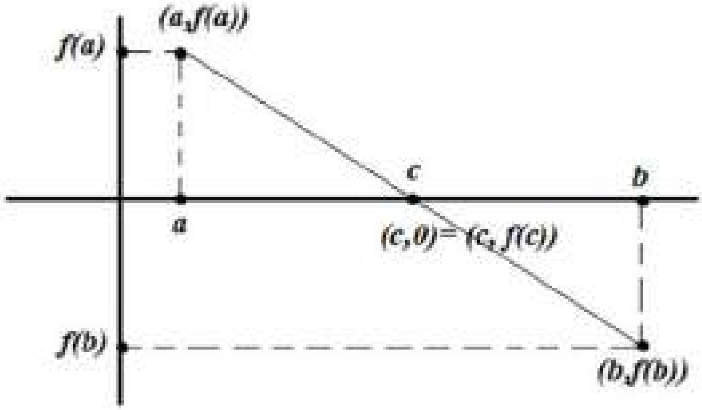
Since $f(b) < 0 \Rightarrow b \notin A$ and from the above theorem, there is an open ball $B(b;r)$ such that $f(x)$ has the same sign as $f(b)$ in $B(b;r) \cap [a,b]$.

$\Rightarrow f\left(b - \frac{r}{2}\right) < 0 \Rightarrow b - \frac{r}{2} \notin A$ and it is also an upper bound of $A$.

$\Rightarrow c = Sup\ A < b$, since $b - \frac{r}{2}$ is an upper bound of $A$ with $b - \frac{r}{2} < b$.

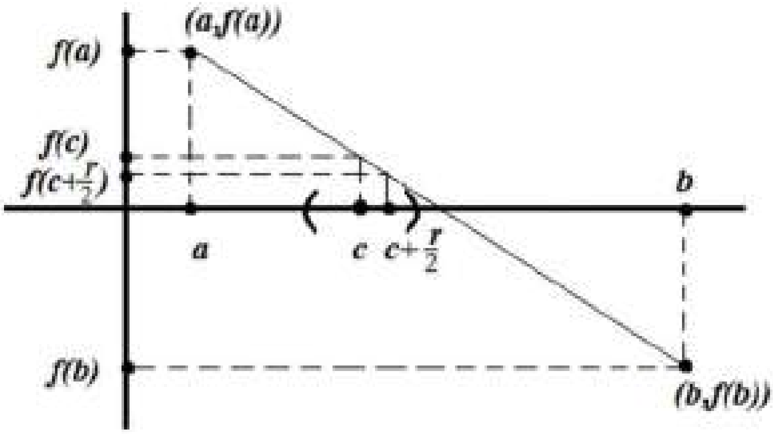
$\Rightarrow a < c$ (since $a \in A$) and $c < b$.

$\Rightarrow a < c < b \quad \Rightarrow \; c \in (a\,,b)$. We will show that, $f(c) = 0$.




If $f(c) \neq 0$, then from the above result, there is an open ball $B(c;\delta)$ such that $f(x)$ has the same sign as $f(c)$ in $B(c;\delta) \cap [a,b]$.

If $f(c) > 0$, then there are points $x \in A$ such that $x > c$ at which $f(x) > 0$ and this is a contradiction since $c = Sup\ A$.





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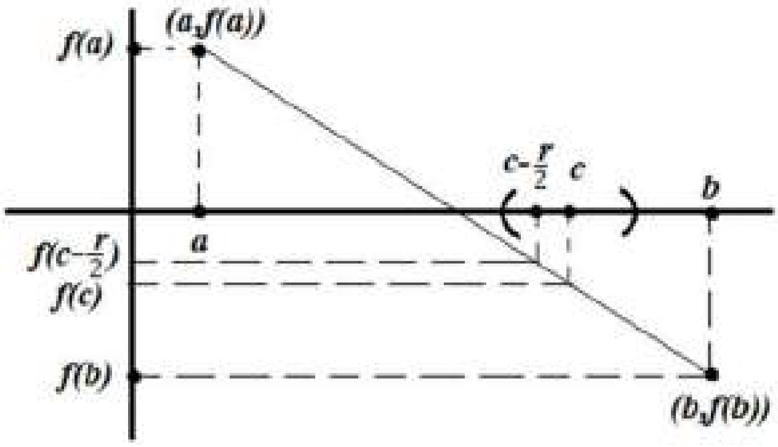
If $f(c) < 0$, then $c - \frac{\delta}{2}$ is an upper bound for $A$ since $f(c - \frac{\delta}{2}) < 0$. But $c = Sup\ A$, hence $c < c - \frac{\delta}{2}$ (contradiction).




