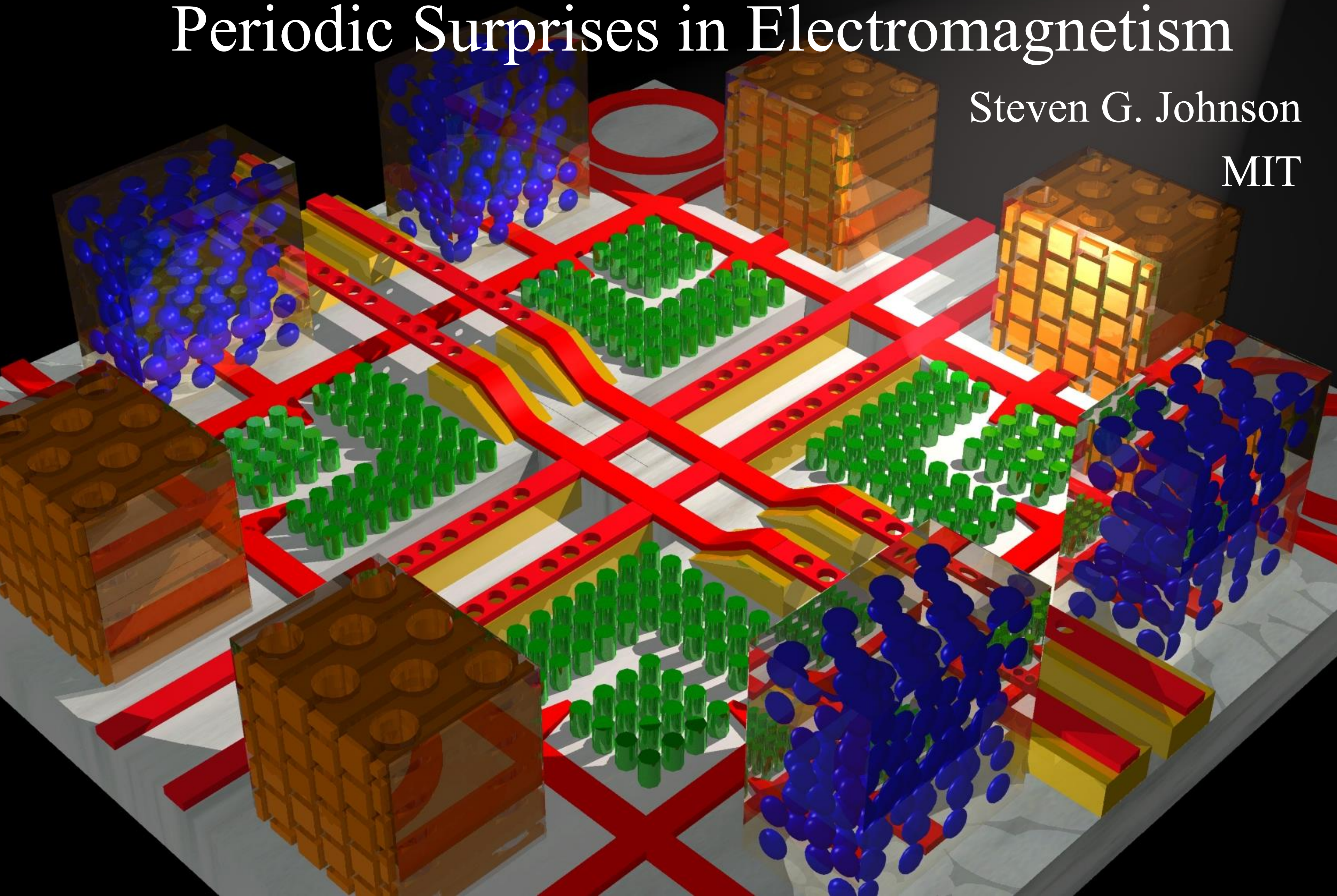
Photonic Crystals:

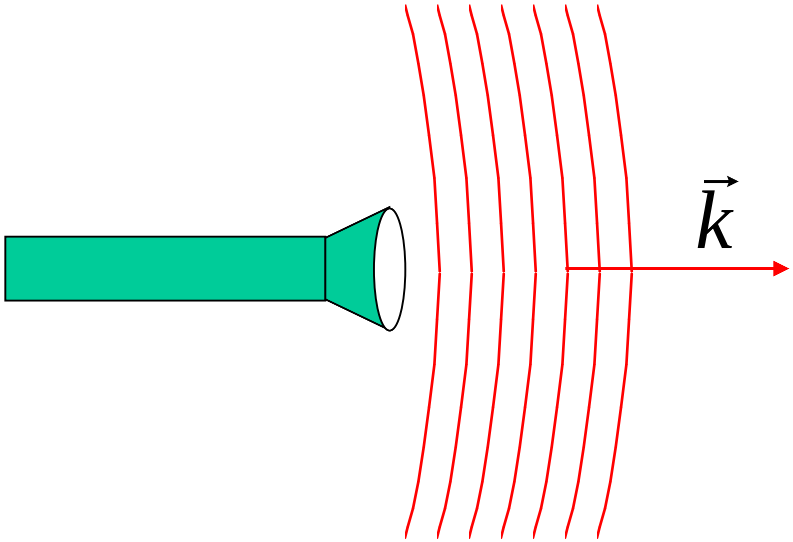
Periodic Surprises in Electromagnetism

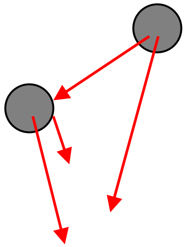


Steven G. Johnson

MIT

To Begin: A Cartoon in 2d



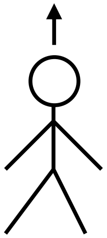


scattering

planewave

$$\vec{E},\vec{H} \sim e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\left|\vec{k}\right| = \omega / c = \frac{2\pi}{\lambda}$$



