Q1/1

$$\boldsymbol{3x(x+1) \leq x(x+5)}$$

$3x(x+1) - x(x+5) \leq 0$

$\Rightarrow x(3x + 3 - x - 5) \leq 0$

$\Rightarrow 2x(x-1) \leq 0$

$S = \{x \in \mathbb{R}: 0 \leq x \leq 1\}$

Q1/2

$$\boldsymbol{\cos\left(2\sec^{-1}\left(\frac{1}{x}\right)\right) - 1 \leq 0.}$$

$\Rightarrow \cos(2\cos^{-1}(x)) - 1 \leq 0$

(since $\sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}(x)$)

$\Rightarrow 2\cos^2\ (\cos^{-1}(x)) - 1 - 1 \leq 0$

(since $\cos(2\cos^{-1}(x))$

$= 2\cos^2\ (\cos^{-1}(x)) - 1)$

$\Rightarrow 2\cos^2\ (\cos^{-1}(x)) - 2 \leq 0\ \Rightarrow$
$\cos^2\ (\cos^{-1}(x)) - 1 \leq 0\ \Rightarrow$

$\cos\ (\cos^{-1}(x)) \cos\ (\cos^{-1}(x)) - 1 \leq 0 \Rightarrow$
$x^2 - 1 \leq 0$ (since $\cos\ (\cos^{-1}(x)) = x$)

$\Rightarrow (x-1)(x+1) \leq 0$

$S = \{x \in \mathbb{R}: -1 \leq x \leq 1\}$

Q2/1

$$\boldsymbol{f(x) = \sqrt{x} - sgn(\sqrt{x}),\ \ at\ x = 1}$$

$$\lim_{x \to 1^+} f(x) = \sqrt{1} - sgn\left(\sqrt{x \to 1^+}\right) = 0$$
$$\lim_{x \to 1^-} f(x) = \sqrt{1} - sgn(\sqrt{x \to 1^-}) = 0$$
$$f(1) = \sqrt{1} - sgn(\sqrt{1}) = 0$$

∴ Continuous!

Q2/2

$$\boldsymbol{f(x) = \lfloor x \rfloor - \lfloor -x \rfloor,\qquad at\ x = 0}$$

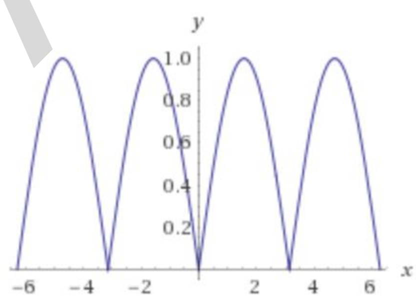
$$\lim_{x \to 0^+} f(x) = \lfloor \boldsymbol{x \to 0^+} \rfloor - \lfloor \boldsymbol{-x \to 0^+} \rfloor = 0$$
$$\lim_{x \to 0^-} f(x) = \lfloor \boldsymbol{x \to 0^-} \rfloor - \lfloor \boldsymbol{-x \to 0^-} \rfloor = 2$$
$$f(0) = \lfloor \boldsymbol{0} \rfloor - \lfloor \boldsymbol{-0} \rfloor = 0$$

∴ Not Continuous!

Q3/1

$$\boldsymbol{y = |\sin(x)|}$$

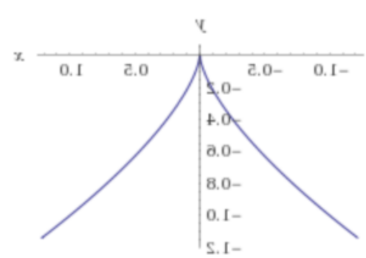
$D_f = \mathbb{R}$, $R_f = [0,1]$



Q3/2

$$\boldsymbol{y = -\sqrt[3]{x^2}}$$

$D_f = \mathbb{R}$, $R_f = (-\infty, 0]$



Q4/1

$$\int sin(5x)cos(2x)dx$$

$$= \int \frac{1}{2}\left(sin(5x-2x)+sin(5x+2x)\right)dx$$

$$= \frac{1}{2}\int (\sin(3x)+\sin(7x))\,dx$$

$$= \frac{1}{2}\int \sin(3x)\,dx + \frac{1}{2}\int \sin(7x)\,dx$$

$$= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + \boldsymbol{C}$$

Q4/2

$$\int \frac{\mathbf{x^3+x}}{\mathbf{x-1}}\mathbf{dx}$$

$$\int \frac{x^3+x}{x-1}dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2ln|x-1|+c$$

Q5/1

$$\lim_{x\to\frac{\pi}{2}} \sec^{99}(x)$$

$$\lim_{x\to\frac{\pi}{2}^+}\left(\frac{1}{\cos(x)}\right)^{99} = \left(\frac{1}{\cos(x)\to 0^-}\right)^{99} = -\infty$$

$$\lim_{x\to\frac{\pi}{2}^-}\left(\frac{1}{\cos(x)}\right)^{99} = \left(\frac{1}{\cos(x)\to 0^+}\right)^{99} = \infty$$