Thus, there is at least a point $c \in (a, b)$. Such that $f(c) = 0$.

## *Uniform continuity:*

***Remark:***

Firstly, let us recall the definition of continuity:

Let $f: S \to T$ be a function from a metric space $(S, d_S)$ into a metric space $(T, d_T)$ and let $A \subseteq S$. Then, $f$ is called continuous on $A$ if, the following condition is hold:

$\forall\ p \in A$ and $\forall\ \epsilon > 0\ \exists\ a\ \ \delta > 0$ (depending on $p$ and on $\epsilon$) such that if $x \in A$

$$\text{and } d_S(x, p) < \delta \quad \Rightarrow d_T\big(f(x), f(p)\big) < \epsilon.$$

In general, we cannot expect that for a fixed $\epsilon > 0$ the same $\delta > 0$ will serve for every point $p$ in .

***Definition (Uniform continuity):***

Let $f: S \to T$ be a function from a metric space $(S, d_S)$, into a metric space $(T, d_T)$. Then $f$ is said to be uniformly continuous on a subset $A$ of $S$, if the following condition holds:

$$\forall\ \epsilon > 0\ \exists\ a\ \ \delta > 0 \text{ (depending on } \epsilon\text{), such that if } x, y \in A \text{ and,}$$

$$d_S(x, y) < \delta \quad \Rightarrow d_T\big(f(x), f(y)\big) < \epsilon.$$

***Theorem:***

Let $f: S \to T$ be a function from a metric space $(S, d_S)$, into a metric space $(T, d_T)$. If $f$ is uniformly continuous on $S$, then $f$ is continuous on $S$. But the converse needs not to be true in general.

***Proof:***

Suppose $f$ is uniformly continuous on $S$. Wanted: $f$ is continuous on $S$. Let $\epsilon > 0$ and $p \in S$, wanted: $\exists\ a\ \delta > 0$ (depending on $p$ and on $\epsilon$) such that if $x \in S$ and $d_S(x,p) < \delta \Rightarrow d_T\big(f(x), f(p)\big) < \epsilon$.

Since $\epsilon > 0$ and $\exists\ a\ \delta > 0$ (depending on $\epsilon$) such that if $x, y \in S$ and $d_S(x,y) < \delta \Rightarrow d_T\big(f(x), f(y)\big) < \epsilon \ldots (*)$. Thus, if we take $y = p$, then $(*)$ becomes, if $x \in S$ and $d_S(x,p) < \delta \Rightarrow d_T\big(f(x), f(p)\big) < \epsilon \Longrightarrow f$ is continuous at $p \in S \Longrightarrow f$ is continuous on $S$.

***Example:***

Let $f$ be real-valued function define on $\mathbb{R}$ by $f(x) = x^2$, $\forall x \in \mathbb{R}$. We will show that $f$ is continuous on $\mathbb{R}$ and $f$ is not uniformly continuous on $\mathbb{R}$:

**For $f$ is continuous on** $\mathbb{R}$: Let $p \in \mathbb{R}$. Wanted: $f$ is continuous at $p$. Let $\epsilon > 0$.

Wanted: $\exists$ a $0 < \delta \leq 1$ such that if;

$$|x - c| < \delta \quad \Rightarrow \quad |f(x) - f(p)| < \epsilon.$$

As we know, $|f(x) - f(p)| = |x^2 - p^2| = |(x-p)(x+p)|$

$$= |x - p|\,|x + p|$$

If we suppose, $|x - p| < \delta \ \Rightarrow |f(x) - f(p)| < \delta|x + p|$

$$\Rightarrow |f(x) - f(p)| < \delta(|x| + |p|) \ldots (*1)$$

Since $\delta < 1 \Rightarrow |x - p| < 1$ . But $\big||x| - |p|\big| \leq |x - p|$

$$\Rightarrow \big||x| - |p|\big| < 1 \Rightarrow -1 < |x| - |p| < 1$$

From $|x| - |p| < 1 \Rightarrow |x| < |p| + 1 \ldots (*2)$

From $(*1)$ and $(*2)$ we have,

$$\Rightarrow |f(x) - f(p)| < \delta(1 + |p| + |p|) = \delta(1 + 2|p|)$$

$$\Rightarrow |f(x) - f(p)| < \delta(1 + 2|p|)$$

So we can choose $\delta = Min\{\frac{\epsilon}{(1+2|p|)}, 1\}$.

