UNIT FOUR

EXPONENTIAL AND LOGARITHM FUNCTIONS

4.1 Exponential Functions

In this chapter we study a new class of functions called exponential functions. For example,

$f(x) = 2^x$

is an exponential function (with base 2). Notice how quickly the values of this function increase.

$f(3) = 2^3 = 8$, $f(10) = 2^{10} = 1024$, $f(30) = 2^{30} = 1{,}073{,}741{,}824$

Compare this with the function $g(x) = x^2$, where $g(30) = 30^2 = 900$. The point is that when the variable is in the exponent, even a small change in the variable can cause a dramatic change in the value of the function

Definition ■ Exponential Functions

The exponential function with base a is defined for all real numbers x by

f(x) **=** a^x

Where a > 1 **a** $\neq 1$

Remark We assume that a $\neq$ 1 because the function f(x) = 1^x =1 is just a constant function .

Here are some examples of exponential functions:

f(x) **=** 2^x **,** ***g*(x)** **=** 3^x **,** **h(x) =** 10^x

base 2 **base 3** **base 10**

Example 1 ■ Evaluating Exponential Functions

Let f(x) = 3^x , and evaluate the following:

a) f(5) , b) $f(-\frac{2}{3})$, c) f(π) , d) $f(\sqrt{2})$

Solution We use a calculator to obtain the values of f .

a) $f(5) = 3^5 = 24$, b) $f(-\frac{2}{3}) = 3^{-\frac{2}{3}} \approx 0.4807$,

c) $f(\pi) = 3^{\pi} \approx 31.544$, d) $f(\sqrt{2}) = 3^{\sqrt{2}} \approx 4.7288$

■ Graphs of Exponential Functions

We first graph exponential functions by plotting points. We will see that the graphs of such functions have an easily recognizable shape.

Example 2 ■ **Graphing Exponential Functions by Plotting Points**

Draw the graph of each function .

a) $f(x) = 3^x$, **b)** $g(x) = (\frac{1}{3})^x$

Solution We calculate values of **f(x)** and **g(x)** and plot points to sketch the graphs in Figure 1 .

x	$f(x) = 3^x$	$g(x) = (\frac{1}{3})^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

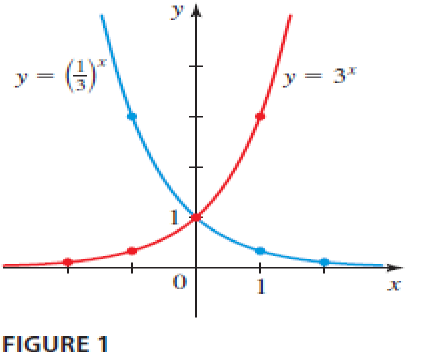


FIGURE 1

Notice that $g(x) = (\frac{1}{3})^x = \frac{1}{3^x} = 3^{-x} = f(-x)$

so we could have obtained the graph of **g** from the graph of **f** by reflecting in the y-axis.

Homework Graph both functions on one set of axes.

F(x) = 2^x and g(x) = 2^{-x}

Family of Exponential Functions

Figure 2 shows the graphs of the family of exponential functions **f(x)=a^x** for various values of the base **a** . All of these graphs pass through the point **(0, 1)** because **a^0=1** for **a≠0**. You can see from **Figure2** that there are two kinds of exponential if **0**< **a** < **1** , the exponential function decreases rapidly. If **a**> **1** , the function increases rapidly (see the margin note).

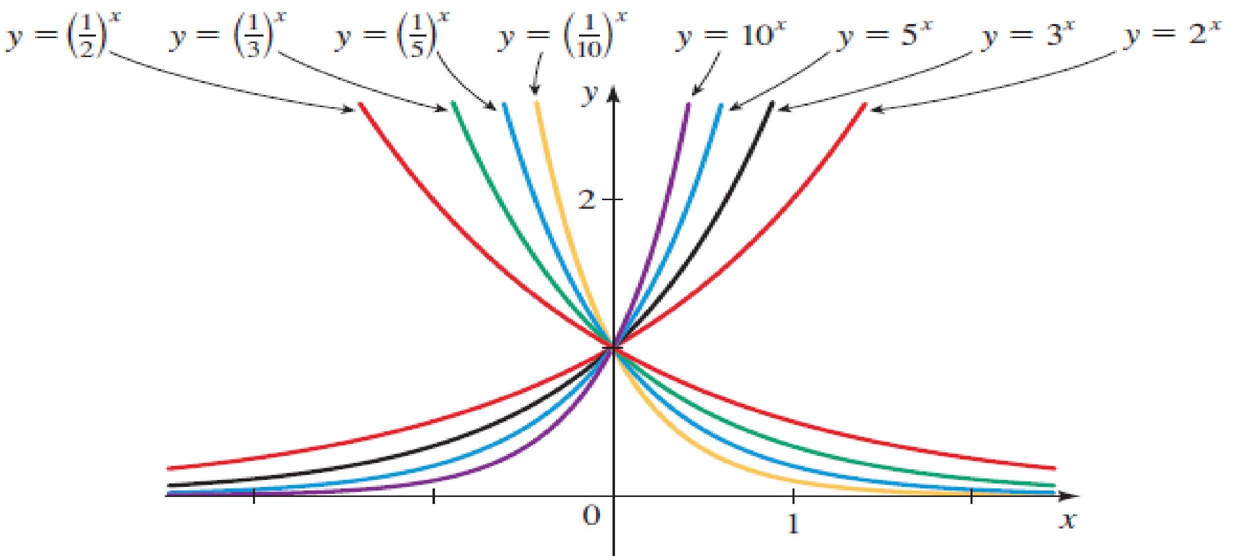


Figure 2 family of exponential functions

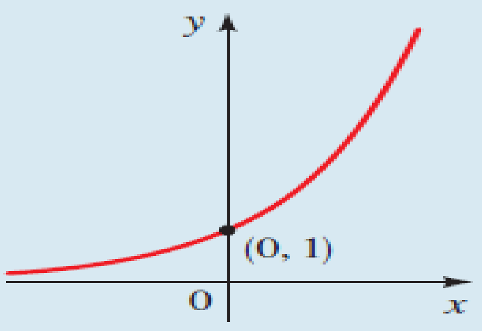
The x-axis is a horizontal asymptote for the exponential function **f(x)=a^x** . This is because when **a**>**1 ,** we have a^x →**0** as **x**→ $-\infty$, and when **0**< **a** < **1 ,** we have a^x →**0** as **x**→ ∞(see Figure 2). Also , a^x >**0 for all x** ∈ R , so the function **f(x)=a^x** has domain R and range and **(0 , ∞)** . These observations are summarized in the following box .

Graphs of Exponential Functions

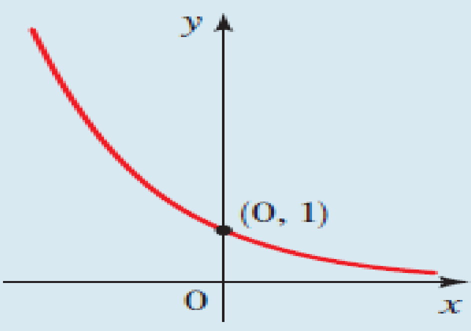
The exponential function

$$f(x) = a^x \qquad a > 0 \quad , \ a \neq 1$$

has domain **R** and range (∞ , 0) . The line *y* = 0 (the x-axis) is a horizontal asymptote of f . The graph of f has one of the following shapes.



