UNIT 2
Matrices

1-MATRICES

2- SOLUTION OF LINEAR SYSETM By USING MATRICES

2-4 (Matrices and Systems of Linear Equations)

A matrix is simply a rectangular array of numbers. Matrices* are used to organize information into categories that correspond to the rows and columns of the matrix. For Example , a scientist might organize information on a population of endangered whales as follows:

$$\begin{array}{l} \\ \text{Male} \\ \text{Female} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Immature} & \text{Juvenile} & \text{Adult} \end{array} \\ \begin{bmatrix} 12 & 52 & 18 \\ 15 & 42 & 11 \end{bmatrix} \end{array}$$

This is a compact way of saying that there are 12 immature males, 15 immature females, 18 adult males, and so on.

In this section we represent a linear system by a matrix, called the augmented matrix of the system .

Linear system / **Augmented matrix**

$$\begin{cases} 2x - y = 5 & \text{Equation 1} \\ x + 4y = 7 & \text{Equation 2} \end{cases} \qquad \begin{array}{c} \begin{bmatrix} 2 & -1 & 5 \\ 1 & 4 & 7 \end{bmatrix} \\ \begin{array}{ccc} x & y & \end{array} \end{array}$$

The augmented matrix contains the same information as the system but in a simpler form. The operations we learned for solving systems of equations can now be performed on the augmented matrix.

❖ Matrices

We begin by defining the various elements that make up a matrix.

<u>Definition:</u> **(matrix)**

An **m × n** matrix is a rectangular array of numbers with **m** rows and **n** columns.

$$\left.\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}}\right\} m \text{ rows}$$

We say that the matrix has dimension **m × n** . The numbers a_{ij} are the entries of the matrix. The subscript on the entry a_{ij} indicates that it is in the **ith** row and the **jth** column.

(The Augmented Matrix of a Linear System)

We can write a system of linear equations as a matrix, called the augmented matrix of the system, by writing only the coefficients and constants that appear in the equations.
Here is an example .

Linear system	Augmented matrix
$\begin{cases} 3x - 2y + z = 5 \\ x + 3y - z = 0 \\ -x + 4z = 11 \end{cases}$	$\begin{bmatrix} 3 & -2 & 1 & 5 \\ 1 & 3 & -1 & 0 \\ -1 & 0 & 4 & 11 \end{bmatrix}$

Notice that a missing variable in an equation corresponds to a 0 entry in the augmented matrix.

Example 1 ■ Finding the Augmented Matrix of a Linear System

Write the augmented matrix of the following system of equations:

$$\begin{cases} 6x - 2y - z = 4 \\ x + 3z = 1 \\ 7y + z = 5 \end{cases}$$

SOLUTION First we write the linear system with the variables lined up in columns.

$$\begin{cases} 6x - 2y - z = 4 \\ x \qquad\quad + 3z = 1 \\ \qquad 7y + z = 5 \end{cases}$$

The augmented matrix is the matrix whose entries are the coefficients and the constants in this system.

$$\begin{bmatrix} 6 & -2 & -1 & 4 \\ 1 & 0 & 3 & 1 \\ 0 & 7 & 1 & 5 \end{bmatrix}$$

Homework :

Write the augmented matrix for the system of linear equations.

$$\begin{cases} 3 + y - z = 2 \\ 2x - y \qquad = 1 \\ x - \qquad z = 3 \end{cases}$$

Types of matrices

(1) Row matrix : if matrix has only one row and any number of columns , is called row matrix or row vector.

For Example , $A = [2 \quad 6 \quad -1 \quad 5]$ and$B = [1 \quad 7 \quad -2]$ are row matrices of orders 1× 4 and 1×3 $respectinely$.

(**2) Column matrix** : if matrix has only one column and any number of rows , is called column matrix or column vector.

For example , $A = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$ are column matrices of orders 2×1 and 3×1 respectively

(3) Square Matrix: A matrix in which the number of rows is equal to the number of the columns is called square matrix .

For example , the matrix A= $\begin{bmatrix} 2 & 5 & -2 \\ 3 & 1 & 9 \\ 4 & 0 & 5 \end{bmatrix}$ is a square matrix of order 3

(4) Diagonal Matrix: A square matrix is called a diagonal matrix if all its non-diagonal entreies are zero and diagonal entries are not all equal .

For example , the matrix A $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is diagonal matrix .

(5) Trace of matrix : the sum of the diagonal elements of a square matrix A (say) is called the trace of a matrix A .

For example , A= $\begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$ Then Trace of A $= 2 + 3 + 4 = 9, or\ tr(A) = 9$

(6) Null matrix or Zero Matrix : Any matrix in which all the elements are zero is called a zero matrix or null matrix.

For example $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices of orders 3×3 and 2×3 respectively and also denoted by $O_{3\times3}$ and $O_{2\times3}$.

(7) Scalar Matrix : A square matrix is called a scalar matrix if all its non-diagonal elements are zero and diagonal elements are equal .

For example , the matrix A = $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$ is a scalar matrix , (k$\neq 0$).

(8) Upper Triangular Matrix : A square matrix in which all elements below the leading diagonal are zero , is called upper triangular matrix .

For example , A= $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & 8 \end{bmatrix}$

(9) Lower **Triangular Matrix :** A square matrix in which all elements above the leading diagonal are zero , is called lower triangular matrix. For example A= $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ 2 & 8 & 6 \end{bmatrix}$

2-5 The Algebra of Matrices

Matrices have many uses in mathematics and the sciences, and for most of these applications a knowledge of matrix algebra is essential. Like numbers, matrices can be added, subtracted, multiplied, and divided. In this section we learn how to perform these algebraic operations on matrices.

■ Equality of Matrices

Two matrices are equal if they have the same entries in the same positions.

Definition : (Equality of Matrices)

The matrices **A =** $[a_{ij}]$ and **B = [**b_{ij} **]** are equal if and only if they have the same dimension **m × n** , and corresponding entries are equal, that is,

$$a_{ij} = b_{ij}$$

for i = 1, 2, . . . , m and j = 1, 2, . . . , n .

Example 1 ■ Equal Matrices

Find a, b, c, and d, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$

SOLUTION :

Since the two matrices are equal, corresponding entries must be the same.

So we must have a = 1 , b = 3, c = 5, and d = 2 .