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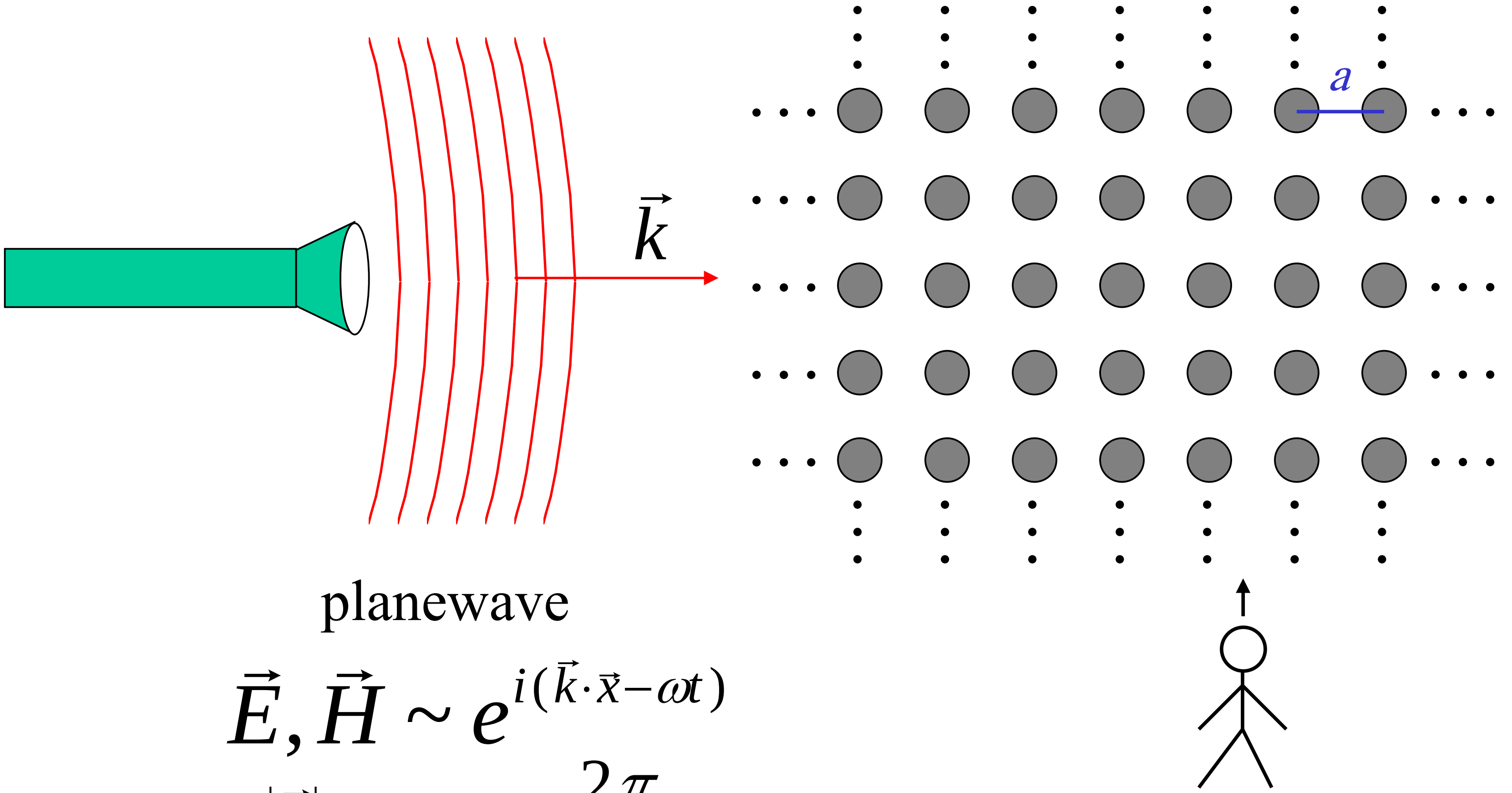


planewave

$$\vec{E}, \vec{H} \sim e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

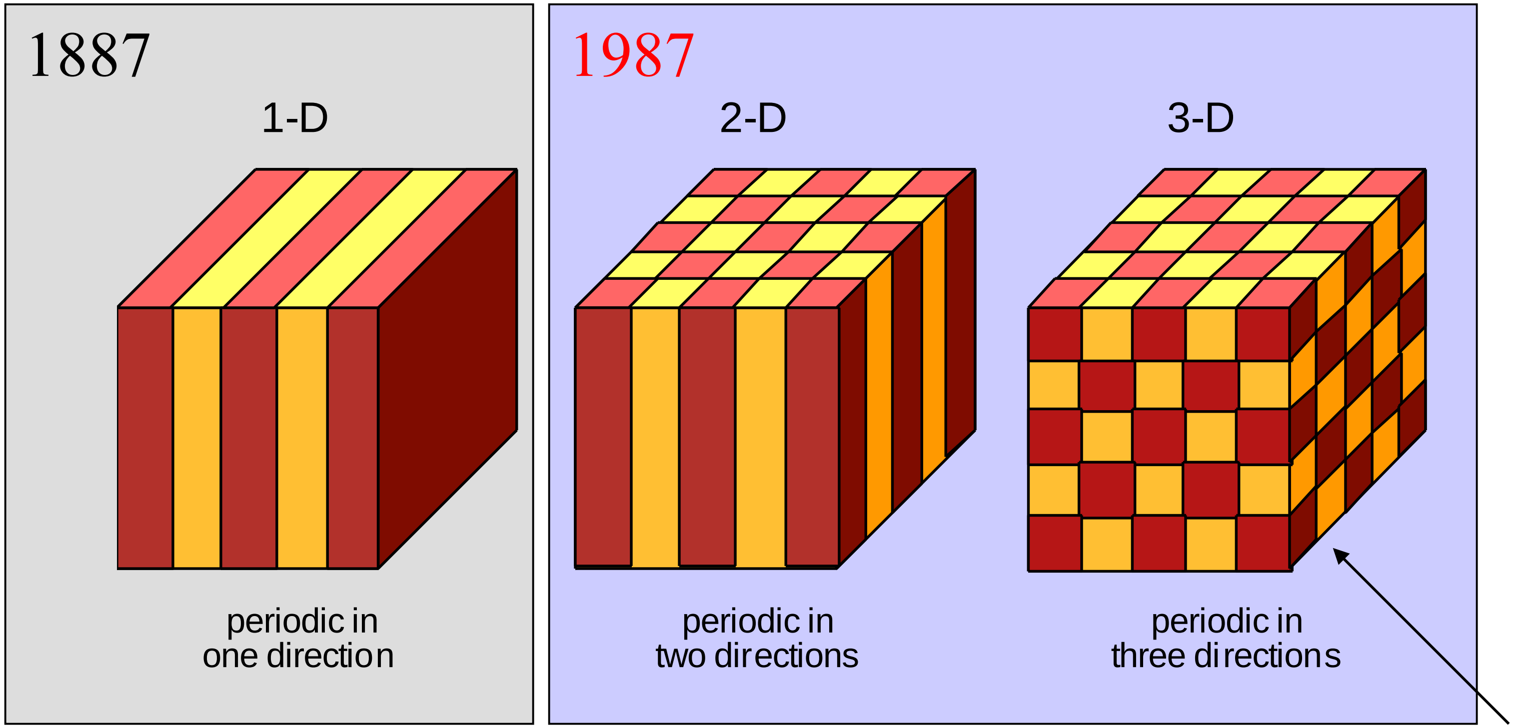
$$\left|\vec{k}\right| = \omega / c = \frac{2\pi}{\lambda}$$