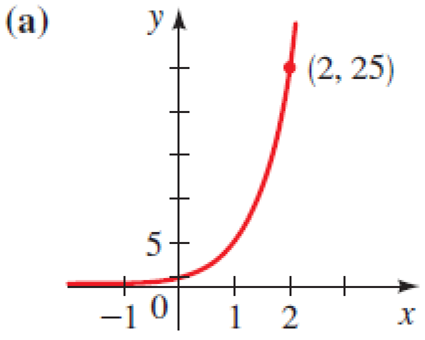
$f(x) = a^x$ for $a > 1$

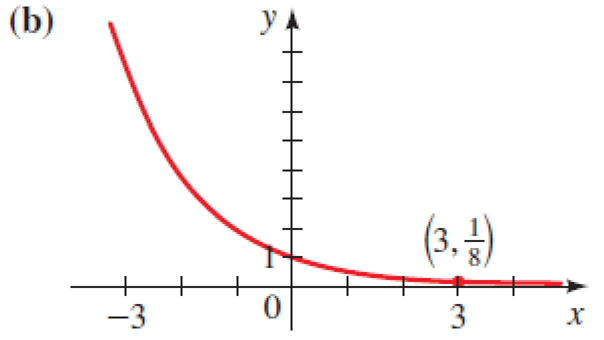


$f(x) = a^x$ for $0 < a < 1$

Example 3 ■ Identifying Graphs of Exponential Functions

Find the exponential function **f(x) =** a^x whose graph is given .





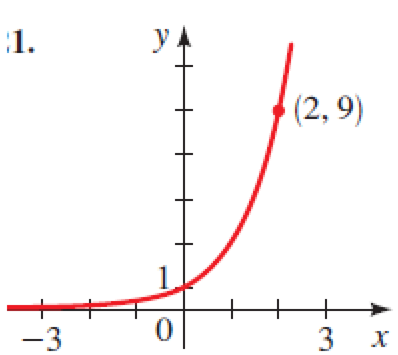
Solution

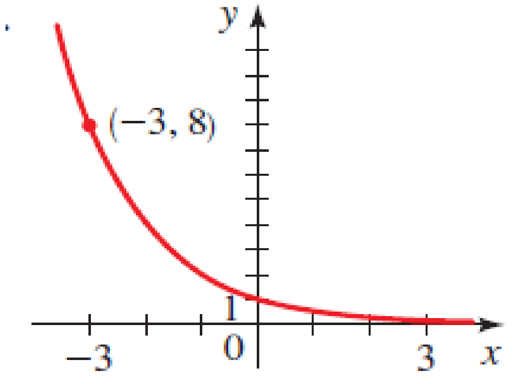
(a) Since f(2) = a^2 = 25 , we see that the base is *a* = *5* . So f(x) = 5^x .

(b) Since f(3) = a^3 = $\frac{1}{8}$, we see that the base is *a* = $\frac{1}{2}$. So f(x) = $(\frac{1}{2})^x$.

Homework : ■

Exponential Functions from a Graph Find the exponential function f(x) = a^x whose graph is given .





Example 4 ■ Transformations of Exponential Functions

Use the graph of f(x) = 2^x to sketch the graph of each function. State the domain , range, and asymptote.

a) g(x) = 1 + 2 ^{x} **b) h(x)= - 2** ^{x} **c) k(x) =** 2^{x-1}

Solution

(a) To obtain the graph of **g(x) = 1 + 2** ^{x} **,** we start with the graph of **g(x) = 2** ^{x} and shift it upward 1 unit to get the graph shown in Figure 3(a). From the graph we see that the domain of **g** is the set **R** of real numbers, the range is the interval **(1 , ∞)** and the line **y =1** is a horizontal asymptote .

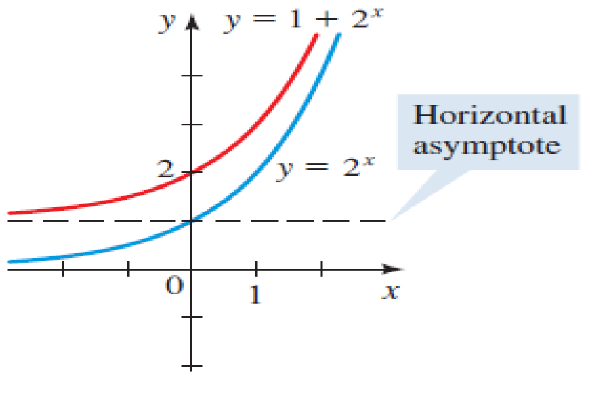


Figure 3(a)

(b) Again we start with the graph of f(x) = 2^x , but here we reflect in the *x*-axis to get the graph of **h(x)=** -2^x shown in **Figure 3(b**). From the graph we see that the domain of **h**is the set **R** of all real numbers, the range is the interval **(0 ,** $-\infty$) and the line ***y* = 0** is a horizontal asymptote.

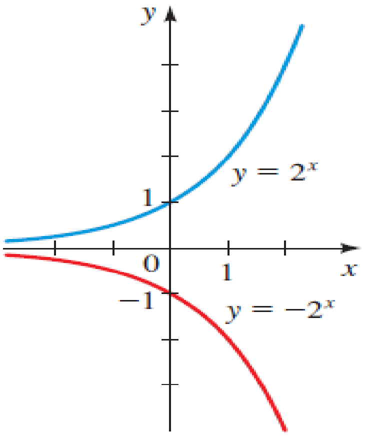


Figure 3(b)

(c) This time we start with the graph of **f(x)** = 2^x and shift it to the right by **1** unit to get the graph of **k(x) =** 2^{x-1}shown in Figure **3(c**). From the graph we see that the domain of **k**is the set **R** of all real numbers, the range is the interval **(0 ,** ∞) and the line ***y* = 0** is a horizontal asymptote.

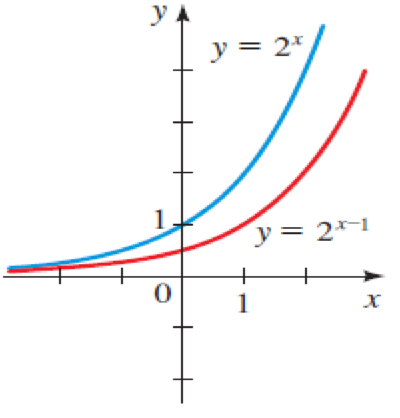


Figure 3(c)

Remark: In the **example (4)**we see how to graph certain functions, not by plotting points, but by taking the basic graphs of the exponential functions in Figure 2 and applying the shifting and reflecting transformations .

Homework■ Graphing Exponential Functions Graph the function, not by plotting points, but by starting from the graphs in **Figure 2**. State the domain, range, and asymptote.

1) g(x) = $2^x - 3$ 2) f(x) = -3^x 3) f(x)= 10^{x+3}

■ Compound Interest

Exponential functions occur in calculating compound interest. If an amount of money **P**, called the principal, is invested at an interest rate **i** per time period, then after one time period the interest is **Pi** , and the amount **A** of money is

A=p + pi =p(1 + i)

If the interest is reinvested, then the new principal is **p(1 + i) ,** and the amount after another time period is A =**p**(1 + i) (1 + i)= $p(1+i)^2$. Similarly, after a third time period the amount is **A**= $p\,(1+i)^3$. In general, after **k**periods the amount is

A $= p\,(1+i)^k$

Notice that this is an exponential function with base **1 + i** .

If the annual interest rate is **r**and if interest is compounded **n**times per year, then in each time period the interest rate is **i** $= \frac{r}{n}$, and there are **nt** time periods in **t** years. This leads to the following formula for the amount after *t* years .