Homework (1) Find the values of *a* and *b* that make the matrices *A* and *B* equal.

$$A = \begin{bmatrix} 3 & 4 \\ -1 & a \end{bmatrix} \qquad B = \begin{bmatrix} b & 4 \\ -1` & 6 \end{bmatrix}$$

(2)Find the value of **x** ,**y** , **z** and **w** which satisfy the matrix equation

$$\begin{bmatrix} \mathbf{x+3} & \mathbf{2y+x} \\ \mathbf{z-1} & \mathbf{4w-8} \end{bmatrix} = \begin{bmatrix} \mathbf{-x-1} & \mathbf{0} \\ \mathbf{3} & \mathbf{2w} \end{bmatrix}$$

■ Addition, Subtraction, and Scalar Multiplication of Matrices

Two matrices can be added or subtracted if they have the same dimension. (Otherwise, their sum or difference is undefined.) We add or subtract the matrices by adding or subtracting corresponding entries. To multiply a matrix by a number, we multiply every element of the matrix by that number. This is called the scalar product

SUM, DIFFERENCE, AND SCALAR PRODUCT OF MATRICES

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same dimension $m \times n$, and let c be any real number.

1. The **sum** $A + B$ is the $m \times n$ matrix obtained by adding corresponding entries of A and B.
$$A + B = [a_{ij} + b_{ij}]$$
2. The **difference** $A - B$ is the $m \times n$ matrix obtained by subtracting corresponding entries of A and B.
$$A - B = [a_{ij} - b_{ij}]$$
3. The **scalar product** cA is the $m \times n$ matrix obtained by multiplying each entry of A by c.
$$cA = [ca_{ij}]$$

Example 2 ■ **Performing Algebraic Operations on Matrices**

A = $\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix}$ B = $\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix}$

C = $\begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix}$ D = $\begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix}$

Carry out each indicated operation, or explain why it cannot be performed.

(a) A + B **(b) C - D** **(c) C + A** **d) 5A**

Solution

(a) $A + B = \begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 6 \\ 9 & \frac{3}{2} \end{bmatrix}$

(b) $C - D = \begin{bmatrix} 7 & -3 & 0 \\ 0 & 1 & 5 \end{bmatrix} - \begin{bmatrix} 6 & 0 & -6 \\ 8 & 1 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 6 \\ -8 & 0 & -4 \end{bmatrix}$

(c) $C + A$ is undefined because we can’t add matrices of different dimensions.

(d) $5A = 5\begin{bmatrix} 2 & -3 \\ 0 & 5 \\ 7 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 10 & -15 \\ 0 & 25 \\ 35 & -\frac{5}{2} \end{bmatrix}$

Homework : Find the values of a and b that make the matrices A and B equal.

A = $\begin{bmatrix} 3 & 5 & 7 \\ -4 & a & 2 \end{bmatrix}$ B = $\begin{bmatrix} 3 & 5 & b \\ -4 & -5 & 2 \end{bmatrix}$

PROPERTIES OF ADDITION AND SCALAR MULTIPLICATION OF MATRICES

Let A, B, and C be $m \times n$ matrices and let c and d be scalars.

$A + B = B + A$	Commutative Property of Matrix Addition
$(A + B) + C = A + (B + C)$	Associative Property of Matrix Addition
$c(dA) = cdA$	Associative Property of Scalar Multiplication
$(c + d)A = cA + dA$ $c(A + B) = cA + cB$	Distributive Properties of Scalar Multiplication

Example 3 ■ Solving a Matrix Equation

Solve the matrix equation **2X-A=B**

for the unknown matrix X, where

$$A=\begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix} \qquad B=\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}$$

SOLUTION : We use the properties of matrices to solve for X .

2X - A = B	Given equation
2X = B+ A	Add the matrix A to each side
$X=\frac{1}{2}(B + A)$	Multiply each side by the scalar **1/2**
So $X=\frac{1}{2}\left(\begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}+\begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}\right)$	substitute the matrices A and B
$=\frac{1}{2}\begin{bmatrix} 6 & 2 \\ -4 & 4 \end{bmatrix}$	Add matrices
$=\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$	Multiply by the scalar **1/2**

Solve the following equation for X and Y.

$$2X-Y=\begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$

$$2Y+X=\begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

■ Multiplication of Matrices

Multiplying two matrices is more difficult to describe than other matrix operations. In later examples we will see why multiplying matrices involves a rather complex procedure, which we now describe .

First, the product **AB** (or **A** . **B**) of two matrices **A** and **B** is defined only when the number of columns in **A** is equal to the number of rows in **B**. This means that if we write their dimensions side by side, the two inner numbers must match :

Matrices	A	B
Dimensions	m × n	n × k
	Columns in A	Rows in B

If the dimensions of **A** and **B** match in this fashion , then the product **AB** is a matrix of dimension **m × k** . Before describing the procedure for obtaining the elements of **AB** , we define the inner product of a row of **A** and a column of **B** .

If $[\,\boldsymbol{a_1} \quad \boldsymbol{a_2} \quad \dots \quad \boldsymbol{a_n}]$ is a row of A , and if $\begin{bmatrix} \boldsymbol{b_1} \\ \boldsymbol{b_2} \\ \cdot \\ \cdot \\ \cdot \\ \boldsymbol{b_n} \end{bmatrix}$ is a column of B , then their inner product of is

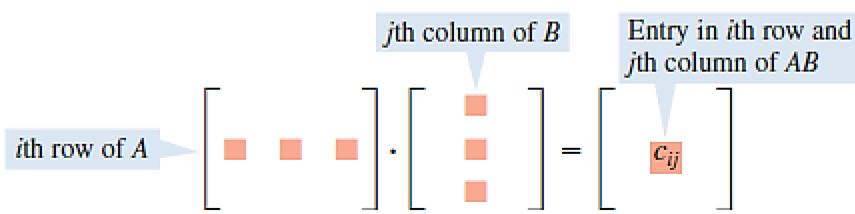
the number $\boldsymbol{a_1}\,\boldsymbol{b_1}+\boldsymbol{a_2}\boldsymbol{b_2} + \dots + \boldsymbol{a_n}\boldsymbol{b_n}$. For example , taking the inner product of

$[\,2 \quad -1 \quad 0 \quad 4]$ and $\begin{bmatrix} 5 \\ 4 \\ -3 \\ \frac{1}{2} \end{bmatrix}$ gives $2.5 + (-1).4 + 0 . (-3) + 4 . \frac{1}{2} = 8$

Definition : (Matrix multiplication)

> **MATRIX MULTIPLICATION**
>
> If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ an $n \times k$ matrix, then their product is the $m \times k$ matrix
>
> $$C = [c_{ij}]$$
>
> where c_{ij} is the inner product of the ith row of A and the jth column of B. We write the product as
>
> $$C = AB$$

This definition of matrix product says that each entry in the matrix **AB** is obtained from a row of **A** and a column of **B** as follows: The entry c_{ij} in the **ith** row and **jth** column of the matrix **AB** is obtained by multiplying the entries in the **ith** row of **A** with the corresponding entries in the **jth** column of **B** and adding the results .