for most λ, beam(s) propagate through crystal without scattering (scattering cancels coherently)

...but for some λ (~ 2*a*), no light can propagate: a photonic band gap

Photonic Crystals

periodic electromagnetic media



1887

1-D

periodic in
one direction

1987

2-D

periodic in
two directions

3-D

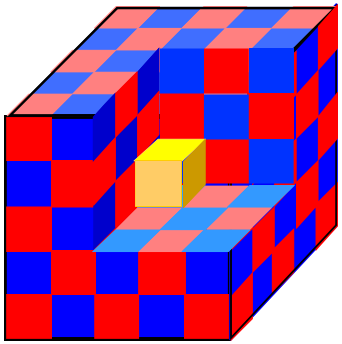
periodic in
three directions

(need a
more
complex
topology)

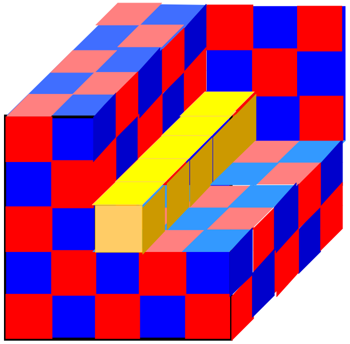
with photonic band gaps: “optical insulators”

Photonic Crystals

periodic electromagnetic media



can trap light in cavities



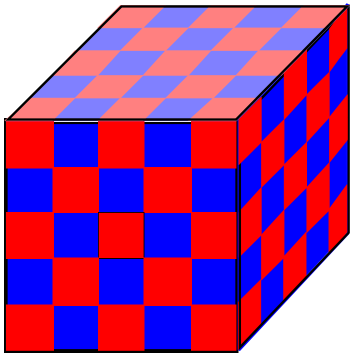
and waveguides ("wires")

magical oven mitts for
holding and controlling light

with photonic band gaps: "optical insulators"

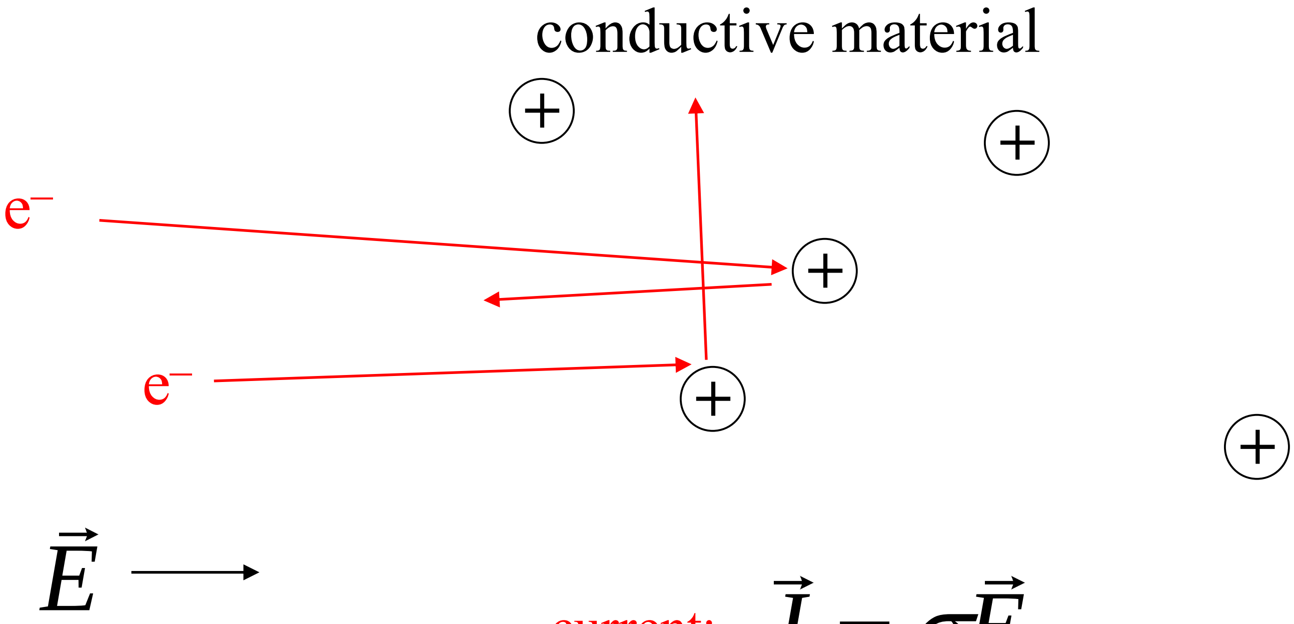
Photonic Crystals

periodic electromagnetic media



But how can we understand such complex systems?
Add up the infinite sum of scattering? Ugh!