Compound Interest

Compound interest is calculated by the formula

$$A(t) = p\,(1+\frac{r}{n})^{nt}$$

where $A(t) =$ amount after t years

$P=$ principal

$r =$ interest rate per year

$n =$ number of times interest is compounded per year

$t=$ number of years

Example 6 ■ Calculating Compound Interest

A sum of $**1000** is invested at an interest rate of **12%** per year . Find the amounts in the account after **3** years if interest is compounded annually , semiannually, quarterly monthly, and daily .

Solution We use the compound interest formula with p=$ 1000 , r= 0.12 , and t =3 .

Compounding	n	A mount after 3 years
Annual	1	$1000\,(1+\frac{0.12}{1})^{1(3)}$ =$ 1404.93
Semiannual	2	$1000\,(1+\frac{0.12}{2})^{2(3)}$ =$ 1418.52
Quarterly	4	$1000\,(1+\frac{0.12}{4})^{4(3)}$ =$ 1425.76
Monthly	12	$1000\,(1+\frac{0.12}{12})^{12(3)}$ =$ 1430.77
Daily	365	$1000\,(1+\frac{0.12}{365})^{365(3)}$ =$ 1433.24

Homework :

Compound Interest If **$10,000** is invested at an interest rate of **3%** per year, compounded semiannually, find the value of the investment after the given number of years.

a) 5 years b) 10 years c) 15 years

4.2 The Natural Exponential Function

Any positive number can be used as a base for an exponential function. In this section we study the special base e, which is convenient for applications involving calculus .

■ The Number *e*

The number e is defined as the value that $\mathbf{(1+\frac{1}{n})^{n}}$ approaches as **n** becomes large. (In calculus this idea is made more precise through the concept of a limit.) . The table shows the values of the expression $\mathbf{(1+\frac{1}{n})^{n}}$for increasingly large values of **n** .

It appears that, rounded to five decimal places **e** ≈ **2.71828** ; in fact, the approximate value to **20** decimal places is

e ≈2.71828182845904523536

n	$(1+\frac{1}{n})^{n}$
1	2.00000
2	2.48832
10	2.59374
100	2.70481
10000	2.71815
100000	2.71827
1000000	2.71828

Remark : It can be shown that **e** is an irrational number, so we cannot write its exact value in decimal form.

Definition (The Natural Exponential Function)

The natural exponential function is the exponential function

$$\mathbf{f(x)} = e^x$$

with base e . It is often referred to as *the* exponential function .

Remark

Since **2**< **e** <**3** , the graph of the natural exponential function lies between the graphs of y= 2^x and y= 3^x, as shown in **Figure 1** . Scientific calculators have a special key for the function **f(x)** $= e^x$. We use this key in the next example.

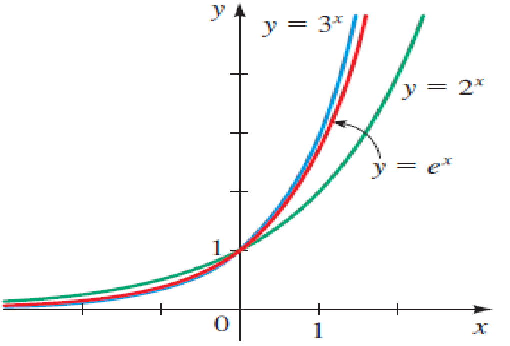


FIGURE 1 Graph of the natural exponential function

Example 1■ Evaluating the Exponential Function

Evaluate each expression rounded to five decimal places .

a) e^3 b) $2e^{-0.53}$ c) $e^{4.8}$

Solution We use the key on a calculator to evaluate the exponential function e^x .

a) $e^3 \approx 20.08554$ **b)** $2e^{-0.53} \approx 1.17721$ **c)** $e^{4.8} \approx 121.51042$

Example 2 ■ Graphing the Exponential Functions

Sketch the graph of each function. State the domain , range, and asymptote .

a) f(x)= e^{-x} **b)** g(x) $= e^{0.5x}$

(a) We start with the graph of **y**= e^x and reflect

(b) in the *y*-axis to obtain the graph of **f(x)** $= e^{-x}$ as in **Figure 2**. From the graph we see that the domain of **f** is the set **R** of all real numbers, the range is the interval **(0 , ∞)** and the line *y* =0 is a horizontal asymptote.

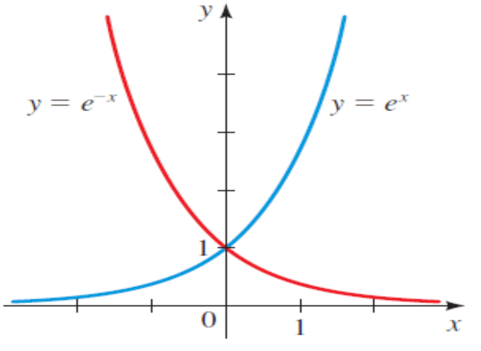


FIGURE 2

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve. The graph is shown in **Figure 3 .** From the graph we see that the domain of **g** is the set **R** of all real numbers, the range is the interval **(0 , ∞) ,** and the line ***y*** =**0** is a horizontal asymptote.

n	g(x) $= e^{0.5x}$
− 3	0.67
− 2	1.10
− 1	1.82
0	3.00
1	4.95
2	8.15
3	13.45

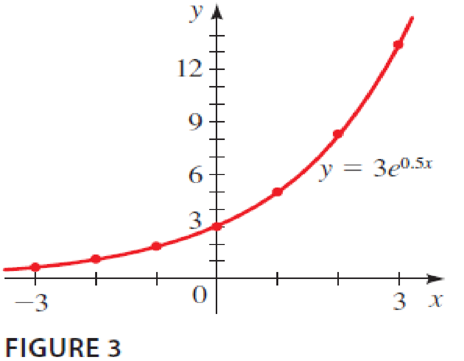


FIGURE 3

Homework : (1) Graphing Exponential Functions Complete the table of values, rounded to two decimal places, and sketch a graph of the function.

x	f(x) $= 1.5\, e^{x}$
-2	
-1	
-0.5	
0	
0.5	
1	
2	

(2) Graphing Exponential Functions Graph the function , not by plotting points, but by starting from the graph of **f(x) $=$1.5 e^{x}in Figure 1** . State the domain, range, and asymptote **. f(x)** $=$**2+e^{x}**

■ C continuously Compounded Interest

In Example **6** of Section **4.1** we saw that the interest paid increases as the number of compounding periods n increases. Let's see what happens as n increases indefinitely. If we let m$= \frac{n}{r}$, then

$A = p\,(1+\frac{r}{n})^{nt} = p[\,(1+\frac{r}{n})^{\frac{n}{r}}\,]^{rt} = p[\,(1+\frac{1}{m})^{m}\,]^{rt}$

Recall that as m becomes large, the quantity $\mathbf{(1+\frac{1}{m})^{m}}$approaches the number $\boldsymbol{e}$. Thus the amount approaches **A** $=$**pe^{rt}** . This expression gives the amount when the interest is compounded at "every instant ."

Definition (Continuously Compounded Interest)

Continuously compounded interest is calculated by the formula

$$A(t) = pe^{rt}$$

where A(t) $=$ amount after t years

P$=$ principal

r $=$ interest rate per year

t $=$number of years

Example 4 ■ Calculating Continuously Compounded Interest

Find the amount after 3 years if **\$1000** is invested at an interest rate of **12%** per year , compounded continuously .

Solution We use the formula for continuously compounded interest with P $=$\$1000 , r $=$0.12 and t $=$3 to get

A(3) $= 1000\, e^{(0.12)3} = 1000\, e^{0.36} = \$\ 1433.33$

Compare this amount with the amounts in Example **6** of Section **4.1.**

Homework : Compound Interest If **\$2000** is invested at an interest rate of **3.5%** per year, compounded continuously, find the value of the investment after the given number of years.