Example 4 : ■ Multiplying Matrices

Let

$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$

Calculate, if possible, the products **AB** and **BA** .

SOLUTION : Since A has dimension **2 × 2** and **B** has dimension **2 × 3** , the product **AB** is defined and has dimension **2 × 3**. We can therefore write

$$AB = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

where the question marks must be filled in using the rule defining the product of two matrices. If we define **C = AB =** $[\, c_{ij} \,]$, then the entry c_{11} is the inner product of the first row of **A** and the first column of **B** :

$$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad 1.(-1) + 3 \,.\, 0 = -1$$

Similarly, we calculate the remaining entries of the product as follows

Entry	Inner product of :	Value	Product matrix
C12	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	1.5 + 3 . 4 = 17	$\begin{bmatrix} -1 & 17 & \\ & & \end{bmatrix}$
C13	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	1 .2 + 3 . 7 = 23	$\begin{bmatrix} -1 & 17 & 23 \\ & & \end{bmatrix}$
C21	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	(-1) .(-1)+ 0 . 0 = 1	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & & \end{bmatrix}$
C22	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	(-1) . 5 + 0 . 4 = -5	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & \end{bmatrix}$
C23	$\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 5 & 2 \\ 0 & 4 & 7 \end{bmatrix}$	(-1) . 2 + 0 . 7 = -2	$\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

Thus we have AB = $\begin{bmatrix} -1 & 17 & 23 \\ 1 & -5 & -2 \end{bmatrix}$

The product BA is not defined, however, because the dimensions of **B** and **A** are **2 × 3** and **2 × 2**

The inner two numbers are not the same, so the rows and columns won't match up when we try to calculate the product .

<u>Homework :</u> The matrices *A*, *B*, *C*, *D*, are defined as follows.

A = $\begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix}$, B = $\begin{bmatrix} 3 & \frac{1}{2} & 5 \\ 1 & -1 & 3 \end{bmatrix}$, C = $\begin{bmatrix} 2 & -\frac{5}{3} & 5 \\ 1 & -1 & 3 \end{bmatrix}$, D = $\begin{bmatrix} 7 & 3 \end{bmatrix}$, E= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

F= $\begin{bmatrix} 5 & -3 & 10 \\ 6 & 1 & 0 \\ -5 & 2 & 2 \end{bmatrix}$

Carry out the indicated algebraic operation, or explain why it cannot be performed.

Find AD and DA , BC , EF , FE and F^2

■ Properties of Matrix Multiplication

Although matrix multiplication is not commutative, it does obey the Associative and Distributive Properties .

PROPERTIES OF MATRIX MULTIPLICATION

Let A, B, and C be matrices for which the following products are defined. Then

$A(BC) = (AB)C$	Associative Property
$A(B + C) = AB + AC$ $(B + C)A = BA + CA$	Distributive Property

Example 5 ■ **Matrix Multiplication Is Not Commutative**

Let $A=\begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}$

Calculate the products AB and BA .

SOLUTION : Since both matrices A and B have dimension **2 × 2**, both products **AB** and **BA** are defined, and each product is also a **2 × 2** matrix .

$$AB = \begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix} = \begin{bmatrix} 5.1+7.9 & 5.2+7.(-1) \\ (-3).1+0.9 & (-3).2+0.(-1) \end{bmatrix} = \begin{bmatrix} 68 & 3 \\ -3 & -6 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 1.5+2.(-3) & 1.7+2.0 \\ 9.5+(-1).(-3) & 9.7+(-1).0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 48 & 63 \end{bmatrix}$$

Homework : The matrices A, B, are defined as follows . $A=\begin{bmatrix} 2 & -5 \\ 0 & 7 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$

Find AB and BA

Remark :

The above example shows that even when both **AB** and **BA** are defined, they aren't necessarily equal. This proves that matrix multiplication is not commutative.

i . e. AB ‡ BA .

■ Applications of Matrix Multiplication

We now consider some applied examples that give some indication of why mathematicians chose to define the matrix product in such an apparently bizarre fashion. Example 6 shows how our definition of matrix product allows us to express a system of linear equations as a single matrix equation .

Example 6 ■ **Writing a Linear System as a Matrix Equation**

Show that the following matrix equation is equivalent to the system of equations

$$\begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & -1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

SOLUTION :

If we perform matrix multiplication on the left-hand side of the equation , we get

$$\begin{bmatrix} x-y+3z \\ x+2y-2z \\ 3x-y+5z \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ 14 \end{bmatrix}$$

Because two matrices are equal only if their corresponding entries are equal, we equate entries to get

$$\begin{cases} x-y+3z=4 \\ x+2y-2z=10 \\ 3x-y+5z=14 \end{cases}$$

This is exactly the system of equations

Various Kinds Of Matrices

(1) Idempotent Matrix : A square matrix is called idempotent provided it satisfies the relation $A^2 = A$

Example 1: Show that the matrix A = $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent

Solution : $A^2 = A.A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2.2 + (-2)(-1) + (-4).1 & 2(-2) + (-2).3 + (-4).(-2) & 2.(-4) + (-2).4 + (-4).(-3) \\ (-1).2 + 3(-1) + 4.1 & (-1).(-2) + 3.3 + 4.(-2) & (-1).(-4) + 3.4 + 4(-3) \\ 1.2 + (-2).(-1) + (-3).1 & 1.(-2) + (-2).3 + (-3).(-2) & 1.(-4) + (-2).4 + (-3).(-3) \end{bmatrix}$$

$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ =A .

Hence the matrix A is idempotent

(2) Involutory Matrix :A Square matrix A is called involutory provided it satisfies the relation $A^2 = I, \ where \ I \ is \ identity \ matrix$.

Example : Show that the matrix A = $\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory

Solution: $A^2 = A.A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25 - 24 + 0 & 40 - 40 + 0 & 0 + 0 + \\ -15 + 15 + 0 & -24 + 24 + 0 & 0 + 0 + \\ -5 + 6 - 1 & -8 + 10 - 2 & 0 + 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence the matrix A is involutory

2-6 ((Inverses of Matrices and Matrix Equations))

■ The Inverse of a Matrix

First, we define identity matrices, which play the same role for matrix multiplication as the number **1** does for ordinary multiplication of numbers; that is, **1 . a = a . 1 = a** for all numbers a . **A square matrix** is one in which the number of rows is equal to the number of columns .The main diagonal of a square matrix consists of the entries whose row and column numbers are the same. These entries stretch diagonally down the matrix, from top left to bottom right.