A mystery from the 19th century



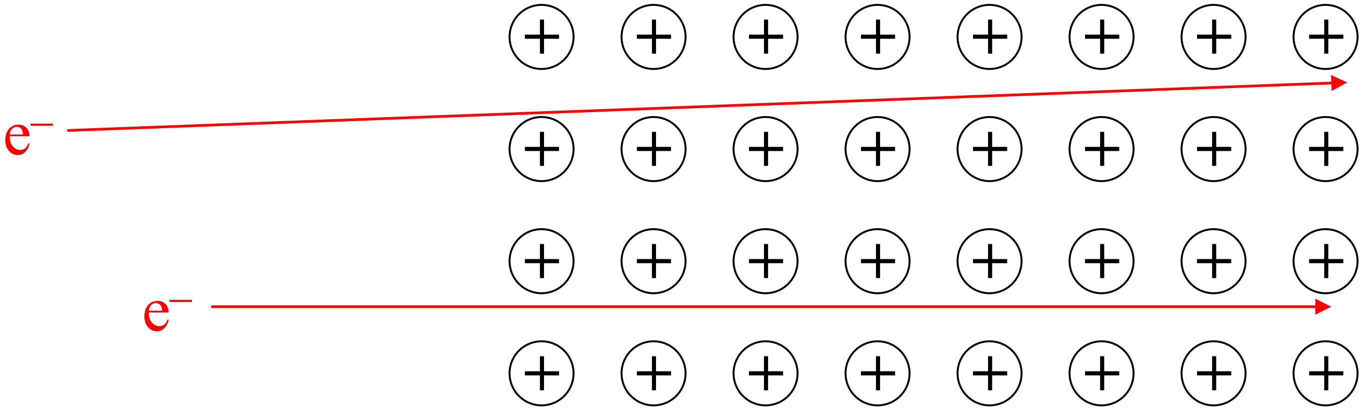
current: $\vec{J} = \sigma\vec{E}$

conductivity (measured)

mean free path (distance) of electrons

A mystery from the 19th century

crystalline conductor (e.g. copper)



10's of periods!

$\vec{E} \longrightarrow$

current: $\vec{J} = \sigma\vec{E}$

conductivity (measured)

↓

mean free path (distance) of electrons

A mystery solved…

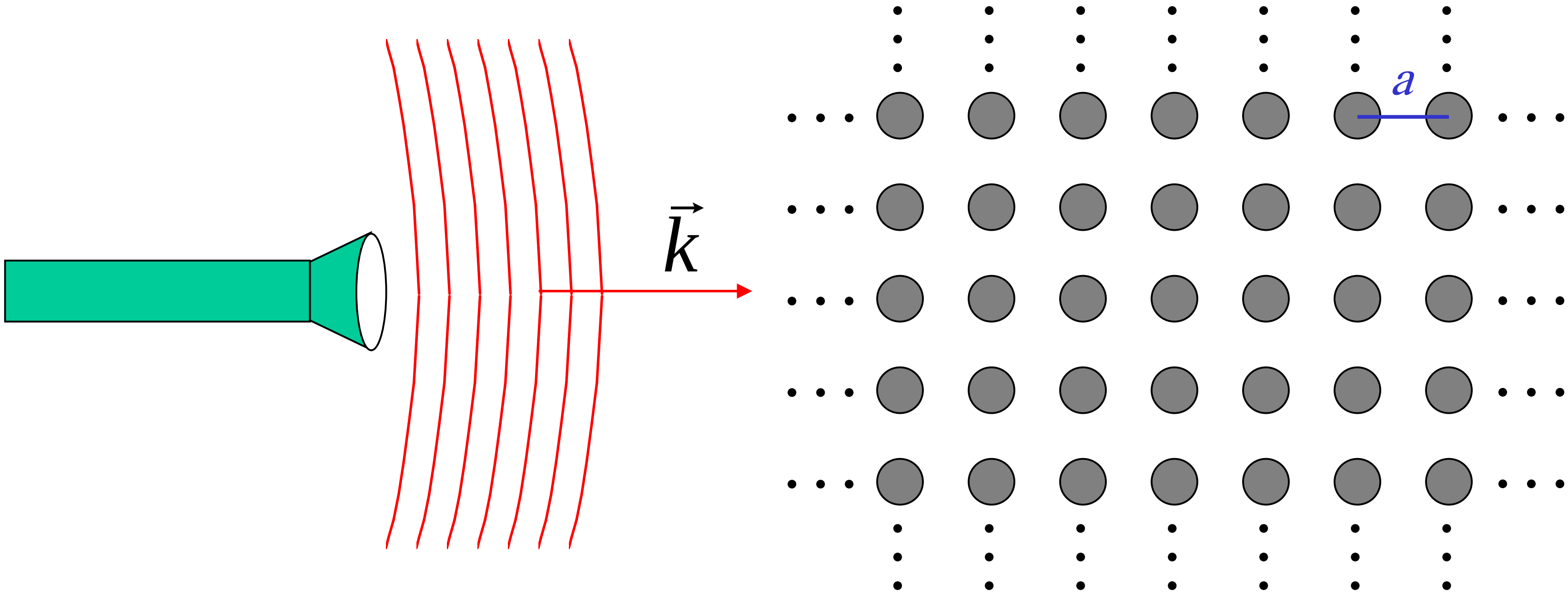
① electrons are waves (quantum mechanics)

② waves in a periodic medium can propagate without scattering:

Bloch's Theorem (1d: Floquet's)

The foundations do not depend on the specific wave equation.

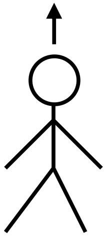
Time to Analyze the Cartoon



planewave

$$\vec{E},\vec{H} \sim e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$\left|\vec{k}\right| = \omega / c = \frac{2\pi}{\lambda}$$



for most λ, beam(s) propagate through crystal without scattering (scattering cancels coherently)

...but for some λ (~ 2a), no light can propagate: a photonic band gap

Fun with Math

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{H} = i\frac{\omega}{c}\vec{H}$$

First task:

get rid of this mess

$$\vec{\nabla} \times \vec{H} = \varepsilon\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \cancelto{0}{\vec{J}} = i\frac{\omega}{c}\varepsilon\vec{E}$$

dielectric function $\varepsilon(\mathbf{x}) = n^2(\mathbf{x})$

$$\underset{\text{eigen-operator}}{\underline{\nabla \times \frac{1}{\varepsilon}\nabla \times}}\,\vec{H} = \underset{\text{eigen-value}}{\underline{\left(\frac{\omega}{c}\right)^2}}\,\vec{H}$$

eigen-state

+ constraint

$$\nabla \cdot \vec{H} = 0$$

Hermitian Eigenproblems

$$\underbrace{\nabla \times \frac{1}{\varepsilon} \nabla \times}_{\text{eigen-operator}} \vec{H} = \underbrace{\left(\frac{\omega}{c}\right)^2}_{\text{eigen-value}} \underbrace{\vec{H}}_{\text{eigen-state}}$$