(a) 2 years

(b) 4 years

(c) 12 years

4.3 Logarithmic Functions

Every exponential function f(x) $= a^x$, with **a** $>$ **0** and **a**$\neq$ **1** , is a **one-to-one** function by the Horizontal Line Test (see **Figure 1** for the case **a** $>$**1**) and therefore has an inverse function . The inverse function f^{-1} is called the *logarithmic function with base a* and is denoted by $\log_a$. Recall that $f^{-1}$is defined by

f^{-1}(x) =**y** $\Leftrightarrow$ f(y) =**x**

This leads to the following definition of the logarithmic function .

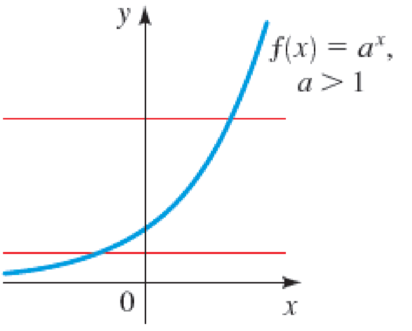


FIGURE 1 $f(x) = a^x$ is one-to-one.

Definition of the Logarithmic Function

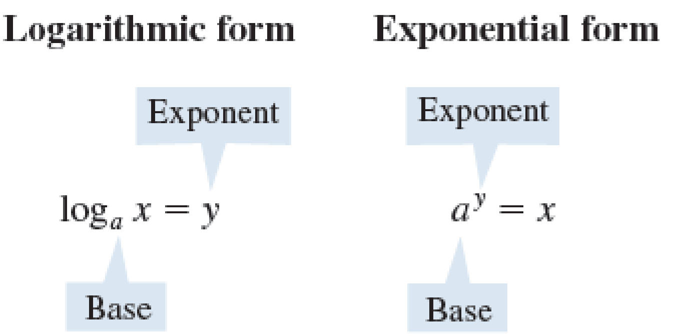
Let a be a positive number with **a** $\neq$ **1** . The logarithmic function with base **a** , denoted By $\log_a$is defined by

$\log_a x$ =**y** $\Leftrightarrow a^y$ =**x**

So $\log_a$is the *exponent* to which the base **a**must be raised to give **x**.

Remark :

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ andthe **exponential form**a^y =**x ,** it is helpful to notice that, in both forms, the base is the same .



Example 1 ■ Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations .

Logarithmic form	Exponential form
$\log_{10} 100,000 = 5$	$10^5 = 100,000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2(\frac{1}{8}) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

Homework:■ Logarithmic and Exponential Forms Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1 .

a)

Logarithmicform	Exponential form
$log_8 8 = 1$	------------
$log_8 8 = 1$	----------
-------------	$8^{2/3} = 4$
-------------	$8^3 = 512$
$log_8(\frac{1}{8}) = -1$	-----------
------------	$8^{-2} = \frac{1}{64}$

b)

Logarithmicform	Exponential form
-----------	$4^3 = 64$
$log_4 2 = \frac{1}{2}$	-----------
-----------	$4^{3/2} = 8$
$log_4(\frac{1}{16}) = -2$	-----------
$log_4(\frac{1}{2}) = -\frac{1}{2}$	-----------
------------	$8^{-5/2} = \frac{1}{32}$

Remark : It is important to understand that $\log_a x$ is an exponent. For example, the numbers in the right-hand column of the table in the margin are the logarithms (**base 10**) of the numbers in the left-hand column. This is the case for all bases, as the following example illustrates.

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

Remark : When we apply the Inverse Function Property **f(x)** $= a^x$ and f^{-1}**(x)**$= \log_a x$, we get

$\log_a(a^x) = x$ x∈ R

$a^{\log_a x} = x$ x> 0

Remark : Inverse Function Property:

$f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x$

Properties of Logarithms

Property	Reason
1- $\log_a 1 = 0$	We must raise a to the power **0** to get **1** .
2- $\log_a a = 1$	We must raise a to the power **1** to get a.
3- $\log_a a^x = x$	We must raise a to the power x to get a^x .
4- $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x .

Example 3 ■ Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is **5** .

$\log_5 1 = 0$ property 1 $\log_5 5 = 1$ property 2

$\log_5 5^8 = 8$ property 3 $5^{\log_5 12} = 12$ property 4

Homework ■ Evaluating Logarithms

1) (a) $\log_2 2$ (b) $\log_5 1$ (c) $\log_6 6^5$

2) (a) $3^{\log_3 5}$ (b) $5^{\log_5 27}$ (c) $e^{ln 10}$

■ Graphs of Logarithmic Functions

Recall that if a **one-to-one** function **f** has domain **A**and range **B** , then its inverse function f^{-1}has domain **B**and range **A** . Since the exponential function **f(x)**= a^x with **a** ≠**1** has domain **R** and range (**0** , ∞**)** , we conclude that its inverse function , f^{-1}**(x)** $= \log_a x$, has domain **(0** , ∞**)** and range **R** .

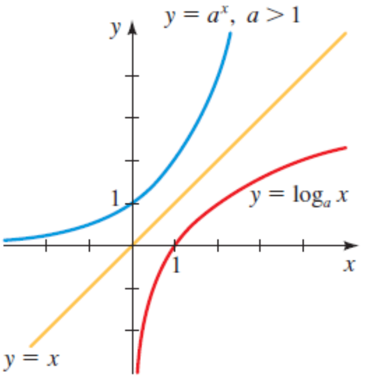


FIGURE 2 Graph of the logarithmic function $f(x) = \log_a x$

The graph of f^{-1}**(x)** $= \log_a x$is obtained by reflecting the graph of **f(x)**= a^x in the line ***y*** =***x*** . **Figure 2** shows the case **a** > **1** . The fact that **y** = a^x (for **a** > **1**) is a very rapidly increasing function for **x** >**0** implies that ***y***= $\log_a x$ is a very slowly increasing function for **x** >**1**

Since $\log_a 1$ =**0** , the ***x*-intercept** of the function ***y***= $\log_a x$ is **1** . The ***y*-axis** is a vertical asymptote of ***y***= $\log_a x$because $\log_a x \to -\infty$ as **x**→ 0^+.

Example 4 ■ Graphing a Logarithmic Function by Plotting Points

Sketch the graph of f(x) $= \log_2 x$.

SOLUTION: To make a table of values, we choose the **x-values** to be powers of **2**so that we can easily find their logarithms . We plot these points and connect them with a smooth curve as in **Figure 3** .

x	$log_2 x$
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4

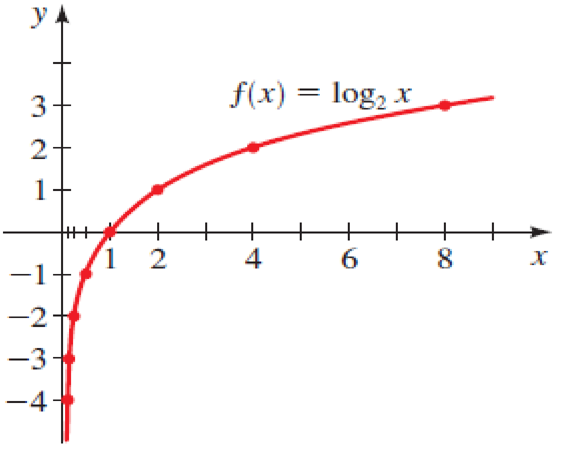


FIGURE 3

■ Graphing Logarithmic Functions Sketch the graph of the function by plotting points .

$$f(x) = \log_3 x$$

■ Family Logarithmic Functions

Figure 4 shows the graphs of the family of logarithmic functions with bases **2** , **3** , **5** , and 10. These graphs are drawn by reflecting the graphs of y= 2^x , y= 3^x , y= 5^x , y= 10^x (see Figure 2 in Section 4.1) in the line *y* =*x* . We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.

