M437

Partial Differential Equations

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Chapter 1

Partial Differential Equations

Definition:

Let $u = u(x_1, ..., x_n)$ be a function of n independent variables $x_1, ..., x_n$. A Partial Differential Equation (PDE for short) is an equation that contains the independent variables $x_1, ..., x_n$, the dependent variable or the unknown function u and its partial derivatives up to some order. It has the form

$$F(x_1, ..., x_n, u, u_{x_1}, ..., u_{x_n}, u_{x_1 x_1}, ..., u_{x_i x_j}, ...) = 0, \qquad (1.1)$$

where F is a given function and $u_{x_j} = \partial u / \partial x_j$, $u_{x_i x_j} = \partial^2 u / \partial x_i \partial x_j$, $i, j = 1, ..., n$ are the partial derivatives of u.

Definition: The *order* of a PDE is the order of the highest derivative which appears in the equation.

Definition: A *solution* of the equation (1.1) we mean a function u such that the substitution of u and its derivatives up to the order in (1.1) makes it an identity

Some examples of PDEs (all of which occur in Physics) are:

1. $u_x + u_y = 0$ (transport equation)

2. $u_x + u u_y = 0$ (shock waves)

3. $u_x^2 + u_y^2 = 1$ (eikonal equation)

4. $u_{tt} - u_{xx} = 0$ (wave equation)

5. $u_t - u_{xx} = 0$ (heat or diffusion equation)

6. $u_{xx} + u_{yy} = 0$ (Laplace equation)

7. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$ (biharmonic equation)

8. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)

Each one of these equations has two independent variables denoted either by x, y or x, t. Equations 1, 2 and 3 are of first-order. Equations numbered as 4, 5, 6, 8 , 9 are of second-order; 7 is of fourth-order.

1.1Elimination of arbitrary functions

The partial differential equations can be obtained by eliminating the arbitrary functions as explain in the following examples:

Example: find the partial differential equation by eliminating the arbitrary function

1) $u = f(x+y) \Rightarrow \frac{\partial u}{\partial x} = f' \;\; and \;\; \frac{\partial u}{\partial y} = f' \;\; \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

2) $u = f(xy) \Rightarrow \frac{\partial u}{\partial x} = yf' \;\; and \;\; \frac{\partial u}{\partial y} = xf' \;\; \Rightarrow x\frac{\partial u}{\partial x} = y\frac{\partial u}{\partial y}$

3) $u(x,y) = yf(x) + xg(y) \Rightarrow \frac{\partial u}{\partial x} = yf' + g \;\; \Rightarrow f' = \frac{1}{y}(\frac{\partial u}{\partial x} - g)$

and

$\frac{\partial u}{\partial y} = f + xg' \;\; \Rightarrow g' = \frac{1}{x}(\frac{\partial u}{\partial y} - f)$

$\frac{\partial^2 u}{\partial x \partial y} = f' + g' = \frac{1}{y}(\frac{\partial u}{\partial x} - g) + \frac{1}{x}(\frac{\partial u}{\partial y} - f),$

$Thus,$

$xy\frac{\partial^2 u}{\partial x \partial y} = x(\frac{\partial u}{\partial x} - g) + y(\frac{\partial u}{\partial y} - f) = x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} - (xf + yg)$

$Therefore,$

$xy\frac{\partial^2 u}{\partial x \partial y} - x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = u$

H.W.

1) $u(x,y) = f(x+y) + g(x-y)$
2) 1) $u(x,y) = f(x-3y) + g(2x-y)$

1.2Elimination of arbitrary constants

The partial differential equations can be obtained by eliminating the arbitrary constants as explain in the following examples:

Example: find the partial differential equation by eliminating the arbitrary constants

1) $u = ax^2 + by^2 + ab$ (*)

$u_x = 2ax \Rightarrow a = \frac{1}{2x} u_x$

$u_y = 2by \Rightarrow b = \frac{1}{2y} u_y$

by substituting in (*), *we have*

$$u = \frac{1}{2} x u_x + \frac{1}{2} y u_y + \frac{1}{4xy} u_x u_y$$

2) $z(x, y) = (x^2 + a)(y^2 + b)$ (**)

$z_x = 2x(y^2 + b) \Rightarrow (y^2 + b) = \frac{1}{2x} z_x$

$z_y = 2y(x^2 + a) \Rightarrow (x^2 + a) = \frac{1}{2y} z_y$

by substituting in (**), *we have*

$$z = (\frac{1}{2y} z_y)(\frac{1}{2x} z_x) = \frac{1}{4xy} z_x z_y$$

Definition: (degree of PDEs) is the degree of the highest order partial derivative occurring in the equation.

Definition: A PDE is linear if it is linear in the unknown function and its derivatives**.**

Note: The PDEs is called linear PDEs if it is satisfied the following conditions:

1- All the derivatives from first order and do not occur as products.

2- The dependent variable does not occur as product with derivative, raised to power or in non-linear function.

Example:

a) $x u_{xx} + y^2 u_{xy} = \tan x$; Linear/ Second-order/ First-degree.

b) $x y u_{xy} = y^3 (u_{yy})^2$; non-Linear/ Second-order/ Second-degree.

c) $y u_{xxx} + x u_{xy} = u^2$; Non-Linear/ Third-order/ First-degree.