IDENTITY MATRIX

The **identity matrix** I_n is the $n \times n$ matrix for which each main diagonal entry is a 1 and for which all other entries are 0.

Thus the **2 × 2, 3 × 3**, and **4 ×4** identity matrices are

$$\mathbf{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{I_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity matrices behave like the number **1** in the sense that

$\mathbf{A} \,.\, \mathbf{I_n} = \mathbf{A}$ and $\mathbf{I_n} \,.\, \mathbf{B} = \mathbf{B}$ **,** whenever these products are defined.

Example 1 ■ Identity Matrices

The following matrix products show how multiplying a matrix by an identity matrix of the appropriate dimension leaves the matrix unchanged.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 6 \\ -1 & 2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 7 & \frac{1}{2} \\ 12 & 1 & 3 \\ -2 & 0 & 7 \end{bmatrix}$$

Definition : (Inverse of Matrix)

Let A be a square n × n matrix. If there exists an n × n matrix A^{-1} with the property that

$$A\,A^{-1} = A^{-1}\,A = I_n$$

then we say that A^{-1} is the inverse of A .

Example 2 ■ Verifying That a Matrix Is an Inverse

Verify that B is the inverse of A, where

A = $\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ and B= $\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

Solution: We perform the matrix multiplications to show that *AB* = *I* and *BA* = *I* .

$$\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 2.3 + 1(-5) & 2(-1) + 1.2 \\ 5.3 + 3(-5) & 5(-1) + 3.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3.2 + (-1)5 & 3.1 + (-1).3 \\ (-5)2 + 2.5 & (-5)1 + 2.3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Homework: Calculate the products AB and BA to verify that B is the inverse of A.

A= $\begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \\ -1 & -3 & 2 \end{bmatrix}$ B= $\begin{bmatrix} 8 & -3 & 4 \\ -2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

■ Finding the Inverse of a 2 × 2 Matrix

The following rule provides a simple way for finding the inverse of a 2 x 2 matrix , when it exists. For larger matrices there is a more general procedure for finding inverses, which we consider later in this section.

INVERSE OF A 2 × 2 MATRIX

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A has no inverse.

Example 3 ■ Finding the Inverse of a 2 ×2 Matrix

Let A = $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ find $\boldsymbol{A^{-1}}$ and verify that $A\,\boldsymbol{A^{-1}} = \boldsymbol{A^{-1}}\,A = \boldsymbol{I_2}$.

SOLUTION : Using the rule for the inverse of a 2 × 2 matrix, we get

$$\boldsymbol{A^{-1}} = \frac{1}{4.3 - 5.2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}$$

To verify that this is indeed the inverse of A $\boldsymbol{A^{-1}}$, we calculate AA^{-1} and $\boldsymbol{A^{-1}}$ A:

$$A\,\boldsymbol{A^{-1}} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4 \times \frac{3}{2} + 5(-1) & 4\left(-\frac{5}{2}\right) + 5.2 \\ 2.\frac{3}{2}3(-1) & 2(-\frac{5}{2}) + 3.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A^{-1}}\,A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \times 4 + \left(-\frac{5}{2}\right)2 & \frac{3}{2}.5 + \left(-\frac{5}{2}\right)3 \\ (-1)4 + 2.2 & (-1)5 + 2.3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Remark

The quantity **ad - bc** that appears in the rule for calculating the inverse of a **2** × **2** matrix is called the determinant of the matrix. If the determinant is **0**, then the matrix does not have an inverse (since we cannot divide by **0**).

Homework : Find the inverse of the matrix and verify that A^{-1}A = A A^{-1} = $\mathbf{I_2}$

$$\begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}$$

■ Finding the Inverse of an n × n Matrix

For **3 × 3** and larger square matrices the following technique provides the most efficient way to calculate their inverses. If **A** is an **n × n** matrix, we first construct the **n × 2n** matrix that has the entries of **A** on the left and of the identity matrix I_n on the right :

$$\left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right]$$

We then use the elementary row operations on this new large matrix to change the left side into the identity matrix. (This means that we are changing the large matrix to reduced row-echelon form.) The right side is transformed automatically into A^{-1} .

Example 4 ■ Finding the Inverse of a 3 × 3 Matrix

Let A be the matrix $$A=\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$$

(a) Find A^{-1} .

(b) Verify that A A^{-1} = A^{-1} A = I_3 .

SOLUTION

(a) We begin with the **3 ×6** matrix whose left half is A and whose right half is the identity matrix .

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 2 & -3 & -6 & 0 & 1 & 0 \\ -3 & 6 & 15 & 0 & 0 & 1 \end{array}\right]$$

We then transform the left half of this new matrix into the identity matrix by performing the following sequence of elementary row operations on the entire new matrix .

$$\xrightarrow[R_3 + 3R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & -4 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_1 + 2R_2 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 0 \\ 0 & 1 & 0 & -4 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 & 0 & \frac{1}{3} \end{array}\right]$$

We have now transformed the left half of this matrix into an identity matrix. (This means that we have put the entire matrix in reduced row-echelon form.) Note that to do this in as systematic a fashion as possible, we first changed the elements below the main diagonal to zeros, just as we would if we were using Gaussian elimination. We then changed each main diagonal element to a **1** by multiplying by the appropriate constant(s). Finally, we completed the process by changing the Remaining entries on the left side to zeros. The right half is now $\boldsymbol{A^{-1}}$.

$$\boldsymbol{A^{-1}} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$$

(b) We calculate $\boldsymbol{A\,A^{-1}}$ and $\boldsymbol{A^{-1}\,A}$ and verify that both products give the identity matrix $\boldsymbol{I_3}$ **.**

$$AA^{-1} = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Homework : Find the inverse of the matrix if it exists , A= $\begin{bmatrix} 2 & 4 & 1 \\ -1 & 1 & -1 \\ 1 & 4 & 0 \end{bmatrix}$

Example 5 ■ A Matrix That Does Not Have an Inverse

Find the inverse of the matrix $\begin{bmatrix} 2 & -3 & -7 \\ 1 & 2 & 7 \\ 1 & 1 & 4 \end{bmatrix}$

SOLUTION We proceed as follows.

$$\left[\begin{array}{ccc|ccc} 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 2 & 7 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 2 & -3 & -7 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1 \to R_3]{R_2 - 2R_1 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & -7 & -21 & 1 & -2 & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{7}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 7 & 0 & 1 & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & -1 & -3 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 - 2R_2 \to R_1]{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & 3 & -\frac{1}{7} & \frac{2}{7} & 0 \\ 0 & 0 & 0 & -\frac{1}{7} & -\frac{5}{7} & 1 \end{array}\right]$$

At this point we would like to change the 0 in the **(3, 3)** position of this matrix to a **1** without changing the zeros in the **(3, 1)** and **(3, 2)** positions. But there is no way to accomplish this, because no matter what multiple of rows **1** and/or **2** we add to row **3** , we can't change the third zero in row **3** without changing the first or second zero as well. Thus we cannot change the left half to the identity matrix, so the original matrix doesn't have an inverse .