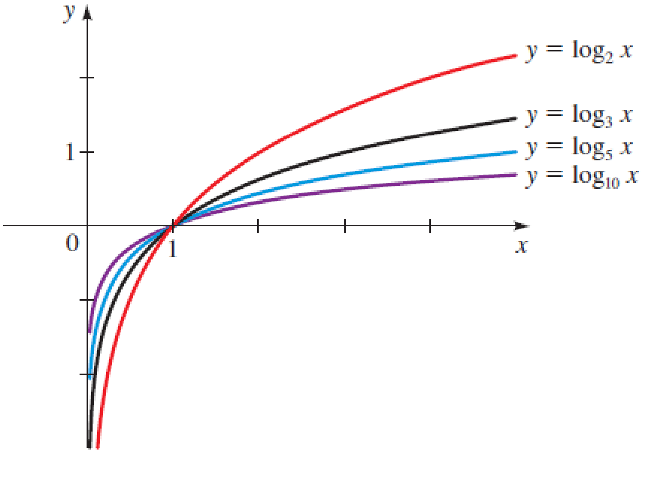


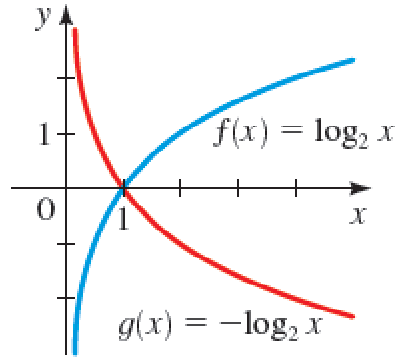
Figure 4A family of logarithmic function

Example 5 ■ Reflecting Graphs of Logarithmic Functions

Sketch the graph of each function. State the domain, range, and asymptote .

(a) $g(x) = -\log_2 x$ **(b)** $h(x) = \log_2(-x)$

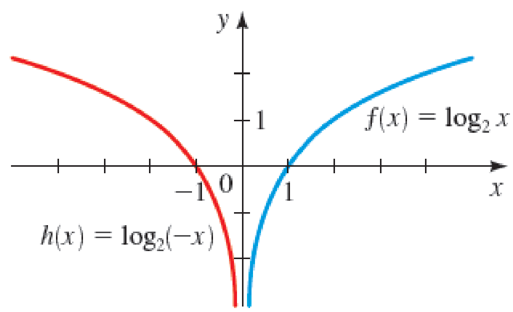
(a) We start with the graph of **f(x)** $= \boldsymbol{log_2 x}$ andreflect in the **x-axis**to get the graph $g(x) = -\boldsymbol{log_2 x}$in **Figure 5(a)**. From the graph we see that the domain of **g**is **(0 , ∞)** , the range is the set **R** of all real numbers, and the line **x =0** is a vertical asymptote .



(a)

Figure (5)

(b) We start with the graph of **f(x)** $= \boldsymbol{log_2 x}$ and reflect in the ***y*–axis** to get the graph of **h(x)** $= \boldsymbol{log_2(-x)}$ in **Figure 5(b)**. From the graph we see that the domain of **h** is **(0 , ∞)** , the range is the set **R** of all real numbers , and the line **x =0** is a vertical asymptote.



(b)

Homework : Graphing Logarithmic Functions

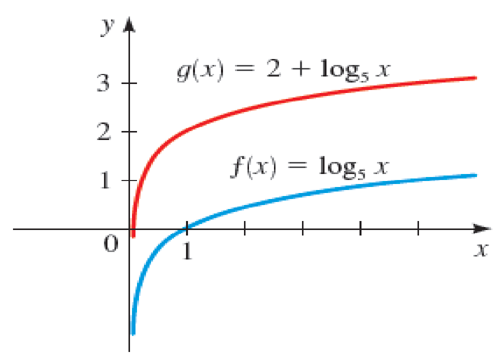
Graph the function, not by plotting points, but by starting from the graphs in Figures 4 . State the domain, range, and asymptote . $g(x) = \boldsymbol{log_5(-x)}$

Example 6 ■ Shifting Graphs of Logarithmic Functions

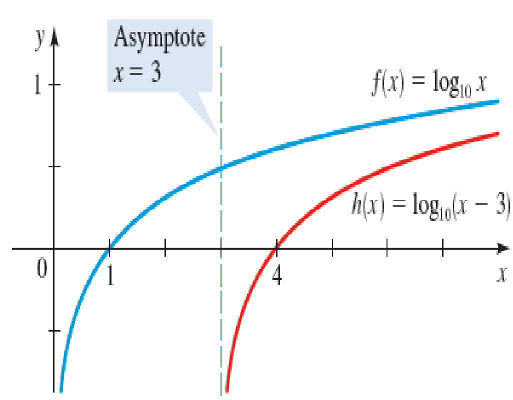
Sketch the graph of each function. State the domain, range, and asymptote .

(a) $g(x) = 2 + \boldsymbol{log_5 x}$ **(b)** $h(x) = \boldsymbol{log_{10}(x - 3)}$

(a) The graph of g is obtained from the graph of **f(x)** $= \boldsymbol{log_5 x}$**(Figure 4)** by shifting upward **2** units, as shown in **Figure 6** . From the graph we see that the domain of g is **(0 , ∞)** , the range is the set **R** of all real numbers, and the line $\boldsymbol{x=0}$ is a vertical asymptote.



(b) The graph of h is obtained from the graph of **f(x)** $= \boldsymbol{log_{10} x}$(Figure 4) by shifting to the right **3** units, as shown in **Figure 7**. From the graph we see that the domain of h is **(3 , ∞)** , the range is the set **R** of all real numbers, and the line **x = 3** is a vertical asymptote.



Homework

Graph the function , not by plotting points, but by starting from the graphs in Figures 4 . State the domain, range, and asymptote .

(a) f(x) = $\log_2(x-4)$ **(b)** y = $2+\log_3 x$

Definition: (Common Logarithm)

The logarithm with base **10** is called the common logarithm and is denoted by omitting the base:

$$\log x = \boldsymbol{log_{10}\, x}$$

Remark

From the definition of logarithms we can easily find that

$\log 10 = 1$, $\log 100 = 2$, $\log 1000 = 3$ and so on

Definition: (Natural Logarithms)

The logarithm with base e is called the natural logarithm and is denoted by ln:

Ln x = $\boldsymbol{\log_e x}$

Remark:

The natural logarithmic function **y = Ln x** is the inverse function of the natural exponential function **y=** e^x. Both functions are graphed in **Figure 9**. By the definition of inverse functions we have

Ln x = y $\Leftrightarrow e^y$ **= x**

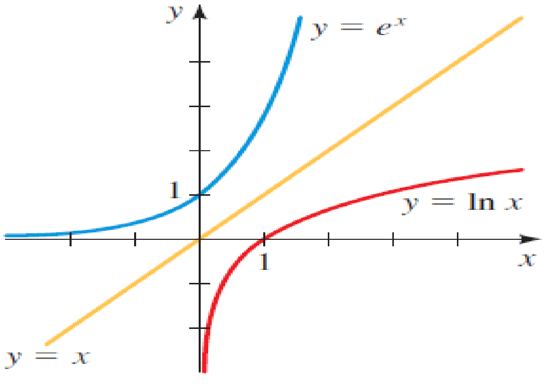


FIGURE 9 Graph of the natural logarithmic function

Remark:

If we substitute **a** =**e** and write **" Ln "** for **"** $\log_e$ **"** in the properties of logarithms mentioned earlier , we obtain the following properties of natural logarithms .