Therefore $|x-p| < \delta \Rightarrow |f(x)-f(p)| = |x^2-p^2| = |(x-p)(x+p)|$

$$= |(x-p)||(x+p)| < \delta|(x+p)| \leq \delta(|x|+|p|) < \delta(1+|p|+|p|)$$

$$= \delta(1+2|p|)$$

$$\Rightarrow |f(x)-f(p)| < \delta(1+2|p|) \ldots (*)$$

Now, if $= 1 \Rightarrow \delta < \frac{\epsilon}{(1+2|p|)}$.

Therefore from (*)$\Rightarrow$ $|f(x)-f(p)| < \frac{\epsilon}{(1+2|p|}.\left(\frac{\epsilon}{(1+2|p|}\right) = \epsilon$.

And , if $\delta = \frac{\epsilon}{(1+2|p|)}$ from (*) $\Rightarrow$ $|f(x)-f(p)| < \epsilon$ .

Therefore, $f$ is continuous at $p \in \mathbb{R} \Rightarrow f$ is continuous on $\mathbb{R}$.

***Exercises***

**(1):** Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

**(2):** Prove that $f(x) = x^2$ is uniformly continuous on $A = (0,1]$.

***<u>Proof (1):</u>***

We need to prove, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, i.e.

wanted: $\exists\, p \in A$ and $\exists\ \epsilon > 0, \forall\ \delta > 0$ if $x, y \in A$ and,

$$|x-y| < \delta \text{ but } |f(x)-f(p)| > \epsilon \ldots (*).$$

Let $\epsilon = 1$, and suppose we could find a $\delta > 0$ to satisfy the condition of $(*)$. Taking $x = \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$, then;

$$|x-p| = \left|\frac{1}{\delta} - (\frac{1}{\delta} + \frac{\delta}{2})\right| = \frac{1}{\delta} + \frac{\delta}{2} < \delta.$$

But $|f(x)-f(p)| = \left|(\frac{1}{\delta})^2 - (\frac{1}{\delta} + \frac{\delta}{2})^2\right| = \left|-(\frac{1}{\delta})^2 - 1\right| = (\frac{1}{\delta})^2 + 1 > 1$.

$$\Rightarrow |f(x)-f(y)| < \epsilon.$$

Thus, $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

***<u>Proof (2):</u>***

Let $\epsilon > 0$, take $\delta = \frac{\epsilon}{2}$. Therefore, if we suppose that $|x-y| < \delta$

$\Rightarrow |f(x) - f(y)| = |x^2 - y^2| = |(x-y)||(x+y)| < \delta|(x+y)| \leq 2\delta,$

since $x, y \in A = (0,1]$ and $x + y \leq 2 \Rightarrow |f(x) - f(y)| < 2\delta = 2.\frac{\epsilon}{2} = \epsilon.$

$$\Rightarrow |f(x) - f(y)| < \epsilon.$$

Since $\delta = \frac{\epsilon}{2}$ depends on $\epsilon$ only, therefore $f(x) = x^2$ is uniformly continuous on $A = (0,1]$.

***Example:***

Let $f$ be a real-valued function defined on $A = (0,1]$ by;

$$f(x) = \frac{1}{x}, \ \forall \ x \in A = (0,1].$$

Clearly, $f$ is continuous on $A$ (as an exercise: show that). We will show that, $f$ is not uniformly continuous at $A$. To prove this, let $\epsilon = 10$ and suppose that we could find a $0 < \delta < 1$, to satisfy the condition of uniform continuity. Take $x = \delta,\ p = \frac{\delta}{11}$. Therefore, $|x - p| = \left|\delta - \frac{\delta}{11}\right| < \delta.$

But $|f(x) - f(p)| = \left|\frac{1}{\delta} - \frac{11}{\delta}\right| = \left|-\frac{10}{\delta}\right| = \frac{10}{\delta} > 10 = \epsilon$, (since $0 < \delta < 1$). Thus $f$ is not uniformly continuous on $A = (0,1]$.