Remark :

If we encounter a row of zeros on the left when trying to find an inverse, as in Example 5, then the original matrix does not have an inverse.

■ Matrix Equations

A system of linear equations can be written as a single matrix equation. For example, the system

$$\begin{cases} x - 2y - 4z = 7 \\ 2x - 3y - 6z = 5 \\ -3x + 6y + 15z = 0 \end{cases}$$

is equivalent to the matrix equation

$$\begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$$

If we let $\quad A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -3 & -6 \\ -3 & 6 & 15 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix}$

then this matrix equation can be written as

$$AX = B$$

The matrix A is called the **coefficient matrix.**

We solve this matrix equation by multiplying each side by the inverse of A (provided that this inverse exists).

$$AX = B$$

$\boldsymbol{A^{-1}}(AX) = \boldsymbol{A^{-1}} B$ Multiply on left by $\boldsymbol{A^{-1}}$

$(\boldsymbol{A^{-1}}A)\, X = \boldsymbol{A^{-1}} B$ Associative property

$\boldsymbol{I_3}\, X = \boldsymbol{A^{-1}} B$ Property of inverses

$X = \boldsymbol{A^{-1}} B$ Property of identity matrix

In example 4 we showed that

$\boldsymbol{A^{-1}} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}$, So from $X = \boldsymbol{A^{-1}} B$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ -4 & 1 & -\frac{2}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 7 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -11 \\ -23 \\ 7 \end{bmatrix}$

Thus x = -11 , y = - 23, z = 7 is the solution of the original system. We have proved that the matrix equation **AX = B** can be solved by the following method .

solving a matrix equation

If **A** is a square **n x n** matrix that has an inverse A^{-1} and if **X** is a variable matrix and **B** a known matrix, both with **n** rows, then the solution of the matrix equation

A X = B is given by **X** $= A^{-1}$ **B**

Example 6 ■ Solving a System Using a Matrix Inverse

A system of equations is given , $\begin{cases} 2x - 5y = 15 \\ 3x - 6y = 36 \end{cases}$

(a) Write the system of equations as a matrix equation.

(b) Solve the system by solving the matrix equation.

SOLUTION

(a) We write the system as a matrix equation of the form ***AX = B*** .

$\begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 36 \end{bmatrix}$, where A = $\begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}$, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} 15 \\ 36 \end{bmatrix}$

(b) Using the rule for finding the inverse of a **2 x 2** matrix, we get

$$A^{-1} = \begin{bmatrix} 2 & -5 \\ 3 & -6 \end{bmatrix}^{-1} = \frac{1}{2(-6)-(-5)3}\begin{bmatrix} -6 & -(-5) \\ -3 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix}$$

Multiplying each side of the matrix equation by this inverse matrix, we get

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & 5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 15 \\ 36 \end{bmatrix} = \begin{bmatrix} 30 \\ 9 \end{bmatrix}$, x = 30 and y = 9 .

Homework :

Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix .

$$\begin{cases} -3x - 5y = 4 \\ 2x + 3y = 0 \end{cases}$$

2- 7((Determinants and Cramer 's Rule))

If a matrix is square (that is, if it has the same number of rows as columns), then we can assign to it a number called its determinant. Determinants can be used to solve systems of linear equations, as we will see later in this section. They are also useful in determining whether a matrix has an inverse.

■ Determinant of a 2 × 2 Matrix

We denote the determinant of a square matrix **A** by the symbol **det(A)** or $|A|$. We first define **det (A)** for the simplest cases. If **A = [a]** is a **1 × 1** matrix, then **det(A) = a** . The following box gives the definition of a **2 × 2** determinant.

Definition (Determinant of a 2 × 2 Matrix)

The determinant of the 2× 2 matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **is Det(A) =** $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ **= ad – bc**

Example 1 ■ Determinant of a 2 × 2 Matrix

Evaluate $|A|$ for A = $\begin{bmatrix} 6 & -3 \\ 2 & 3 \end{bmatrix}$

Solution : $\begin{vmatrix} 6 & -3 \\ 2 & 3 \end{vmatrix}$ = 6 . 3 – (-3) 2 = 18 -(- 6) =24

■ Determinant of an n × n Matrix

To define the concept of determinant for an arbitrary **n × n** matrix, we need the following terminology.

Definition (Minors and Cofactors)

Let ***A*** be an ***n*** × ***n*** matrix.

1. The **minor** $\boldsymbol{M_{ij}}$ of the element $\boldsymbol{a_{ij}}$ is the determinant of the matrix obtained by deleting the *i*th row and *j*th column of *A*.

2. The **cofactor** $\boldsymbol{A_{ij}}$ of the element $\boldsymbol{a_{ij}}$ is

$\boldsymbol{A_{ij}} = (\boldsymbol{-1})^{i+j}\ \boldsymbol{M_{ij}}$

For example, if **A** is the matrix $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$

then the minor $\boldsymbol{M_{12}}$ is the determinant of the matrix obtained by deleting the first row and second column from ***A***. Thus

$$\boldsymbol{M_{12}} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} \boldsymbol{0} & \boldsymbol{4} \\ \boldsymbol{-2} & \boldsymbol{6} \end{vmatrix} = 0\ (6) - 4\ (-\ 2) = 8$$

So the cofactor $\boldsymbol{A_{12}}$ = = $(\boldsymbol{-1})^{\boldsymbol{1+2}}\ \boldsymbol{M_{12}}$ = - 8 . Similarly

$$\boldsymbol{M_{33}} = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = \begin{vmatrix} \boldsymbol{2} & \boldsymbol{3} \\ \boldsymbol{0} & \boldsymbol{2} \end{vmatrix} = 2\ .\ 2 - 3\ .\ 0 = 4$$

