1.3.2 Solution through characteristic curves

To find the general solution of Lagrange's equation (1.4) through the curve Γ, we define the characteristic curves

$$\Gamma : \begin{cases} x = x(t) \\ y = y(t) \qquad t \in [a,b], \\ u = u(t) \end{cases} \tag{1.6}$$

the integral curves in Ω of the *characteristic system*

$$\begin{cases} \dfrac{dx}{dt} = a(x,y,u) \\ \dfrac{dy}{dt} = b(x,y,u) \\ \dfrac{du}{dt} = c(x,y,u). \end{cases} \tag{1.7}$$

The last system can be rewritten shortly as

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}, \tag{1.8}$$

which is an *autonomous system* of ODEs. Here, we need to find the relationship between c_1 and c_2 in the general solution ($F(c_1, c_2) = 0$).

Example: Find the solution of the equation

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} + z^2 = 0,$$

through the curve $xy = x + y, \quad z = 1.$

The characteristic equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

Thus,

$$\frac{dx}{x^2} = -\frac{dz}{z^2} \Rightarrow \frac{1}{x} + \frac{1}{z} = c_1 \Rightarrow \frac{x+z}{xz} = c_1$$

$$\frac{dy}{y^2} = -\frac{dz}{z^2} \Rightarrow \frac{1}{y} + \frac{1}{z} = c_2 \Rightarrow \frac{y+z}{yz} = c_2$$

Now, we can write the curves by using *t* as

$$z = 1, \quad x = t, \quad y = \frac{t}{t-1}.$$

Thus,

$$\frac{1+t}{t}=c_1 \Rightarrow \frac{1}{t}+1=c_1 \Rightarrow t=\frac{1}{c_1-1}$$

and

$$\frac{1+\frac{t}{t-1}}{\frac{t}{t-1}}=c_2 \Rightarrow \frac{t-1}{t}+1=c_2 \Rightarrow 1-\frac{1}{t}+1=c_2 \Rightarrow 2-\frac{1}{t}=c_2 \Rightarrow \frac{1}{\frac{1}{c_1-1}}=c_2$$

$$\Rightarrow 2-(c_1-1)=c_2 \quad \Rightarrow \quad c_1+c_2=3$$

Therefore,

$$\frac{x+z}{xz}+\frac{y+z}{yz}=3$$

Example: Find the solution of the equation

$$(x-y)\frac{\partial z}{\partial x}+(y-x-z)\frac{\partial z}{\partial y}=z,$$

through the curve $x^2+y^2=1$, $z=1$.

The characteristic equations are

$$\frac{dx}{(x-y)}=\frac{dy}{(y-x-z)}=\frac{dz}{z}$$

Thus,

$$\frac{dx}{(x-y)}=\frac{dz}{z} \qquad \Rightarrow \quad dx=\frac{x-y}{z}dz$$

$$\frac{dy}{(y-x-z)}=\frac{dz}{z} \quad \Rightarrow \quad dy=\frac{y-x-z}{z}dz$$

Thus,

$$dx+dy+dz=(\frac{x-y}{z}+\frac{y-x-z}{z}+1)dz=(\frac{x-y+y-x-z+z}{z})dz=0,$$

$$dx+dy+dz=0 \quad \Rightarrow \quad x+y+z=c_1.$$

Also,

$$\frac{dx-dy+dz}{2(x-y+z)}=\frac{dz}{z} \quad \Rightarrow \quad \ln(x-y+z)=2\ln(z)+c$$

$$\Rightarrow \ln(x-y+z)-\ln(z^2)=c \quad \Rightarrow \; x-y+z=c_2 z^2$$

Now, we can write the curves by using *t* as

$$z=1, \quad x=t, \quad y=\sqrt{1-t^2}\,.$$

Thus,

$$t+\sqrt{1-t^2}+1=c_1, \qquad (*)$$

$$t-\sqrt{1-t^2}+1=c_2, \qquad (**)$$

From (*) and (**), we have

$$2t+2=c_1+c_2 \quad \Rightarrow \quad t=\frac{c_1+c_2-2}{2}.$$

By substitution into (*)