Properties of Natural Logarithms

Property	**Reason**
1- Ln 1= 0	We must raise *e* to the power **0** to get **1** .
2- Ln e = 1	We must raise **e** to the power **1** to get ***e***.
3- Ln e^x = x	We must raise **e** to the power **x** to get $\mathbf{e^x}$.
4- $e^{Ln\, x}$ = x	**Ln *x*** is the power to which **e** must be raised to get **x** .

Example 10 ■ Finding the Domain of a Logarithmic Function

(a) Ln e^8 =8 Definition of natural logarithm

(b) Ln$(\frac{1}{e^2})$ =Ln $e^{-2} = -2$ Definition of natural logarithm

(c) Ln 5 =1.609

Example 10 ■ Finding the Domain of a Logarithmic Function

Find the domain of the function **f(x)** =**Ln (4** $-x^2$ **)** .

Solution As with any logarithmic function, **Ln x** is defined when **x** > **0** . Thus the domain of **f** is

$\{x \mid 4-x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\}$

$= \{x \mid -2 < x < 2\} = (-2, 2)$

Homework

Find the domain of the function.

(a) f(x)= $\log_{10}(x+3)$ **(b)** g(x)= $\log_3(x^2-1)$

Example 11 ■ Drawing the Graph of a Logarithmic Function

Draw the graph of the function **y** =**x Ln** (**4** $-x^2$) , and use it to find the asymptotes and local maximum and minimum values .

Solution As in **Example 10** the domain of this function is the interval **(− 2 , 2)** , so we choose the viewing rectangle **[− 3, 3]** by **[− 3, 3]** . The graph is shown in **Figure 10** , and from it we see that the lines $x = -2$ and $x = 2$ are vertical asymptotes .

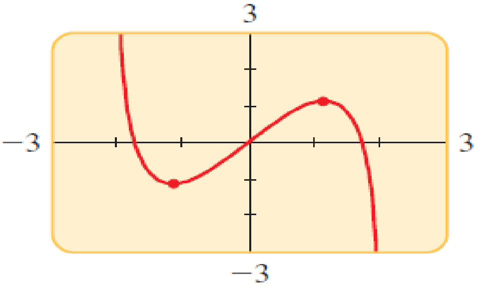


Figure 10

The function has a local maximum point to the right of **x** =**1** and a local minimum point to the left of **x**= − **1**. By zooming in and tracing along the graph with the cursor , we find that the local maximum value is approximately **1.13** and this occurs when **x** ≈ **1.15** . Similarly (or by noticing that the function is odd), we find that the local minimum value is about **−1.13** , and it occurs when **x** ≈ **−1.15** .

4 - 4 (Laws of Logarithms)

Let *a* be a positive number , with a ≠ **1** Let *A*, *B*, and *C* be any real numbers with A > 0 and B > 0

Law Description

1- $\log_a(AB) = \log_a A + \log_a B$ The logarithm of a product of numbers is the sum of the logarithms of the numbers .

2- $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$ The logarithm of a quotient of numbers is the difference of the logarithms of the numbers .

3- $\log_a(A^C) = C\log_a A$ The logarithm of a power of a number is the exponent times the logarithm of the number .

Proof : We make use of the property $\log_a \boldsymbol{a^x} = x$ from Section 4 – 3 .

Law 1 Let $\log_a \boldsymbol{A} = u$ and $\log_a \boldsymbol{B} = v$. When written in exponential form, these equations become $\boldsymbol{a^u} = A$ and $\boldsymbol{a^v} = B$

Thus $\log_a(AB) = \log_a(\boldsymbol{a^u a^v}) = \log_a(\boldsymbol{a^{u+v}})$

$= u + v = \log_a A + \log_a B$

Law 2 Using Law 1 we have

$\log_a A = \log_a\left[\left(\frac{A}{B}\right) B\right] = \log_a\left(\frac{A}{B}\right) + \log_a B$

So $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$

Law 3 Let $\log_a \boldsymbol{A} = u$. Thus $\boldsymbol{a^u} = A$, so

$\log_a(A^C) = \log_a (\boldsymbol{a^u})^C = \log_a \left(\boldsymbol{a^{uC}}\right) = uC = C\log_a A$

Example 1 ■ U sing the Laws of Logarithms to Evaluate Expressions

Evaluate each expression

(a) $\log_4 2 + \log_4 32$**(b)** $\log_2 80 + \log_2 5$ **(c)** $-\frac{1}{3}\log\ 8$

Solution

(a) $\log_4 2 + \log_4 32 = \log_4(2\,.32)$ **Law 1**

$= \log_4(64)$ =3 **Because 64 $= 4^3$**

(b) $\log_2 80 + \log_2 5 = \log_2(\frac{80}{30})$ **Law 2**

$= \log_2 16$ =4 **Because 16 $= 2^4$**

$(\mathbf{c}) -\frac{1}{3}\log\ 8 = \log\ 8^{-\frac{1}{3}}$ **Law 3**

$= \log\ 2^{3^{-\frac{1}{3}}} = \log\ (\frac{1}{2})$ **Property of negative exponents**

$=$ −0.301

■ Expanding and Combining Logarithmic Expressions

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called expanding a logarithmic expression, is illustrated in the next example.

Example 2 : Use the Laws of Logarithms to expand each expression.

$(a)\log_2(6\mathrm{x})$ **(b)** $\log_2(x^3y^6)$ **(c)** Ln$(\frac{ab}{\sqrt[3]{c}})$

Solution :

$(a)\log_2(6\mathrm{x}) = \log_2 6 + \log_2 \mathrm{x}$**Law 1**

$(b)\log_2(x^3y^6) = \log_2(x^3\ ) + \log_2(\ \mathrm{y}^6)$ **Law 1**

$= 3\log_2\ x\ + 6\log_2 y$ **Law 3**

(c) Ln$(\frac{ab}{\sqrt[3]{c}})$= Ln(ab) −Ln $\sqrt[3]{c}$ **Law 2**

=Ln a+ Ln b − Ln $\sqrt[3]{c}$**Law 1**

= Ln a+ Ln b $-\frac{1}{3}$Ln c **Law 3**

Remark: The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called expanding a logarithmic expression, is illustrated in the next example.

Example 3■ Combine Logarithm Expression

Use the Laws of Logarithms to combine each expression into a single logarithm.

(a) 3 Ln x= $\frac{1}{2}$ Ln (x+1) **(b)** 3Ln s+$\frac{1}{2}$ Ln t −4Ln(t^2+1)

Solution :

(a) 3 log x + $\frac{1}{2}$ log (x +1) = log x^3 +log $(x+1)^{\frac{1}{2}}$ **Law 3**

= log$(x^3(x+1)^{\frac{1}{2}})$ **Law 1**

b) 3 Ln s + $\frac{1}{2}$ Ln t − 4Ln(t^2+1)=log$s^3 + Ln\ t^{\frac{1}{2}}$ − Ln $(t^2+1)^4$ **Law 3**

=log$(s^3t^{\frac{1}{2}})$ − Ln $(t^2+1)^4$ = Ln $(\frac{s^3\sqrt{t}}{(t^2+1)^4})$ **Law 2**

Remark : Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, there is no corresponding rule for the logarithm of a sum or a difference. For instance,

$$\log_a(x + y) \cancel{=} \log_a x + \log_a y$$

In fact, we know that the right side is equal to loga1xy2 . Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \cancel{=} \log\left(\frac{6}{2}\right) \qquad \text{and} \qquad (\log_2 x)^3 \cancel{=} 3\log_2 x$$

■ Change of Base Formula

For some purposes we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$. Let

$$y = \log_b x$$

We write this in exponential form and take the logarithm, with base ***a***, of each side.