The important point to note here, the sequence $\langle\frac{1}{n}\rangle$ is a Cauchy sequence in $\mathbb{R}$, but the sequence $\langle f(\frac{1}{n})\rangle = \langle n\rangle$ is not a Cauchy sequence in $\mathbb{R}$.

Thus, if $f: S \to T$ is a continuous function on a subset $A$ of $S$ and $\langle x_n\rangle$ is a Cauchy sequence in $A$, then $\langle f(x_n)\rangle$ need not to be a Cauchy sequence in $T$.

***Theorem***:

Let $f: S \to T$ be a function from a metric space $(S, d_S)$, into a metric space $(T, d_T)$. If $f$ is uniformly continuous on $S$ and $\langle x_n\rangle$ is a Cauchy sequence in $S$, then $\langle f(x_n)\rangle$ is a Cauchy sequence in $T$.

***Proof:***

Wanted: $\langle f(x_n)\rangle$ is a Cauchy sequence in $T$. Let $\epsilon > 0$, wanted: $N \in \mathbb{Z}^+ \ni$

$$d_T(f(x_m), f(x_n)) < \epsilon, \forall m, n \geq N.$$

Since $f$ is uniformly continuous on $S$ and $\epsilon > 0$, hence $\exists$ a $\delta > 0$ (depending on $\epsilon$ only) such that if $x, y \in A$ and,

$$d_S(x,y) < \delta \Rightarrow d_T\big(f(x), f(y)\big) < \epsilon \ldots (*).$$

Since $\delta > 0$ and $\langle x_n \rangle$ is a Cauchy sequence in $S$, then $\exists N \in \mathbb{Z}^+ \ni$

$$d_T(x_m, x_n) < \delta, \forall m, n \geq N.$$

From $(*)$ above $\Longrightarrow d_T(f(x_m), f(x_n)) < \epsilon, \forall m, n \geq N.$

$\langle f(x_n) \rangle$ is a Cauchy sequence in $T$.

***Theorem (Heine theorem):***

Let $f: S \rightarrow T$ be a function from a metric space $(S, d_S)$, into a metric space $(T, d_T)$. If $f$ is continuous on a compact subset $A \subseteq S$, then $f$ is uniformly continuous on $A$.

***Proof:***

Let $\epsilon > 0$. Wanted: $\exists\ a\ \ \delta > 0$ (depending on $\epsilon$) such that if $x, p \in A$ and,

$$d_S(x,p) < \delta \Rightarrow d_T\big(f(x), f(p)\big) < \epsilon.$$

Since $f$ is continuous on $A$ and $\epsilon > 0$, then, $\forall\ a \in A\ \exists\ a\ \ \delta_a > 0$ (depending on $a$ and on $\epsilon$) such that if $x \in A$ and;

$$d_S(x,a) < \delta_a \ \ \Rightarrow d_T\big(f(x), f(a)\big) < \tfrac{\epsilon}{2} \ldots (*)$$

The collection $\left\{B_S\left(a; \frac{\delta_a}{2}\right) \mid a \in A\right\}$ forms an open covering of $A$, since;

$$A \subseteq \cup_{a \in A} B_S\left(a; \frac{\delta_a}{2}\right).$$

But $A$ is compact $\Longrightarrow$ $\exists$ a finite subcover of $A$ of $\left\{B_S\left(a; \frac{\delta_a}{2}\right) \mid a \in A\right\}$, say;

$$\left\{B_S\left(a_1; \frac{\delta_{a_1}}{2}\right), B_S\left(a_2; \frac{\delta_{a_2}}{2}\right), \ldots, B_S\left(a_n; \frac{\delta_{a_n}}{2}\right)\right\};$$

i.e. $A \subseteq \cup_{i=1}^{n} B_S\left(a_i; \frac{\delta_{a_i}}{2}\right)$. Choose $\delta = Min\{\frac{\delta_{a_1}}{2}, \frac{\delta_{a_2}}{2}, \ldots, \frac{\delta_{a_n}}{2}\} > 0$. That is, our choice of $\delta$ in this case implies that $\delta \leq \frac{\delta_{a_k}}{2}$, for all $k = 1,2, \ldots, n$ and hence $\delta$ depend on $\epsilon$ only.