There is no limit (Right limit≠Left limit)

Q5/2

$$\lim_{x\to\infty}\frac{3^x+3^{-x}}{3^x-3^{-x}}$$

$$\lim_{x\to\infty}\frac{\frac{3^x}{3^x}+\frac{3^{-x}}{3^x}}{\frac{3^x}{3^x}-\frac{3^{-x}}{3^x}}$$

$$= \lim_{x\to\infty}\frac{1+3^{-2x}}{1-3^{-2x}} = \frac{1+3^{-\infty}}{1-3^{-\infty}} = \frac{1}{1} = 1$$

Q6/

Find all the **asymptotes** of the function $\boldsymbol{f(x)} = \frac{x^2-x+4}{2x+2}$

$$\begin{array}{l|l} \multicolumn{2}{l}{\frac{1}{2}x-1} \\ x^2-x+4 & 2x+2 \\ x^2+x & \\ \hline -2x+4 & \\ -2x-2 & \\ \hline 6 & \end{array}$$

Oblique Asymptote

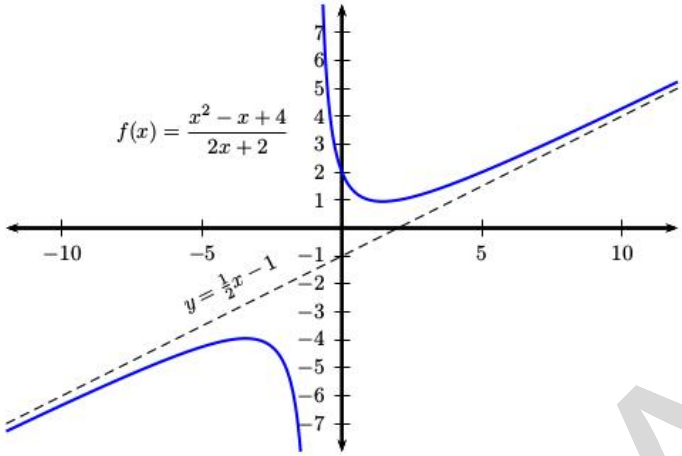
Vertical asymptote:

$\frac{x^2-x+4}{2x+2} \to \pm\infty$ as $x \to -1$

Oblique asymptote:

$\frac{x^2-x+4}{2x+2}$ is asymptotic to $\frac{x}{2}-1$

Also, $2x+2=0 \Longrightarrow x=-1$ is the vertical asymptote



Q7/

Prove that the function $\boldsymbol{f(x) = |x-2|}$ is continuous, but not differentiable at $\boldsymbol{x = 2}$.

$$f(x) = |x-2| = \begin{cases} x-2; & x \geq 2 \\ -(x-2); & x < 2 \end{cases}$$

1. Continuity

a) $f(2) = |2-2| = |0| = 0 \Rightarrow$ f is defined at $x = 2$

b) Let's compute the right and left limits separately:

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} |x-2| = \lim_{x\to 2^+} (x-2) = 2-2 = 0$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} |x-2| = \lim_{x\to 2^-} -(x-2) = -(2-2) = 0$

$\therefore \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 0$

$\therefore \lim_{x\to 2} f(x) = 0$ (the limit exist)

c) $\lim_{x\to 2} f(x) = f(2) = 0$

$\therefore f(x)$ is continuous at $x = 2$

2. Differentiable

$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{|(x+\Delta x)-2| - |x-2|}{\Delta x}$

$\therefore f'(2) = \lim_{\Delta x\to 0} \frac{|(2+\Delta x)-2| - |2-2|}{\Delta x} = \lim_{\Delta x\to 0} \frac{|0+\Delta x| - |0|}{\Delta x}$

$= \lim_{\Delta x\to 0} \frac{|\Delta x|}{\Delta x}$

Let's compute the right and left limits separately:

$\lim_{\Delta x\to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{\Delta x}{\Delta x} = \lim_{\Delta x\to 0^+} 1 = 1$

$\lim_{\Delta x\to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{\Delta x} = \lim_{\Delta x\to 0^+} (-1) = -1$

$\therefore \lim_{\Delta x\to 0^+} \frac{|\Delta x|}{\Delta x} \neq \lim_{\Delta x\to 0^-} \frac{|\Delta x|}{\Delta x}$

$\therefore \lim_{\Delta x\to 0} \frac{|\Delta x|}{\Delta x}$ dose not exist

$\therefore f'(2)$ dose not exist

$\therefore f(x)$ is not differentiable at $x = 2$.

