1.3.3 Existence and uniqueness of solution

In this section we need to present the conditions which are used to show that whether there may not exist solution, there may exist solution, or there exist infinitely may distinct solutions. In other words, either there is no existence of solution or no uniqueness.

Consider the first-order PDE

$$a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}=c. \tag{1.9}$$

Here, we want to find the solution of equation (1.9) through the curve which is defined by

$x=x(t), \quad y=y(t), \quad z=z(t).$

Now,

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy, \quad \Rightarrow \quad \frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Thus, we have

$$x'\frac{\partial z}{\partial x}+y'\frac{\partial z}{\partial y}=z'. \tag{1.10}$$

Then, from equations (1.9) and (1.10), we have the following system

$$\begin{pmatrix} a & b \\ x' & y' \end{pmatrix}\begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix}=\begin{pmatrix} c \\ z' \end{pmatrix}. \tag{1.11}$$

Now,

1) If $\begin{vmatrix} a & b \\ x' & y' \end{vmatrix} \neq 0 \quad \Rightarrow \quad ay'-bx' \neq 0$

Then, the system (1.11) has a unique solution.

2) $\begin{vmatrix} a & b \\ x' & y' \end{vmatrix} = 0 \quad \Rightarrow \quad ay'-bx' = 0$, then

2a) $\begin{vmatrix} a & c \\ x' & z' \end{vmatrix} = 0 \quad \Rightarrow \quad az'-cx' = 0.$

Then, there exist infinitely distinct solutions.

2b) $\begin{vmatrix} a & c \\ x' & z' \end{vmatrix} \neq 0 \quad \Rightarrow \quad az' - cx' \neq 0.$

Then, there may not exist a solution.

Example: $(x-y)\dfrac{\partial z}{\partial x} + (y-x-z)\dfrac{\partial z}{\partial y} = z$, through the curve $x^2 + y^2 = 1$, $z = 1$.

$x = t, \quad \Rightarrow \quad x' = 1$

$y = \sqrt{1-t^2} \;\Rightarrow\; y' = \dfrac{-t}{\sqrt{1-t^2}}$

$z = 1,$

and,

$a = x - y = t - \sqrt{1-t^2}$, $b = \sqrt{1-t^2} - t - 1 \quad c = z = 1.$

Thus,

$$\begin{vmatrix} a & b \\ x' & y' \end{vmatrix} = \begin{vmatrix} t-\sqrt{1-t^2} & \sqrt{1-t^2}-t-1 \\ 1 & \dfrac{-t}{\sqrt{1-t^2}} \end{vmatrix} = (\frac{-t^2}{\sqrt{1-t^2}} + t) - (\sqrt{1-t^2} - t - 1) = \frac{-t^2}{\sqrt{1-t^2}} - \sqrt{1-t^2} + 2t + 1$$

$$= \frac{\sqrt{1-t^2}(2t-1) - 1}{\sqrt{1-t^2}}$$

Example: $z\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 1$, through the curve $2x = y^2$, $z = y$.

$y = t, \quad \Rightarrow \quad y' = 1$

$x = \dfrac{1}{2}t^2 \;\Rightarrow\; x' = t$

$z = t, \quad \Rightarrow z' = 1$

and,

$a = z = t$, $b = 1 \quad c = 1.$

Thus,

$$\begin{vmatrix} a & b \\ x' & y' \end{vmatrix} = \begin{vmatrix} t & 1 \\ t & 1 \end{vmatrix} = t - t = 0,$$

and

$$\begin{vmatrix} a & c \\ x' & z' \end{vmatrix} = \begin{vmatrix} t & 1 \\ t & 1 \end{vmatrix} = t - t = 0.$$

Therefore, there exist infinitely distinct solutions.