+ constraint

$$\nabla \cdot \vec{H} = 0$$

Hermitian for real (lossless) ε

➡ well-known properties from linear algebra:

ω are real (lossless)

eigen-states are orthogonal

eigen-states are complete (give all solutions)

Periodic Hermitian Eigenproblems

[G. Floquet, “Sur les équations différentielles linéaries à coefficients périodiques,” *Ann. École Norm. Sup.* **12**, 47–88 (1883).]
[F. Bloch, “Über die quantenmechanik der electronen in kristallgittern,” *Z. Physik* **52**, 555–600 (1928).]

if eigen-operator is periodic, then Bloch-Floquet theorem applies:

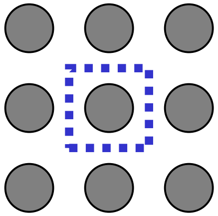
can choose: $$\vec{H}(\vec{x},t) = e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}\vec{H}_{\vec{k}}(\vec{x})$$

planewave

periodic “envelope”

Corollary 1: **k** is conserved, *i.e.* no scattering of Bloch wave

Corollary 2: $\vec{H}_{\vec{k}}$ given by finite unit cell,
so ω are discrete $\omega_n(\mathbf{k})$



Periodic Hermitian Eigenproblems

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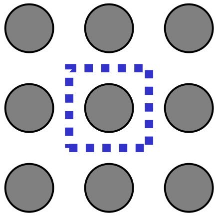
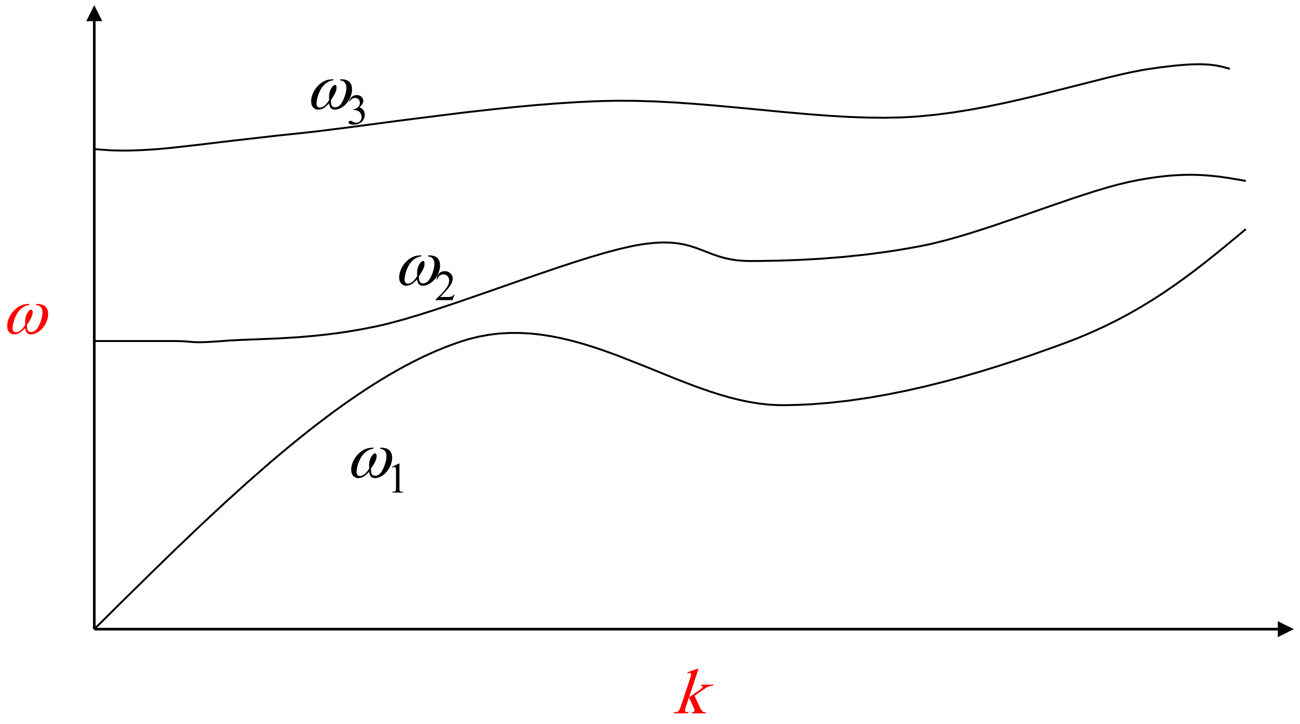


band diagram (dispersion relation)



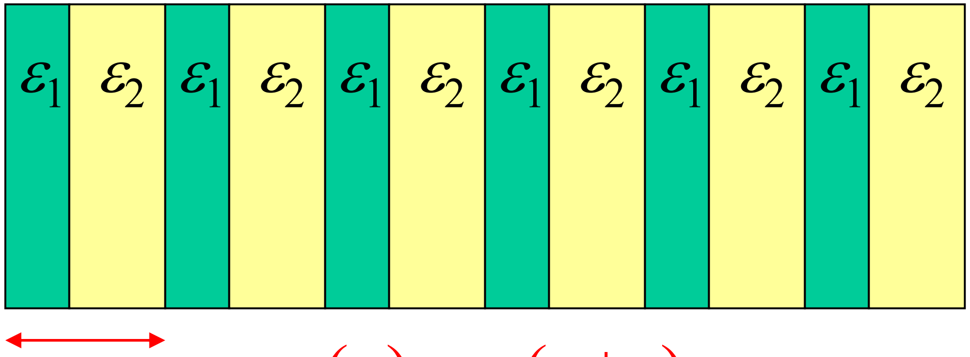
map of
what states
exist &
can interact

?

range of k?