Q8/

Evaluate $\lim\limits_{x\to1}\left(\int_0^x t\sqrt{t+1}dt\right)$

At first we evaluate $\int_0^x t\sqrt{t+1}dt$

Let $t+1=u \Rightarrow x=u-1 \Rightarrow dt=du$

If $t=0 \Rightarrow u=1$, if $t=x \Rightarrow u=x+1$

$$\therefore \int_0^x t\sqrt{t+1}dt = \int_1^{x+1}(u-1)\sqrt{u}\ du = \int_1^{x+1}\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du$$

$$= [\tfrac{2}{5}\ u^{\frac{5}{2}}-\tfrac{2}{3}\ u^{\frac{3}{2}}]_0^{x+1} = \left(\tfrac{2}{5}\ (x+1)^{\frac{5}{2}}-\tfrac{2}{3}\ (x+1)^{\frac{3}{2}}\right)-\left(\tfrac{2}{5}\ (1)^{\frac{5}{2}}-\tfrac{2}{3}\ (1)^{\frac{3}{2}}\right)$$

$$= \left(\tfrac{2}{5}\ (x+1)^{\frac{5}{2}}-\tfrac{2}{3}\ (x+1)^{\frac{3}{2}}\right)-\left(\tfrac{2}{5}-\tfrac{2}{3}\right)$$

$$\therefore \lim_{x\to1}\left(\int_0^x t\sqrt{t+1}dt\right) = \lim_{x\to1}\left(\left(\tfrac{2}{5}\ (x+1)^{\frac{5}{2}}-\tfrac{2}{3}\ (x+1)^{\frac{3}{2}}\right)+\tfrac{4}{15}\right) = \left(\tfrac{2}{5}\ (2)^{\frac{5}{2}}-\tfrac{2}{3}\ (2)^{\frac{3}{2}}+\tfrac{4}{15}\right)$$

$$= \left(\tfrac{8\sqrt{2}}{5}-\tfrac{4\sqrt{2}}{3}\right)+\tfrac{4}{15} = \tfrac{4\sqrt{2}}{15}+\tfrac{4}{15} = \tfrac{4(\sqrt{2}+1)}{15}$$

Q9/

Suppose that $\lim\limits_{x\to1} f(x)=2$ and $\lim\limits_{x\to1} g(x)=4$. Find $\lim\limits_{x\to1}\left(\frac{f(x)}{g(x)}\right)^2$

$$\lim_{x\to1}\left(\frac{f(x)}{g(x)}\right)^2 = \left(\lim_{x\to1}\frac{f(x)}{g(x)}\right)^2 = \left(\frac{2}{4}\right)^2 = \frac{1}{4}$$

Q10/

Use the **definition** of the derivative to prove that $\frac{d}{dx}(x^n) = n\,x^{n-1}$

$$\frac{d}{dx}(x^n) = \lim_{h\to0}\frac{(x+h)^n-x^n}{h}$$

$$= \lim_{h\to0}\frac{x^n+\binom{n}{1}x^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\cdots+\binom{n}{n-1}xh^{n-1}+h^n-x^n}{h}$$

$$= \lim_{h\to0}\frac{\binom{n}{1}x^{n-1}h+\binom{n}{2}x^{n-2}h^2+\binom{n}{3}x^{n-3}h^3+\cdots+\binom{n}{n-1}xh^{n-1}+h^n}{h}$$

$$= \lim_{h\to0}\frac{h\left(\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h^1+\binom{n}{3}x^{n-3}h^2+\cdots+\binom{n}{n-1}xh^{n-2}+h^{n-1}\right)}{h}$$

$$= \lim_{h\to0}\left(\binom{n}{1}x^{n-1}+\binom{n}{2}x^{n-2}h^1+\binom{n}{3}x^{n-3}h^2+\cdots+\binom{n}{n-1}xh^{n-2}+h^{n-1}\right)$$

$=nx^{n-1}+0+\cdots+0=nx^{n-1}$.

Q11/

Find all the points $(\boldsymbol{x}, \boldsymbol{y})$ on the graph of $\boldsymbol{y} = \boldsymbol{x}^{\mathbf{2}}$ with tangent lines passing through the point $(\mathbf{3}, \mathbf{8})$

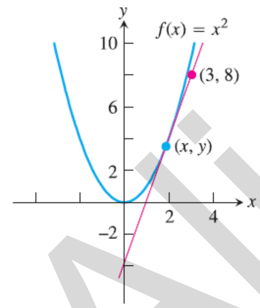
$m = \frac{y-8}{x-3}$

$f(x) = x^2 \Rightarrow f'(x) = 2x; m = f'(x) \Rightarrow m = 2x$

$$\Rightarrow \frac{y-8}{x-3} = 2x \Rightarrow \frac{x^2-8}{x-3} = 2x \Rightarrow x^2 - 8 = 2x^2 - 6x$$

$\Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x = 4 \; or \; x = 2$

$\Rightarrow f(4) = 16, f(2) = 4 \Rightarrow (4,16) \; or \; (2,4).$



Q12/

Suppose that $\boldsymbol{f}(\boldsymbol{x}) = \frac{\boldsymbol{e}^{\boldsymbol{x}} - \boldsymbol{e}^{-\boldsymbol{x}}}{\boldsymbol{e}^{\boldsymbol{x}} + \boldsymbol{e}^{-\boldsymbol{x}}}$. Evaluate $\boldsymbol{f}'(\mathbf{0})$

$$f'(x) = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \mathbf{1} - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^{\mathbf{2}}}$$

$$f'(0) = 1 - \frac{(1-1)^2}{(1+1)^2} = 1$$