$$\frac{c_1+c_2-2}{2}+\sqrt{1-\left(\frac{c_1+c_2-2}{2}\right)^2}+1=c_1,$$

$$\sqrt{1-\left(\frac{c_1+c_2-2}{2}\right)^2}=\left[c_1-\left(1+\frac{c_1+c_2-2}{2}\right)\right] \quad \Rightarrow \quad 1-\left(\frac{c_1+c_2-2}{2}\right)^2=\left[c_1-\left(1+\frac{c_1+c_2-2}{2}\right)\right]^2,$$

$$4-(c_1+c_2-2)^2=4\left[c_1^2-2c_1\left(1+\frac{c_1+c_2-2}{2}\right)+\left(1+\frac{c_1+c_2-2}{2}\right)^2\right]$$

$$c_1^2+c_2^2-2c_1-2c_2=0.$$

Thus,

$$(x+y+z)^2+\frac{(x-y+z)^2}{z^4}-2(x+y+z)-\frac{2(x-y+z)}{z^2}=0$$

Example: Find the solution of the equation

$$4yz\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+zy=0,$$

through the curve $x+z=2$, $y^2+z^2=1$.

The characteristic equations are

$$\frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y}$$

Thus,

$$\frac{dy}{1} = \frac{dz}{-2y} \quad \Rightarrow -y^2 = z + c \quad \Rightarrow \quad y^2 + z = c_1,$$

and,

$$\frac{dx}{4yz} = \frac{dz}{-2y} \Rightarrow -dx = 2zdz \quad \Rightarrow \quad -x = z^2 + c', \quad \Rightarrow \quad x + z^2 = c_2.$$

Now, we can write the curves by using t as

$$x = t, \quad z = 2 - t, \quad y = \sqrt{1-(2-t)^2}\ .$$

Thus,

$$1-(2-t)^2 + (2-t) = c_1, \quad \Rightarrow \quad -t^2 + 3t - 1 = c_1, \qquad (*)$$
$$t + (2-t)^2 = c_2, \quad \Rightarrow \qquad t^2 - 3t + 4 = c_2, \qquad (**)$$

From (*) and (**), we have

$$c_1 + c_2 = 3, \qquad \Rightarrow (y^2 + z) + (x + z^2) = 3.$$

1.3.3 Existence and uniqueness of solution

In this section we need to present the conditions which are used to show that whether there may not exist solution, there may exist solution, or there exist infinitely may distinct solutions. In other words, either there is no existence of solution or no uniqueness.

Consider the first-order PDE

$$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = c. \tag{1.9}$$

Here, we want to find the solution of equation (1.9) through the curve which is defined by

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

Now,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy, \quad \Rightarrow \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\ .$$

Thus, we have

$$x'\frac{\partial z}{\partial x}+y'\frac{\partial z}{\partial y}=z'. \tag{1.10}$$

Then, from equations (1.9) and (1.10), we have the following system

$$\begin{pmatrix} a & b \\ x' & y' \end{pmatrix}\begin{pmatrix} \dfrac{\partial z}{\partial x} \\ \dfrac{\partial z}{\partial y} \end{pmatrix}=\begin{pmatrix} c \\ z' \end{pmatrix}. \tag{1.11}$$

Now,

1) If $\begin{vmatrix} a & b \\ x' & y' \end{vmatrix} \neq 0 \quad \Rightarrow \quad ay'-bx' \neq 0$

Then, the system (1.11) has a unique solution.

2) $\begin{vmatrix} a & b \\ x' & y' \end{vmatrix} = 0 \quad \Rightarrow \quad ay'-bx' = 0$, then

2a) $\begin{vmatrix} a & c \\ x' & z' \end{vmatrix} = 0 \quad \Rightarrow \quad az'-cx' = 0.$

Then, there exist infinitely distinct solutions.

2b) $\begin{vmatrix} a & c \\ x' & z' \end{vmatrix} \neq 0 \quad \Rightarrow \quad az'-cx' \neq 0.$

Then, there may not exist a solution.