Theorem (1-1): Let $u = u(x, y, z)$ and $v = v(x, y, z)$ be independent functions of x, y *and* z, $\phi(u, v) = 0$, and $z = z(x, y)$. Then $Pz_x + Qz_y = R$, where

$$P = u_y v_z - u_z v_y, \quad Q = u_z v_x - u_x v_z \quad \text{and} \quad R = u_x v_y - u_y v_x$$

Proof:

$$\phi_x(u,v) = \phi_u u_x + \phi_v v_x = 0,$$

$$u_x = u_x \frac{dx}{dx} + u_y \frac{dy}{dx} + u_z \frac{dz}{dx} = u_x + u_z z_x,$$

$$v_x = v_x \frac{dx}{dx} + v_y \frac{dy}{dx} + v_z \frac{dz}{dx} = v_x + v_z z_x.$$

Thus,

$$\phi_u(u_x + u_z z_x) + \phi_v(v_x + v_z z_x) = 0. \quad (1)$$

Similarly,

$$\phi_u(u_y + u_z z_y) + \phi_v(v_y + v_z z_y) = 0. \quad (2)$$

Therefore, we have the system

$$\begin{bmatrix} u_x + u_z z_x & v_x + v_z z_x \\ u_y + u_z z_y & v_y + v_z z_y \end{bmatrix} \begin{bmatrix} \phi_u \\ \phi_v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

To solve this system, we should have,

$$\begin{vmatrix} u_x + u_z z_x & v_x + v_z z_x \\ u_y + u_z z_y & v_y + v_z z_y \end{vmatrix} = 0$$

$$(u_x + u_z z_x)(v_y + v_z z_y) - (v_x + v_z z_x)(u_y + u_z z_y) = 0,$$

$$u_x v_y + u_x v_z z_y + v_y u_z z_x + u_z z_x v_z z_y - v_x u_y - v_x u_z z_y - u_y v_z z_x - v_z z_x u_z z_y = 0,$$

$$(v_y u_z - u_y v_z) z_x + (u_x v_z - v_x u_z) z_y = v_x u_y - u_x v_y,$$

$$(u_y v_z - v_y u_z) z_x + (v_x u_z - u_x v_z) z_y = u_x v_y - v_x u_y,$$

$$P z_x + Q z_y = R.$$

Example: Find the PDE, which has a general solution $\phi(\frac{z}{x^3}, \frac{y}{x}) = 0$, where ϕ is an arbitrary function and $z = z(x,y)$.

First method:

Let $u = \frac{z}{x^3}$ *and* $v = \frac{y}{x}$ $\Rightarrow \phi(u,v) = 0$.

$$\phi_x(u,v) = \phi_u u_x + \phi_v v_x = 0,$$

$$u_x = u_x + u_z z_x \quad \text{and} \quad v_x v_x + v_z z_x$$

$$\phi_u(u_x + u_z z_x) + \phi_v(v_x + v_z z_x) = 0. \qquad (1)$$

$$\phi_u(u_y + u_z z_y) + \phi_v(v_y + v_z z_y) = 0. \qquad (2)$$

and

$$\begin{vmatrix} u_x + u_z z_x & v_x + v_z z_x \\ u_y + u_z z_y & v_y + v_z z_y \end{vmatrix} = 0$$

$$(u_x + u_z z_x)(v_y + v_z z_y) - (v_x + v_z z_x)(u_y + u_z z_y) = 0,$$

$$u_x v_y + u_x v_z z_y + v_y u_z z_x + u_z z_x v_z z_y - v_x u_y - v_x u_z z_y - u_y v_z z_x - v_z z_x u_z z_y = 0,$$

$$(v_y u_z - u_y v_z) z_x + (u_x v_z - v_x u_z) z_y = v_x u_y - u_x v_y,$$

$$(u_y v_z - v_y u_z) z_x + (v_x u_z - u_x v_z) z_y = u_x v_y - v_x u_y. \qquad (3)$$

Since,

$$u_x = \frac{-3z}{x^4}, \quad u_y = 0, \quad u_z = \frac{1}{x^3},$$

$$v_x = \frac{-y}{x^2}, \quad v_y = \frac{1}{x}, \quad u_z = 0$$

By substituting into (3), we have

$$\left((\frac{1}{x^3})(\frac{1}{x}) - (0)(0) \right) z_x + \left((\frac{-3z}{x^4})(0) - (\frac{-y}{x^2})(\frac{1}{x^3}) \right) z_y = (\frac{-y}{x^2})(0) - (\frac{-3z}{x^4})(\frac{1}{x}),$$

$$\frac{1}{x^4} z_x + \frac{y}{x^5} z_y = \frac{3z}{x^5} \quad \Rightarrow \; x z_x + y z_y = 3z.$$

Second method :(Using Theorem (1-1))

Since

$$u_x = \frac{-3z}{x^4}, \quad u_y = 0, \quad u_z = \frac{1}{x^3},$$

$$v_x = \frac{-y}{x^2}, \quad v_y = \frac{1}{x}, \quad u_z = 0$$

Thus,

$$P = u_y v_z - u_z v_y = (0)(0) - (\frac{1}{x^3})(\frac{1}{x}) = \frac{1}{x^4},$$

$$Q = u_z v_x - u_x v_z = (\frac{1}{x^3})(\frac{-y}{x^2}) - (\frac{-3z}{x^4})(0) = \frac{-y}{x^5},$$

$$R = u_x v_y - u_y v_x = (\frac{-3z}{x^4})(\frac{1}{x}) - (0)(\frac{-y}{x^2}) = \frac{-3z}{x^5}.$$

Therefore, by applying theorem (1-1), the PDE is given by

$$-\frac{1}{x^4} z_x - \frac{y}{x^5} z_y = -\frac{3z}{x^5} \quad \Rightarrow \; xz_x + yz_y = 3z.$$