S o $\boldsymbol{A_{33}}$ = = $(\boldsymbol{-1})^{\boldsymbol{3+3}}\ \boldsymbol{M_{33}}$ = 4

Note that the cofactor of $\boldsymbol{a_{ij}}$ is simply the minor of $\boldsymbol{a_{ij}}$ multiplied by either **1** or **-1**, depending on whether **i + j** is even or odd. Thus in a **3 × 3** matrix we obtain the cofactor of any element by prefixing its minor with the sign obtained from the following checkerboard pattern.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Definition : Determinant Of A Square Matrix

If A is an n × n matrix, then the determinant of A is obtained by multiplying each element of the first row by its cofactor and then adding the results. In Symbols **,**

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

Example 2 ■ Determinant of a 3 × 3 Matrix

Evaluate the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{bmatrix}$$

Solution

$$\det(A) = |A| = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 \\ -2 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix}$$

$= 2(2 . 6 – 4 . 5) – 3 [0 .6 - 4 (-2)] – [0 . 5 – 2 (-2)]$

$= - 16 -24 – 4 = - 44$

Homework : Find the determinant of the following matrices . Determine whether the matrix has an inverse, but don't calculate the inverse.

$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 4 \\ 0 & 1 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 & 5 \\ -2 & -3 & 2 \\ 3 & 5 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{bmatrix}$

Remark :

In our definition of the determinant we used the cofactors of elements in the first row only. This is called expanding the determinant by the first row. In fact, we can expand the determinant by any row or column in the same way and obtain the same result in each case .

Example 3 ■ Expanding a Determinant About a Row and a Column

Let A be the matrix of Example 2. Evaluate the determinant of A by expanding

(a) by the second row

(b) by the third column

Verify that each expansion gives the same value.

Solution

(a) Expanding by the second row , we get

$$\det(A) = |A| = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = -0\begin{vmatrix} 3 & -1 \\ 5 & 6 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -2 & 6 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix}$$

$= 0 + 2 [2 . 6 – (-1)(- 2)] – 4[2 .5 – 3 (-2)]$

$= 0 + 20 – 64 = -44$

(b) Expanding by the third column gives

$$\det(A) = |A| = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 2 & 4 \\ -2 & 5 & 6 \end{vmatrix} = -1\begin{vmatrix} 0 & 2 \\ -2 & 5 \end{vmatrix} + -4\begin{vmatrix} 2 & 3 \\ -2 & 5 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix}$$

$= - [0 .5 -2 (-2)] -4 [2 . 5 – 3 (- 2)] + 6 (2 . 2 – 3 .0)$

$= - 4 -64 + 24 = -44$

In both cases we obtain the same value for the determinant as when we expanded by the first row in Example 2 .

❑ Definition (Transpose of Matrix)

A matrix of order $n \times m$ *is* obtained by interchanging the rows and columns of an $m \times n$ matrix A is called transpose of A and is denoted by A^{τ}.

If $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ -2 & -1 & 4 & 8 \\ 7 & 5 & 3 & 1 \end{bmatrix}$ is 3×4 then $A^{\tau} = \begin{bmatrix} 2 & -1 & 7 \\ 3 & -1 & 5 \\ 4 & 4 & 3 \\ 5 & 8 & 1 \end{bmatrix}$

❑ Compute the Inverse of Matrix by Using Determinant and Cofactor

Example: using determinant and cofactor to compute the inverse of the following matrix

A =$\begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

Solution:

1- we find the value determinant A , if$|A|$ = detA$\neq 0$ then the matrix has inverse.

$$|A| = detA = 2\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$=2(-3) -(-2) +0 = -6 +2 =4$

Since the Determinant of A is not equal zero then the matrix has inverse

2- we find new matrix N by compute the nine cofactor A_{ij} ,

$$N = \begin{bmatrix} +\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & 2 & 5 \\ -1 & 2 & -1 \\ 1 & -2 & 3 \end{bmatrix}$$

3- Now we find the transpose of N

$$N^{\tau} = \begin{bmatrix} -3 & -1 & 1 \\ 2 & 2 & -2 \\ 5 & -1 & 3 \end{bmatrix}$$

4- Now we will find A^{-1} as

$$A^{-1} = \frac{1}{|A|}\, N^{\tau} = -\frac{1}{4}\begin{bmatrix} -3 & -1 & 1 \\ 2 & 2 & -2 \\ 5 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{5}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Calculating a Determinant in Different Ways Consider the matrix

$$A = \begin{bmatrix} 4 & 1 & 0 \\ -2 & -1 & 1 \\ 4 & 0 & 3 \end{bmatrix}$$

(a) Evaluate det(A) by expanding by the second row.

(b) Evaluate det(A) by expanding by the third column.

(c) Do your results in parts (a) and (b) agree?

Definition (Invertibility Criterion)

If A is a square matrix, then A has an inverse if and only if det (A) ‡ 0

Remark **:**The following criterion allows us to determine whether a square matrix has an inverse without actually calculating the inverse. This is one of the most important uses of the determinant in matrix algebra, and it is the reason for the name determinant .

Example 4 ■ U sing the Determinant to Show That a Matrix Is Not Invertible

Show that the matrix A has no inverse . $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{bmatrix}$

Solution : We begin by calculating the determinant of A. Since all but one of the elements of the second row is zero, we expand the determinant by the second row. If we do this, we see from the following equation that only the cofactor $\boldsymbol{A_{24}}$ will have to be calculated.

Det (A) = $\begin{vmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 5 & 6 & 2 & 6 \\ 2 & 4 & 0 & 9 \end{vmatrix}$ **=** $-0 \,.\, A_{21} + A_{22} - 0 \; A_{23} + 3\,A_{24} = 3\; A_{24}$

$= 3 \begin{vmatrix} 1 & 2 & 0 \\ 5 & 6 & 2 \\ 2 & 4 & 0 \end{vmatrix}$ Expand this by column

$= 3\,(-2) \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (-2)\,(1 \,.\, 4 - 2 \,.\, 2) = 0$

Since the determinant of A is zero, A cannot have an inverse, by the Invertibility Criterion .

Homework : Find the determinant of the matrix. Determine whether the matrix has an inverse, but don't calculate the inverse.

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & 0 & 8 \\ 0 & 2 & 2 \end{bmatrix}$$

Definition : (Row and Column Transformations of a Determinant)

If **A** is a square matrix and if the matrix **B** is obtained from **A** by adding a multiple of one row to another or a multiple of one column to another, then **det(A**) = **det (B**) .

Example 5 ■ **Using Row and Column Transformations to Calculate a Determinant**

Find the determinant of the matrix A. Does it have an inverse?