Periodic Hermitian Eigenproblems in 1d

$$H(x) = e^{ikx} H_k(x)$$



$\varepsilon(x) = \varepsilon(x+a)$

Consider $k+2\pi/a$: $$e^{i(k+\frac{2\pi}{a})x} H_{k+\frac{2\pi}{a}}(x) = e^{ikx}\left[e^{i\frac{2\pi}{a}x} H_{k+\frac{2\pi}{a}}(x) \right]$$

periodic!

satisfies same equation as H_k

$= H_k$

k is periodic:

$k + 2\pi/a$ equivalent to k

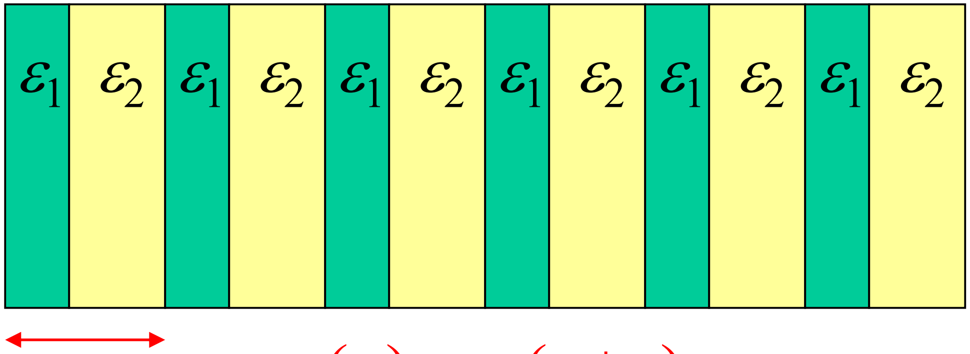
“quasi-phase-matching”

Periodic Hermitian Eigenproblems in 1d

k is periodic:

$k + 2\pi/a$ equivalent to k

"quasi-phase-matching"



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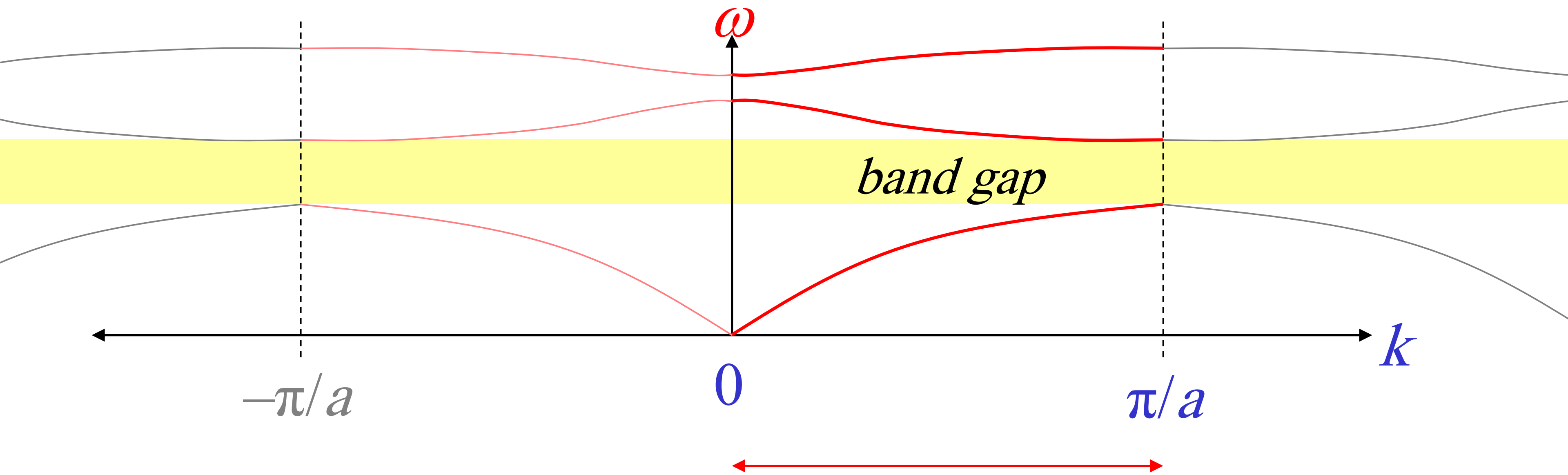
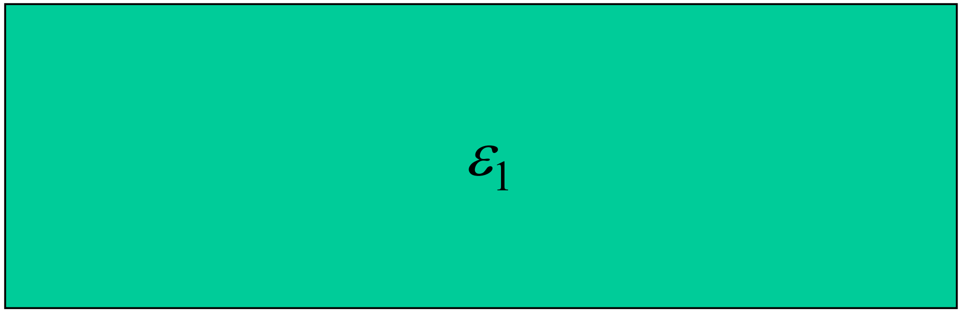


