## 1.3 First-order Partial Differential Equations

The general form of a first-order PDE for a function $u = u(x_1, ..., x_n)$ of $n$ independent variables $(x_1, ..., x_n)$ is

$$F(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_n}) = 0, \tag{1.2}$$

where $F$ is a given function and $u_{x_j} = \partial u/\partial x_j$, $j = 1, ..., n$ are the partial derivatives of the unknown function $u$. In the case of two independent variables $x, y$ the above form is

$$F(x, y, u, u_x, u_y) = 0. \tag{1.3}$$

### 1.3.1 Lagrange's Equation

The general form of first-order partial differential equations with dependent variable *z* and two independent variables *x* and *y* can be expressed in the form

$$a(x,y,z)\frac{\partial z}{\partial x} + b(x,y,z)\frac{\partial z}{\partial y} = c(x,y,z), \tag{1.4}$$

where its coefficients *a*, *b* and *c* are functions of *x*, *y* and *z*. Equation (1.4) is called Lagrange's equation

- If *p* and *q* are functions of *x* and *y* only, then the equation (1.4) is called a linear equation.

**Theorem (1.2):** the general solution of Equation (1.4) is $F(u,v) = 0$, where F is arbitrary function of $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$, which are solution curves of characteristic equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} \tag{1.5}$$

The solution curves defined by $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ are called the family of characteristic curves of equation (1.4).

Example: Find the solution of the following PDEs

1) $(x+1)\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2z$

The characteristic equations are

$$\frac{dx}{(x+1)} = -\frac{dy}{y} = \frac{dz}{2z}$$

Thus,

$$\frac{dx}{(x+1)} = -\frac{dy}{y} \Rightarrow \ln(x+1) = -\ln(y) + c \Rightarrow \ln(x+1) + \ln(y) = c \Rightarrow \ln(y(x+1)) = c$$
$$y(x+1) = c_1 \Rightarrow u(x,y,z) = y(x+1).$$

Also,

$$\frac{dx}{(x+1)} = \frac{dz}{2z} \Rightarrow 2\ln(x+1) = \ln(z) + c' \Rightarrow \ln(x+1)^2 - \ln(z) = c' \Rightarrow \ln(\frac{(x+1)^2}{z}) = c'$$
$$\frac{1}{z}(x+1)^2 = c_2 \Rightarrow v(x,y,z) = \frac{1}{z}(x+1)^2.$$

Therefore the solution of PDE can be written as:

$$F(u,v) = 0 \quad \Rightarrow F(y(x+1), \frac{1}{z}(x+1)^2).$$

2) $(y+z)\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x - y$

The characteristic equations are

$$\frac{dx}{(y+z)} = \frac{dy}{y} = \frac{dz}{(x-y)}.$$

Thus,

$$\frac{dx+dz}{x+z} = \frac{dy}{y} \Rightarrow \frac{d(x+z)}{x+z} = \frac{dy}{y} \Rightarrow \ln(x+z) = \ln(y) + c \Rightarrow \ln(x+z) - \ln(y) = c$$
$$\Rightarrow \ln(\frac{x+z}{y}) = c \Rightarrow \frac{x+z}{y} = c_1 \Rightarrow u(x,y,z) = \frac{x+z}{y}.$$

Also,

$$\frac{dx-dy}{z} = \frac{dz}{x-y} \Rightarrow \frac{d(x-y)}{z} = \frac{dz}{x-y} \Rightarrow (x-y)\,d(x-y) = z\,dz \Rightarrow \frac{1}{2}(x-y)^2 - \frac{1}{2}z^2 = c'$$
$$\Rightarrow z^2 - (x-y)^2 = c_2 \Rightarrow v(x,y,z) = z^2 - (x-y)^2.$$

Therefore the solution of PDE is

$$F(u,v) = 0 \quad \Rightarrow F(\frac{x+z}{y}, z^2 - (x-y)^2).$$

3) $(y-z)\frac{\partial z}{\partial x} + (x-y)\frac{\partial z}{\partial y} = z - x$

The characteristic equations are

$$\frac{dx}{(y-z)} = \frac{dy}{(x-y)} = \frac{dz}{(z-x)} .$$

Thus,

$$\frac{dx}{(y-x)} = \frac{dz}{(z-x)} \;\Rightarrow\; dx = (\frac{y-x}{z-x})dz,$$

$$\frac{dy}{(x-y)} = \frac{dz}{(z-x)} \;\Rightarrow\; dy = (\frac{x-y}{z-x})dz,$$

$$dx + dy + dz = (\frac{y-x}{z-x})dz + (\frac{x-y}{z-x})dz + dz = (\frac{y-x+x-y+z-x}{z-x})dz = 0$$

$$dx + dy + dz = 0 \;\Rightarrow\; x + y + z = c_1 \quad \Rightarrow u(x, y, z) = x + y + z$$

Also,

$$xdx + zdy + ydz = x(\frac{y-x}{z-x})dz + z(\frac{x-y}{z-x})dz + ydz = 0$$

$$xdx + zdy + ydz = 0 \quad \Rightarrow \; xdx + d(yz) = 0 \;\Rightarrow\; \frac{1}{2}x^2 + yz = c_2 \;\Rightarrow\; v(x, y, z) = \frac{1}{2}x^2 + yz$$

Therefore the solution of PDE is

$$F(u,v) = 0 \quad \Rightarrow F(x + y + z, \frac{1}{2}x^2 + yz).$$