A = $\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 24 & 6 & 1 & -12 \\ 2 & 2 & 7 & -1 \end{bmatrix}$

Solution : If we add **- 3** times row **1** to row **3** , we change all but one element of row **3** to zeros.

$$\begin{bmatrix} 8 & 2 & -1 & -4 \\ 3 & 5 & -3 & 11 \\ 0 & 0 & 4 & 0 \\ 2 & 2 & 7 & -1 \end{bmatrix}$$

This new matrix has the same determinant as A , and if we expand its determinant by the third row , we get

$$\det(A) = 4\begin{vmatrix} 8 & 2 & -4 \\ 3 & 5 & 11 \\ 2 & 2 & -1 \end{vmatrix}$$

Now, adding 2 times column 3 to column 1 in this determinant gives us

$$\det(A) = 4\begin{vmatrix} 0 & 2 & -4 \\ 25 & 5 & 11 \\ 0 & 2 & -1 \end{vmatrix} \quad \text{Expand this by column 1}$$

$$= 4(-25)\begin{vmatrix} 2 & -4 \\ 2 & -1 \end{vmatrix}$$

$$= 4(-25)\,[2(-1) - (-4(2)] = -600$$

Since the determinant of *A* is not zero, *A* does have an inverse.

■ Cramer's Rule

The solutions of linear equations can sometimes be expressed by using determinants. To illustrate, let's solve the following pair of linear equations for the variable **x**.

$$\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$$

To eliminate the variable **y** , we multiply the first equation by **d** and the second by **b** and subtract.

$$\begin{array}{c} adx + bdy = rd \\ bcx + bdy = bs \\ \hline adx - bcx = rd - bs \end{array}$$

Factoring the left-hand side, we get (ad - bc) x = rd - bs. Assuming that ad - bc ≠ 0 , we can now solve this equation for x:

$$x = \frac{rd - bs}{ad - bc}$$

Similarly, we find $\quad Y = \dfrac{as - cr}{ad - bc}$

The numerator and denominator of the fractions for ***x*** and ***y*** are determinants of **2 × 2** matrices. So we can express the solution of the system using determinants as follows.

Definition (Cramer 's Rule for Systems in Two Variables)

The linear system $\begin{cases} ax + by = r \\ cx + dy = s \end{cases}$

Has the solution , $x = \dfrac{\begin{vmatrix} r & b \\ s & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ **,** $y = \dfrac{\begin{vmatrix} a & r \\ c & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$ **, Provided that** $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

Remark : using the notation

$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ Coefficient matrix

$D_x = \begin{vmatrix} r & b \\ s & d \end{vmatrix}$ Replace first column of D by r and s

$D_y = \begin{vmatrix} a & r \\ c & s \end{vmatrix}$ Replace second column of D by r and s

We can write the solution of the system as

$$x = \frac{|D_x|}{|D|} \qquad y = \frac{|D_y|}{|D|}$$

Example 6 ■ U sing Cramer's Rule to Solve a System with Two Variables

Use Cramer's Rule to solve the system . $\begin{cases} 2x + 6y = -1 \\ x + 8y = 2 \end{cases}$

Solution : for this system we have

$|D| = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 2 . 8 - 6 . 1 = 10$, $|D_x| = \begin{vmatrix} -1 & 6 \\ 2 & 8 \end{vmatrix} = (-1)\, 8 - 6 . 2 = -20$

$|D_y| = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 2 . 2 - (-1)\, 1 = 5$

The solution is $x = \dfrac{|D_x|}{|D|} = \dfrac{-20}{10} = -2$, $y = \dfrac{|D_y|}{|D|} = \dfrac{5}{10} = \dfrac{1}{2}$

Homework : Use Cramer's Rule to solve the system.

$$\begin{cases} 2x - y = -9 \\ x + 2y = 8 \end{cases}$$

Remark : Cramer's Rule can be extended to apply to any system of *n* linear equations in *n* variables in which the determinant of the coefficient matrix is not zero. As we saw in the preceding section, any such system can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

By analogy with our derivation of Cramer's Rule in the case of two equations in two unknowns, we let D be the coefficient matrix in this system, and D_{xi} be the matrix obtained by replacing the *i*th column of D by the numbers $b_1 , b_2 , . . . , b_n$ that appear to the right of the equal sign. The solution of the system is then given by the following rule.

CRAMER'S RULE

If a system of n linear equations in the n variables $x_1, x_2, \ldots, x_n$ is equivalent to the matrix equation $DX = B$, and if $|D| \neq 0$, then its solutions are

$$x_1 = \frac{|D_{x_1}|}{|D|} \qquad x_2 = \frac{|D_{x_2}|}{|D|} \qquad \cdots \qquad x_n = \frac{|D_{x_n}|}{|D|}$$

where D_{x_i} is the matrix obtained by replacing the ith column of D by the $n \times 1$ matrix B.

<u>Example 7</u> ■ U sing Cramer's Rule to Solve a System with Three Variables

Use Cramer's Rule to solve the system

$$\begin{cases} 2x \; - 3y + 4z = 1 \\ x \qquad\quad\; + 6z = 0 \\ 3x \; - 2y \qquad\; = 5 \end{cases}$$

<u>Solution</u> First, we evaluate the determinants that appear in Cramer's Rule. Note that *D* is the coefficient matrix and that $\boldsymbol{D_x}$, $\boldsymbol{D_y}$, and $\boldsymbol{D_z}$ are obtained by replacing the first, second, and third columns of ***D*** by the constant terms.

$|D| = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 0 & 6 \\ 3 & -2 & 0 \end{vmatrix}$ = -38 $\quad |\boldsymbol{D_x}| = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 0 & 6 \\ 5 & -2 & 0 \end{vmatrix}$ = 78

$|\boldsymbol{D_y}| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & 0 & 6 \\ 3 & 5 & 0 \end{vmatrix}$ = -22 $\quad |\boldsymbol{D_z}| = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 3 & -2 & 5 \end{vmatrix}$ = 13

Now we use Cramer's Rule to get the solution

X = $\dfrac{|\boldsymbol{D_x}|}{|D|} = \dfrac{-78}{-38} \; \dfrac{39}{19}$

y = $\dfrac{|\boldsymbol{D_y}|}{|D|} = \dfrac{-22}{-38} = \dfrac{11}{19}$

Z = y = $\dfrac{|\boldsymbol{D_z}|}{|D|} = \dfrac{13}{-38} = -\dfrac{13}{38}$

<u>Homework</u> : Use Cramer's Rule to solve the system .