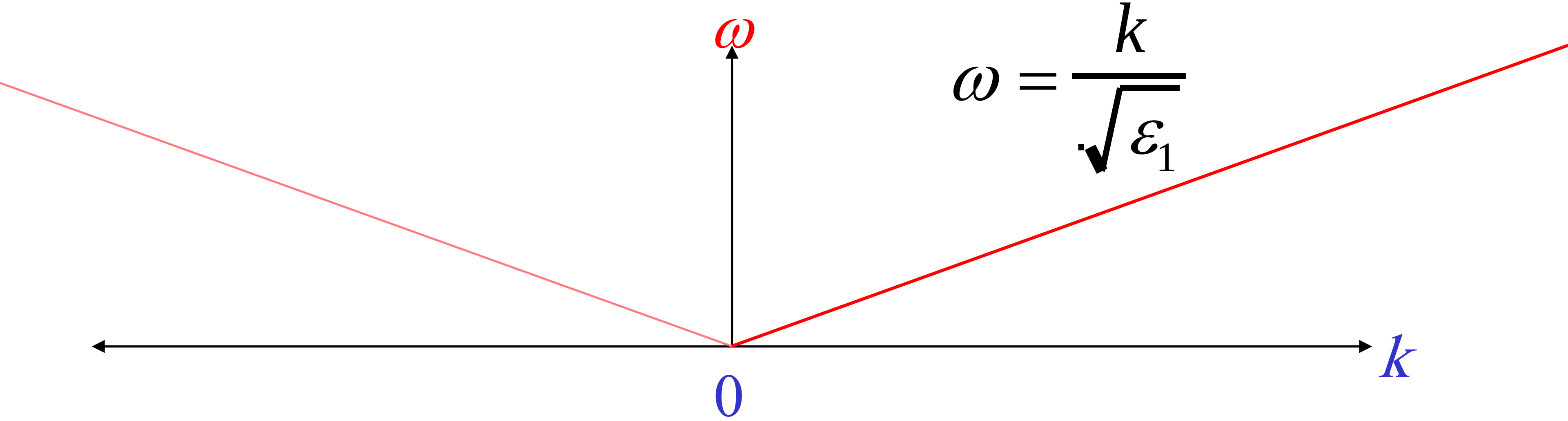
irreducible Brillouin zone

Any 1d Periodic System has a Gap

[Lord Rayleigh, “On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure,” *Philosophical Magazine* **24**, 145–159 (1887).]

Start with
a uniform (1d) medium:





$$\omega = \frac{k}{\sqrt{\varepsilon_1}}$$

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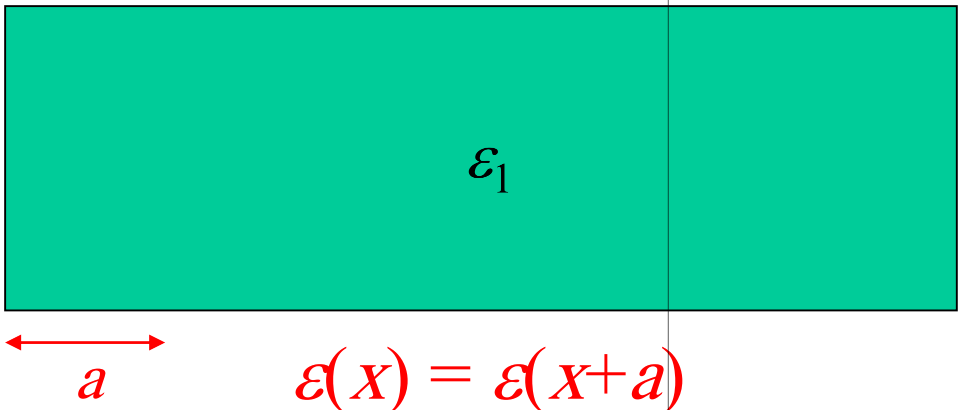
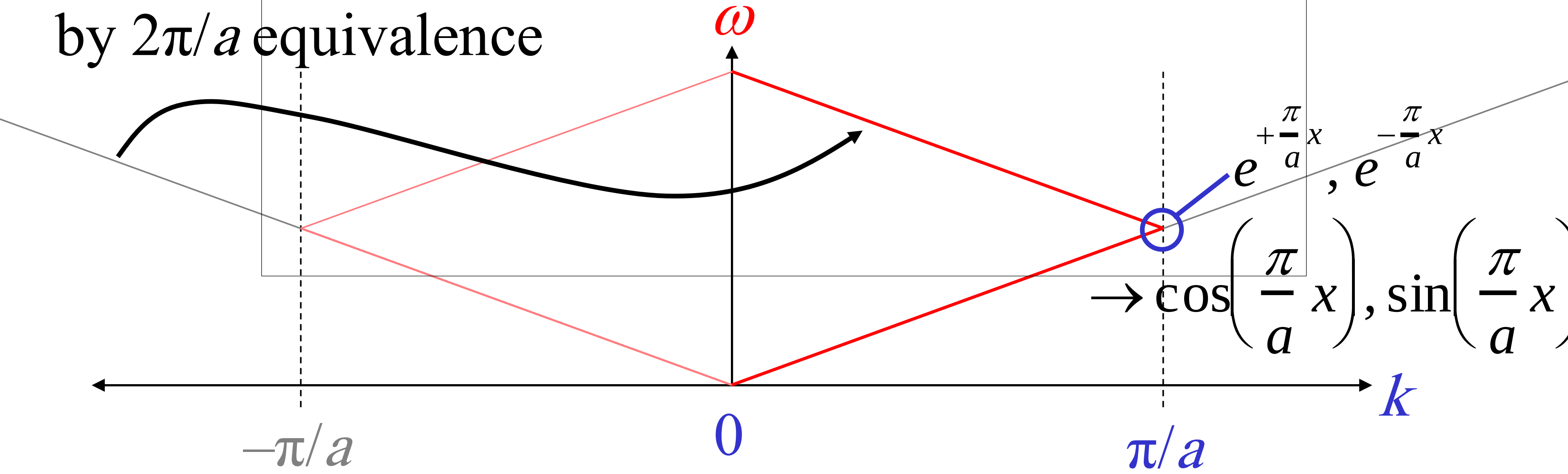
يتعذر عرض الصورة.