Example: Find the solution of the equation

$(x^2+2y^2)\frac{\partial z}{\partial x}-xy\frac{\partial z}{\partial y}=xz$, through the curves $x^2+z^2=\alpha^2$, $y=\beta$, where α and β are constants

$$x=t, \quad \Rightarrow \quad x'=1$$
$$y=\beta \quad \Rightarrow \quad y'=0$$
$$z=\sqrt{\alpha^2-t^2}, \quad \Rightarrow z'=\frac{-t}{\sqrt{\alpha^2-t^2}}$$

and,

$$a=x^2+2y^2=t^2+2\beta^2 \quad , \quad b=-xy=-\beta t \quad c=xz=t\sqrt{\alpha^2-t^2}\ .$$

Thus,

$$\begin{vmatrix} a & b \\ x' & y' \end{vmatrix}=\begin{vmatrix} t^2+2\beta^2 & -\beta t \\ 1 & 0 \end{vmatrix}=\beta t\neq 0,$$

Therefore, there exists a unique solution.

The characteristic equations are

$$\frac{dx}{(x^2+2y^2)}=\frac{dy}{-xy}=\frac{dz}{xz}$$

Thus,

$$\frac{dx}{(x^2+2y^2)}=\frac{dy}{-xy} \quad \Rightarrow \quad (x^2+2y^2)\,dy+xy\,dx=0 \quad \Rightarrow \quad x(x\,dy+y\,dy)+2y^2\,dy=0$$
$$xy\,d(xy)+2y^3\,dy=0 \quad \Rightarrow \quad \frac{1}{2}x^2y^2+\frac{1}{2}y^4=c \quad \Rightarrow \quad x^2y^2+y^4=c_1$$

$$\frac{dy}{-xy}=\frac{dz}{xz} \quad \Rightarrow \frac{dy}{-y}=\frac{dz}{z} \quad \Rightarrow \ \ln(y)+\ln(z)=c' \quad \Rightarrow \quad yz=c_2$$

$$\beta^2t^2+\beta^4=c_1 \qquad (*)$$
$$\beta\sqrt{\alpha^2-t^2}=c_2 \quad \Rightarrow -\beta^2t^2+\beta^2\alpha^2=c_2^2 \qquad (**)$$

From (*) and (**) we have

$c_1 + c_2^2 = \beta^2(\alpha^2 + \beta^2)$

Therefore, the general solution is

$$x^2 y^2 + y^4 + y^2 z^2 = \beta^2(\alpha^2 + \beta^2).$$

1.4 Partial differential equation of high-order

The general form of this type of equation is

$$a_\circ \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = f(x,y) \tag{1.12}$$

where, $a_\circ, a_1, a_2, \cdots, a_n$ are constants.

1.4.1 Partial differential operator

The operator *D* uses in the PDEs to represent the partial derivatives, and define as

$$\begin{aligned} D_x &= \frac{\partial}{\partial x}, \qquad D_y = \frac{\partial}{\partial y} \\ D_x^2 &= \frac{\partial^2}{\partial x^2}, \qquad D_y^2 = \frac{\partial^2}{\partial y^2} \\ &\vdots \\ D_x^n &= \frac{\partial^n}{\partial x^n}, \qquad D_y^n = \frac{\partial^n}{\partial y^n} \end{aligned} \tag{1.13}$$

Therefore, by using (1.13) equation (1.12) can be expressed as

$$(a_\circ D_x^n + a_1 D_x^{n-1} D_y + a_2 D_x^{n-2} D_y^2 + \cdots + a_n D_y^n) z = f(x,y) \tag{1.14}$$

In addition, equation (1.14) can be written also as

$$L(D_x, D_y) z = f(x,y) \tag{1.15}$$

In the case of $f(x,y) = 0$, equation (1.15) is called homogeneous equation, in contrast if $f(x,y) \neq 0$ equation (1.15) is called inhomogeneous equation. In the case of inhomogeneous equation (1.15), firstly we have to find the general solution for the homogeneous equation $L(D_x, D_y) z = 0$, and then the particular solution for inhomogeneous equation $L(D_x, D_y) z = f(x,y)$.