$b^y = x$ Exponential form

$\log_a(b^y) = \log_a x$ Take $\log_a$ of each side

$y \log_a b = \log_a x$ Law 3

$y = \frac{\log_a x}{\log_a b}$ Divide by $\log_a b$

This proves the following formula.

Change of Base Formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Remark : In particular, if we put **x** =**a**, then $\log_a a = 1$, and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

Example : Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, rounded to five decimal places.

(a) $\log_8 5$ **(b)** $\log_9 20$

Solution :

(a) We use the Change of Base Formula with **b** =**8** and **a**=**10** :

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} = 0.77398$$

(b) We use the Change of Base Formula with ***b*** =**9** and ***a***=***e*** :

$$\log_9 20 = \frac{\ln 20}{\ln 9} = 1.36342$$

4.5 (Exponential and Logarithmic Equations)

In this section we solve equations that involve exponential or logarithmic functions. The techniques that we develop here will be used in the next section for solving applied problems.

■ Exponential Equations

An exponential equation is one in which the variable occurs in the exponent. Some exponential equations can be solved by using the fact that exponential functions are one-to-one. This means that $\boldsymbol{a^x = a^y} \Rightarrow \mathbf{x = y}$

We use this property in the next example.

Example 1 : Solve the exponential equation.

(a) $\mathbf{5^x = 125}$ **(b)** $\mathbf{5^{2x} = 5^{x+1}}$

Solution :

(a) We first express **125** as a power of **5** and then use the fact that the exponential function

f(x)= $\mathbf{5^x}$ is one – to – one

$5^x = 125$ given equation

$5^x = 5^3$ Because $125 = 5^3$

x = 3 one – to – one property

The solution is $x = 1$.

(b) We first use the fact that the function ***f*(x)** = $\mathbf{5^x}$ is one-to-one.

$5^{2x} = 5^{x+1}$ Given equation

2x = x +1 one – to – one property

x= 1 solve for x

The solution is $x = 1$.

Homework

Exponential Equations Find the solution of the exponential equation, as in Example 1.

$(1)\, 7^{2x-3} = 7^{6+5x}$ **(2)** $10^{2x^2-3} = 10^{9-x^2}$

Remark:

The equations in **Example 1** were solved by comparing exponents. This method is not suitable for solving an equation like $5^x = 160$ because 160 is not easily expressed as a power of the base 5. To solve such equations, we take the logarithm of each side and use Law 3 of logarithms to "bring down the exponent." The following guidelines describe the process.

Guidelines for Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the logarithm of each side, then use the Laws of Logarithms to "bring down the exponent."
3. Solve for the variable.

Example 2 ■ Solving an Exponential Equation

Consider the exponential equation $3^{x+2} = 7$. Find the exact solution of the equation expressed in terms of logarithms.

SOLUTION:

(a) We take the common logarithm of each side and use Law 3.

$3^{x+2} = 7$ Given equation

$\log(3^{x+2}) = \log 7$ Take log of each side

$(x + 2)\log 3 = \log 7$ Law 3

$x+2 = \frac{\log 7}{\log 3}$ divide by log 3

The exact solution is $x = \frac{\log 7}{\log 3} - 2 \approx 0.228756$

Example 3 ■ Solving an Exponential Equation

Solve the equation $8e^{2x} = 20$.

SOLUTION: We first divide by to isolate the exponential term on one side of the equation.

$8e^{2x} = 20$ Given equation

$e^{2x} = \frac{20}{8}$ divide by 8

$\text{Ln } e^{2x} = \text{Ln } 2.5$ Take Ln of each side

$2x = \text{Ln } 2.5$ property of Ln

$x = \frac{Ln\ 2.5}{2} \approx 0.458$ Divide by 2 (exact solution)

Homework

(a) Find the exact solution of the exponential equation in terms of logarithms.

(1) $2^{1-x} = 3$ **(2)** $3e^{x} = 10$ **(3)** $4(1+10^{5x}) = 9$

Example 4 ■ Solve the equation $e^{3-2x} = 4$ algebraically and graphically.

Solution 1: Algebraic

Since the base of the exponential term is **e**, we use natural logarithms to solve this equation.

$e^{3-2x} = 4$ Given equation

$(e^{3-2x}) = \text{Ln } 4$ Take LN of each side

$3 - 2x = \text{Ln } 4$ Property of Ln

$-2x = -3 + \text{Ln } 4$ Subtract 3

$X = \frac{1}{2}(3 - \text{Ln } 4) \approx 0.807$ Multiply by $\frac{1}{2}$

You should check that this answer satisfies the original equation.

Solution 2: Graphical

We graph the equations $y = e^{3-2x}$ and $y = 4$ in the same viewing rectangle as in **Figure 1**. The solutions occur where the graphs intersect. Zooming in on the point of intersection of the two graphs, we see that $x \approx 0.81$.

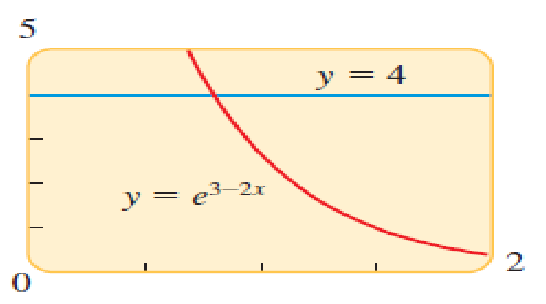


FIGURE 1

Example 5 : ■ An Exponential Equation of Quadratic Type

Solve the equation $e^{2x} - e^{x} -6= 0$

Solution : To isolate the exponential term, we factor.

$e^{2x} - e^{x} -6 = 0$ given equation

$(e^{x})^2 - e^{x} -6 = 0$ Law of exponents

$(e^{x} -3)(e^{x} +2) =0$ Factor (a quadratic in e^{x})

$e^{x} -3=0$ or $e^{x} +2 =0$ Zero – product property

$e^{x} = 3$or $e^{x} = -2$

The equation$e^{x} = 3$leads to x=ln 3. But the equation $e^{x} = -2$has no solution because $e^{x} > 0$for all x. Thus x =ln 3=1.0986 is the only solution. You should check that this answer satisfies the original equation.

Homework : ■ Exponential Equations of Quadratic Type Solve the equation.

(1) $e^{2x} - 3e^{x} + 2 = 0$

Example 6 ■ An Equation Involving Exponential Functions

Solve the equation 3x$e^{x} - x^2e^{x} = 0.$

Solution : First we factor the left side of the equation.

$e^{2x} - 3e^{x} + 2 = 0$Given equation

x(3 +x) $e^{x} = 0$ Factor out common factors

x(3 +x)$= 0$ Divide by e^{x}(because $e^{x} \neq 0$)

x $= 0$ or 3 $+$ x $= 0$ Zero -product property

Thus the solutions are *x* =0 and $x = -3$.