$$\begin{cases} \;\; x - y + 2z = 0 \\ 3x \quad\; + z \quad\; = 11 \\ -x + 2y \;\; = 0 \end{cases}$$

Homework

(1) Using inverse of matrix to solve the following system of linear equations, where

$$\mathbf{x + 2z = 4}$$

$$\mathbf{2x - y + 3z = 2}$$

$$\mathbf{4x + y + 8z = 6}$$

(3) Find the value of **t** which make the following Determinant equal zero

$$\begin{vmatrix} \mathbf{t-2} & \mathbf{3} \\ \mathbf{4} & \mathbf{t-1} \end{vmatrix} = 0$$

(4) (5) Solve the following questions

(6) Find **x** so that $\begin{bmatrix} \mathbf{1} & \boldsymbol{x} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{0} & \mathbf{4} & \mathbf{1} \\ \mathbf{0} & \mathbf{3} & \mathbf{2} \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \boldsymbol{x} \end{bmatrix} = \mathbf{0}$

((**8**)using Cramer's Rule to solve the following Linear system of equations, where

$\mathbf{2x + 3y - z = 1}$

$$\mathbf{3x + 5y + 2z = 8}$$

$$\mathbf{x - 2y - 3z = -1}$$

(9) Use Cramer's Rule to solve the following Linear system of equations, where

$$\mathbf{x - y + 3z = 4}$$

$$\mathbf{x + 2y + 2z = 10}$$

$$\mathbf{3x - y + 5z = 14}$$

(**10**) (**1**) If **A** $= \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{6} \end{bmatrix}$ and **B** $= \begin{bmatrix} \mathbf{-3} & \mathbf{-2} \\ \mathbf{1} & \mathbf{-5} \\ \mathbf{4} & \mathbf{3} \end{bmatrix}$ Find **D** $= \begin{bmatrix} \boldsymbol{p} & \boldsymbol{q} \\ \boldsymbol{r} & \boldsymbol{s} \\ \boldsymbol{t} & \boldsymbol{u} \end{bmatrix}$ such that **A** $+$**B** $-$**D** $= 0$

(**11**) Solve the system of equations by converting to a matrix equation and using the inverse of the coefficient matrix **2x − y = -9**

x + 2y = 8

(12) Solve the matrix equation to find the values of **x** and **y**

$$\begin{bmatrix} \boldsymbol{x} & \mathbf{3} \\ \boldsymbol{y} & \mathbf{4} \end{bmatrix} \begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{-2} \end{bmatrix} = \begin{bmatrix} \mathbf{43} & \mathbf{-23} \\ \mathbf{-2} & \mathbf{-1} \end{bmatrix}$$

(13) Find the magnitude and direction of the vector **u = 4i +4** $\sqrt{\mathbf{3}}$ **j**

(14) Find the value **x** and **y** if

$$\begin{bmatrix} \mathbf{2}\boldsymbol{x} & \mathbf{4} & \mathbf{3}+\boldsymbol{y} \\ \mathbf{8} & \mathbf{-1} & \boldsymbol{x}+\boldsymbol{y} \end{bmatrix} + \begin{bmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{-4}\boldsymbol{y} & \mathbf{7} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{-4} & \mathbf{6} & \mathbf{0} \end{bmatrix}$$

15) Find the inverse of each of the following matrices (if it exists)

A= $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$ B $= \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

16) Find x , y , z and t where

$$\mathbf{3}\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{z} & \boldsymbol{t} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} & \mathbf{6} \\ \mathbf{-1} & \mathbf{2}\boldsymbol{t} \end{bmatrix} + \begin{bmatrix} \mathbf{4} & \boldsymbol{x}+\boldsymbol{y} \\ \boldsymbol{z}+\boldsymbol{t} & \mathbf{3} \end{bmatrix}$$

17)Given that A= $\begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 1 & 2 \end{bmatrix}$, B= $\begin{bmatrix} 2 & 1 & 2 \\ 3 & - & 0 \end{bmatrix}$ find AB and BA

18) Find the value of y if

$$\begin{bmatrix} y & 5y & 1 \end{bmatrix} \begin{bmatrix} y \\ -1 \\ 4 \end{bmatrix}$$

Homework

Q1: If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \\ 5 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & -1 \\ 2 & 3 \\ -4 & 1 \end{bmatrix}$

Calculate $A - B + 2C$

Q2: Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

Compute AB and BA show that matrix multiplication is not commutative .

Q3:Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$

Show that $AB \neq 0$, $BA = 0$

Q4: Compute AB and BA whichever exist

A= $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 0 & 3 \\ 6 & -3 & 4 & -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 6 & 5 \\ 7 & 0 \end{bmatrix}$

Q5: Given that $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ verify that $A^2 - 4A + 5I = 0$

Q6: If $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 7 & 2 & 1 \end{bmatrix}$, B= $\begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ Compute $A^2 - 4B^2$

Q7: Show that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent .

Q8: Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutary .

Q9: Solve for X the matrix equations , (a) $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ (b) $X \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Q10: Compute $AB - BA$ if $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & 1 \\ -4 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

Q11: If $A = \begin{bmatrix} 3 & 5 & 4 \\ -2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$ find AB . If BA defined , if it is defined then find it .

Q12: If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & x \end{bmatrix}$ is an idempotent matrix , then find the value of $\boldsymbol{x}$

Q13: If $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$ $and\ C = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$. Show that

$$AB = AC\ but\ B \neq C$$

Q14: Find x , y , z and t where $\mathbf{3}\begin{bmatrix} \boldsymbol{x} & \boldsymbol{y} \\ \boldsymbol{z} & \boldsymbol{t} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x} & \mathbf{6} \\ \mathbf{-1} & \mathbf{2}\boldsymbol{t} \end{bmatrix} + \begin{bmatrix} \mathbf{4} & \boldsymbol{x}+\boldsymbol{y} \\ \boldsymbol{z}+\boldsymbol{t} & \mathbf{3} \end{bmatrix}$

Q15: Let $A = \frac{1}{2}\begin{bmatrix} 8 & 5 \\ 6 & 5 \end{bmatrix}$, $B = \frac{1}{5}\begin{bmatrix} 5 & -5 \\ -6 & 8 \end{bmatrix}$. Show that $AB = I\ where\ I$ denote the identity matrix .

Q16: Solve the following matrix equation to find the value of $x\ and\ y$

$$\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x & 6 \\ 1 & y \end{bmatrix} = \begin{bmatrix} 10 & -8 \\ -13 & -33 \end{bmatrix}$$