Now, we will show this $\delta > 0$ satisfy the uniform continuity condition of $f$. To do this, let $x$ and $p$ be any two points of $A$ with $d_S(x,p) < \delta$, we need only to show $d_T\big(f(x), f(y)\big) < \epsilon$.

Since $x \in A \subseteq \bigcup_{i=1}^{n} B_S\left(a_i; \frac{\delta_{a_i}}{2}\right)$, hence $\exists k = 1, \dots, n \ni x \in B_S\left(a_k; \frac{\delta_{a_k}}{2}\right)$, i.e. $d_S(x, a_k) < \frac{\delta_{a_k}}{2}$. Since $x, p, a_k \in A$, hence by using the triangle inequality we have;

$$d_S(p, a_k) \leq d_S(p, x) + d_S(x, a_k) < \delta + \frac{\delta_{a_k}}{2} < \frac{\delta_{a_k}}{2} + \frac{\delta_{a_k}}{2} = \delta_{a_k}.$$

From $(*)$ above, since $d_S(x, a_k) < \frac{\delta_{a_k}}{2} < \delta_{a_k}$ and $d_S(p, a_k) \leq \delta_{a_k}$, hence $d_T\big(f(x), f(a_k)\big) < \frac{\epsilon}{2}$ and $d_T\big(f(p), f(a_k)\big) < \frac{\epsilon}{2}$. So, the triangle inequality gives us;

$$d_T\big(f(p), f(x)\big) \leq d_T\big(f(x), f(a_k)\big) + d_T\big(f(p), f(a_k)\big) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$$\Longrightarrow d_T\big(f(p), f(x)\big) < \epsilon.$$

Therefore, $f$ is uniformly continuous on $A$.

## *Fixed-point theorem for contractions:*

***Definition:***

Let $f: S \to S$ be a function from a metric space $(S, d_S)$, into itself. A point $p \in S$ is called ***a fixed point*** of $f$ if $f(p) = p$. The function $f$ is called a ***contraction*** of $S$ if there is a number $0 < x < 1$ (called a contraction constant), such that, $d\big(f(x), f(y)\big) \leq \alpha d(x, y)$, $\forall x, y \in S$ ... $(*)$

***Exercise:***

Let $(S, d)$ be a metric space. If $f: S \to S$ is a contraction of , then $f$:is uniformly continuous in $S$ .

***Proof:***

Let $\epsilon > 0$. Wanted: $\exists\ \delta > 0$ (depending on $\epsilon$) $\ni$ for any $x, y \in S$;

$$d(x, y) < \delta \Rightarrow d\big(f(x), f(y)\big) < \epsilon.$$

Since $x, y \in S$ and $f: S \to S$ is a contraction of $S$, hence;

$$\exists\ 0 < x < 1 \ \ni\ d\big(f(x), f(y)\big) \le \alpha\, d(x, y).$$

Choose $\delta = \frac{\epsilon}{\alpha} > 0$. Therefore, if we suppose that;

$$(x, y) < \delta \quad \Rightarrow\ d\big(f(x), f(y)\big) < \alpha\delta = \alpha\frac{\epsilon}{\alpha} = \epsilon.$$

$$\Rightarrow\ d\big(f(x), f(y)\big) < \epsilon \ \text{ (where } \delta \text{ depending on } \epsilon \text{ only)}$$

Therefore, $f$ is uniformly continuous.

***Theorem (Fixed-point theorem):***

Let $(S, d)$ be a complete metric space. If $f: S \to S$ is a contraction of $S$, then $f$ has a unique fixed point, i.e. there is a unique point $p$ in $S$ such that $f(p) = p$.

***Proof :***

First of all, we show that $\exists\ p \in S \ \ni\ f(p) = p$ .