Homework: ■ Equations Involving Exponential Functions

Solve the equations **(1)** $x^2 2^{x} - 2^{x} = 0$ **(2)** $4x^3e^{-3x} - 3x^4e^{-3x}$ =0

■ Logarithmic Equations

A logarithmic equation is one in which a logarithm of the variable occurs. Some logarithmic equations can be solved by using the fact that logarithmic functions are one-to-one. This means that $\log_a x = \log_a y \Rightarrow$ x =y

We use this property in the next example

Example 7 ■ Solving a Logarithmic Equation

Solve the equation $\log(x^2+1) = \log(x-2) + \log(x+3)$

Solution: First we combine the logarithms on the right-hand side, and then we use the one-to-one property of logarithms.

$\log_5 (x^2+1) = \log_5(x-2) + \log_5(x+3)$ Given equation

$\log_5 (x^2+1) = \log_5[(x-2)(x+3)]$ Law 1: $\log_a$ *AB* $= \log_a$ *A* + $\log_a$ *B*

$\log_5 (x^2+1) = \log_5 (x^2+x-6)$ Expand

$(x^2+1) = (x^2+x-6)$ log is one-to-one (or raise 5 to each side)

x=7 Solve for x

The solution is x = 7. (You can check that x = 7 satisfies the original equation.)

Homework Logarithmic Equations Solve the logarithmic equation for x, as in Example 7.

$$\log_4 (x+2) + \log_4 3 = \log_4 5 + \log_4(2x-3)$$

Remark : The method of Example 7 is not suitable for solving an equation like log5 x ? 13 because the right-hand side is not expressed as a logarithm (base 5). To solve such equations, we use the following guidelines.

Guidelines for Solving Logarithmic Equations

1. Isolate the logarithmic term on one side of the equation; you might first need to combine the logarithmic terms.
2. Write the equation in exponential form (or raise the base to each side of the equation).
3. Solve for the variable.

Example 8 ■ Solving Logarithmic Equations

Solve each equation for x. **(a)** $\mathrm{Ln}\ x = 8$ **(b)** $\log_2(25 - x) = 3$

Solution :(a) $\mathrm{Ln}\ x = 8$ Given equation

$x = e^8$ Exponential form

Therefore $x = e^8 = 2981$.

We can also solve this problem another way.

(b) The first step is to rewrite the equation in exponential form.

$\log_2(25 - x) = 3$ Given equation

$25 - x = 2^3$ Exponential form (or raise 2 to each side)

$25 - x = 8 \Rightarrow x = 25 - 8 = 7$

Homework : ■ Logarithmic Equations Solve the logarithmic equation

(1) $\ln x = 10$ $\log_2(x^2 - x - 2) = 2$

Example 9 ■ Solving a Logarithmic Equation

Solve the equation $4 + 3\log(2x) = 16$

Solution We first isolate the logarithmic term. This allows us to write the equation in exponential form.

$4 + 3\log(2x) = 16$ Given equation

$3\log(2x) = 12$ Subtract 4

$\log(2x) = 4$ Divide by 3

$2x = 10^4$ Exponential form (or raise 10 to each side)

$x = 5000$ Divide by 2

Homework ■ Logarithmic Equations Solve the logarithmic equation for *x*.

(a) $4 - \log(3 - x) = 3$ (b) $\log_2(x^2 - x - 2) = 2$

Example 10 ■ Solving a Logarithmic Equation Algebraically and Graphically

Solve the equation $\log(x + 2) + \log(x - 1) = 1$ algebraically and graphically.

Solution 1: Algebraic

We first combine the logarithmic terms, using the Laws of Logarithms.

$\mathrm{Log}[(x + 2)(x - 1)] = 1$ Law 1

$(x + 2)(x - 1) = 10$ Exponential form (or raise 10 to each side)

$x^2 + x - 2 = 10$ Expand left side

$x^2 + x - 12 = 0$ Subtract 10

$(x+4)(x-3)=0$ Factor

$x=-4$ or $x=3$

We check these potential solutions in the original equation and find that **x = − 4** is not a solution (because logarithms of negative numbers are undefined), but **x = 3** is a solution. (See Check Your Answers.)

$x=-4$, Log$(-4+2)(-4-1)$

Log$(-2)(-5)$ undefined ✗

$x=3$, Log$(3+2)(3-1)=$ Log$(5)(2)=$ Log$(10)=1$ ✓

Solution 2: Graphical

We first move all terms to one side of the equation:

Log$(x+2)(x-1)-1=$ **0**

Then we graph

y=Log$(x+2)(x-1)-1$

as in Figure 2. The solutions are the *x*-intercepts of the graph.

Thus the only solution is **x** ≈***3***

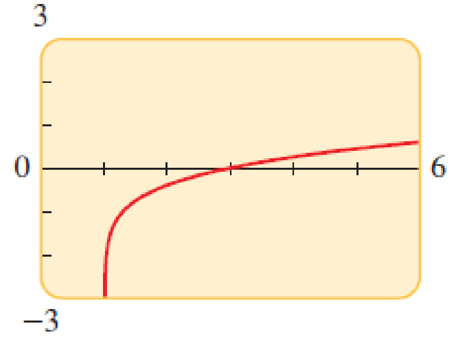


FIGURE 2

■ Compound Interest

Recall the formulas for interest that we found in Section 4.1. If a principal P is invested at an interest rate r for a period of t years, then the amount A of the investment is given by

$A=p(1+r)$ Simple interest (for one year)

$A(t)=p(1+\frac{r}{n})^{nt}$ Interest compounded *n* times per year

$A(t)=pe^{rt}$ Interest compounded continuously

We can use logarithms to determine the time it takes for the principal to increase to a given amount.

Example 13■ Finding the Term for an Investment to Double

A sum of \$5000 is invested at an interest rate of 5% per year. Find the time required for the money to double if the interest is compounded according to the following methods.

(a) Semiannually **(b)** Continuously

Solution: (a) We use the formula for compound interest with **P =\$5000, A(t) =\$10,000, r=0.05, and n = 2**, and solve the resulting exponential equation for t.

$A(t)=p(1+\frac{r}{n})^{nt}$, $5000\,(1+\frac{0.05}{2})^{2t}=10000$

$(1.025)^{2t}=2$ Divide by 5000

Log $(1.025)^{2t}=$ Log2 Take log of each side

2t Log 1.025=Log2 Law 3 (bring down the exponent)

$t=\dfrac{\text{Log}2}{2\ \text{Log}\ 1.025}$ Divide by 2 log 1.025

$t\approx$ 14.04 The money will double in 14.04 years.

(b) We use the formula for continuously compounded interest with P ⍰ \$5000, A(t)=\$10,000, and r =0.05 and solve the resulting exponential equation for t.

$A(t)=pe^{rt}$ $5000e^{0.05t}=10000$

$e^{0.05t}=2$Divide by 5000

Ln $e^{0.05t} = Ln2$ Take ln of each side

0.05 =Ln 2 Property of ln

t= $\frac{Ln\ 2}{0.05}$Divide by 0.05

t ≈ 13.86The money will double in 14.04 years.

Example 14 ■ Time Required to Grow an Investment

A sum of $1000 is invested at an interest rate of 4% per year. Find the time required for the amount to grow to $4000 if interest is compounded continuously.

Solution :We use the formula for continuously compounded interest with P=$1000,

A(t) =$4000, and r=0.04 and solve the resulting exponential equation for t.

A(t) = pe^{rt}1000$e^{0.05t} =$ 4000

$e^{0.04t} = 4$ Divide by 1000

Ln $e^{0.04t} = Ln\ 4$Take ln of each side

0.04 $= Ln\ 4$Property of ln

t= $\frac{Ln\ 4}{0.04}$Divide by 0.04

t ≈ 34.66 The amount will be $4000 in about 34 years and 7 months.