Let $x \in S$ be any point of $S$ and consider the sequence;

$$x, f(x), f\big(f(x)\big), f\Big(f\big(f(x)\big)\Big), \ldots ;$$

This is defining a sequence $\langle p_n \rangle$ in $S$ inductively by:

$$p_0 = x\ ,\ \ p_{n+1} = f(p_n),\ \ n = \ 1\ ,2\ , \ldots ;$$

$$\text{i.e. } \ p_0 = x,\ \ p_1 = f(p_0) = f(x),\ \ p_2 = f(p_1) = f\big(f(x)\big), \ldots.$$

Since;

$$d(p_{n+1}\ , p_n) = d(f(p_n)\ , f(p_{n-1})) \le \alpha\, d(p_n\ , p_{n-1}), \text{ (since } f \text{ is a contraction of S)}$$

$$= \alpha\, d(f(p_{n-1}), f(p_{n-2}))$$

$$\le \alpha^2\, d((p_{n-1}), (p_{n-2}))$$

$$= \alpha^2\, (f(p_{n-2}), f(p_{n-3}))$$

$$\le \alpha^3\, d((p_{n-3}), (p_{n-3}))$$

$$\ldots \le \alpha^n\, d((p_1), (p_0))$$

$$\Rightarrow d(p_{n+1}\ , p_n) \le \alpha^n\, d(p_1, p_0)$$

If we let $(p_1, p_0) = c \ \Rightarrow d(p_{n+1}\ , p_n) \le \alpha^n c$. Using the triangle inequality we find, for $m > n$;

$d(p_m, p_n) \leq d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2}) + \cdots + d(p_{m-1}, p_m);$

$\leq \alpha^n c + \alpha^{n+1} c + \alpha^{n+2} c + \cdots + \alpha^{m-1} c;$

$= c(\alpha^n + \alpha^{n+1} + \alpha^{n+2} + \cdots + \alpha^{m-1});$

$= c((\alpha^{m-1} + \cdots + \alpha^{n+2} + \alpha^{n+1} + \alpha^n + \alpha^{n-1} + \cdots + \alpha) - (\alpha^{n-1} + \cdots + \alpha));$

$= c(\frac{1-\alpha^m}{1-\alpha} - \frac{1-\alpha^n}{1-\alpha})$, (the above geometric series a converge since $\alpha < 1$ );

$= c\left(\frac{1}{1-\alpha} - \frac{\alpha^m}{1-\alpha} - \frac{1}{1-\alpha} + \frac{\alpha^n}{1-\alpha}\right) = c\left(\frac{\alpha^n}{1-\alpha} - \frac{\alpha^m}{1-\alpha}\right) < c(\frac{\alpha^n}{1-\alpha});$

$\Rightarrow d(p_m, p_n) < c\frac{\alpha^n}{1-\alpha}.$

$\Rightarrow d(p_m, p_n) \to 0$ $as\, n \to \infty$ (since $\alpha^n \to 0$ as $n \to \infty$ and hence $\frac{\alpha^n}{1-\alpha} \to 0$ as $n \to \infty$). Thus, the sequence $\langle p_n \rangle$ is a Cauchy sequence in $S$. But $S$ is complete $\Rightarrow \exists\ p \in S \ni p_n \to p$ in $S$. Since $f$ is uniformly continuous on $S$ (as $f$ is a contraction of $S$), hence $f$ is continuous on $S$. But $p \in S$, therefore $f$ is continuous at $p$. Since $p_n \to p$ in $S$ and $f$ is continuous at $p \Rightarrow f(p_n) \to f(p)$, i.e. $\lim_{n\to\infty} f(p_n) = f(p)$, but $\lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} p_{n+1} = p$ .Therefore, $f(p) = p$.

Finally, we need only to show that $p$ is unique. To do this, assume $p$ and $q$ are two fixed-points of $f$, i.e. $f(p) = p$ and $f(q) = q$.

Since $p, q \in S$ and $f$ is a contraction of $S$,

$$\Rightarrow \exists\ \ 0 < \alpha < 1 \ \ni\ d\big(f(p), f(q)\big) \leq \alpha\ d(p, q)$$

$$\Rightarrow \exists\ \ 0 < \alpha < 1 \ni d(p, q) \leq \alpha\ d(p, q)$$

If we assume that, $d(p, q) \neq 0 \Rightarrow \alpha = 1$(contradiction). Therefore, $(p, q) = 0$ $\Rightarrow\ p = q$.