Treat it as
“artificially” periodic



$\varepsilon(x) = \varepsilon(x+a)$

bands are “folded”
by 2π/*a* equivalence



$e^{+\frac{\pi}{a}x}, e^{-\frac{\pi}{a}x}$

$\rightarrow \cos\left(\frac{\pi}{a}x\right), \sin\left(\frac{\pi}{a}x\right)$

Any 1d Periodic System has a Gap

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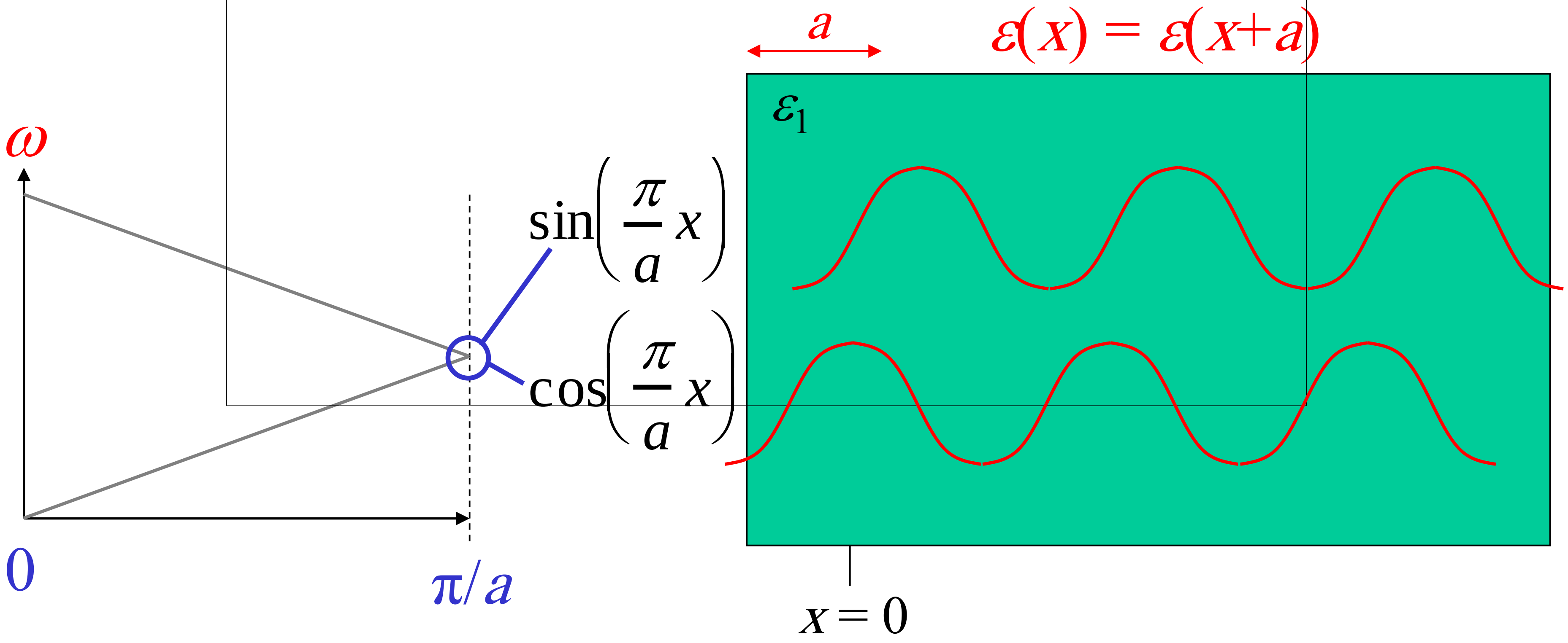
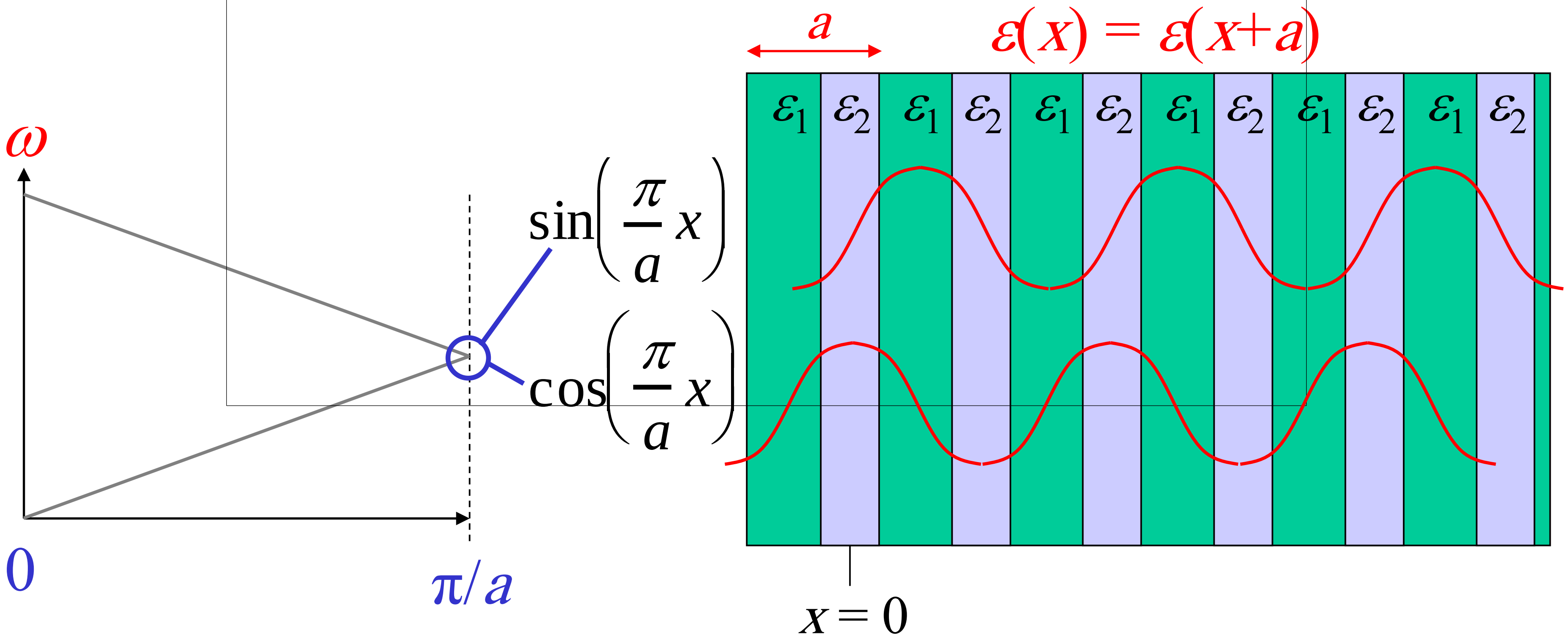


Any 1d Periodic System has a Gap

[Lord Rayleigh, "On the maintenance of vibrations by forces of double frequency, and on the propagation of waves through a medium endowed with a periodic structure," *Philosophical Magazine* **24**, 145–159 (1887).]

Add a small
"real" periodicity

$$\varepsilon_2 = \varepsilon_1 + \Delta\varepsilon$$



Any 1d Periodic System has a Gap

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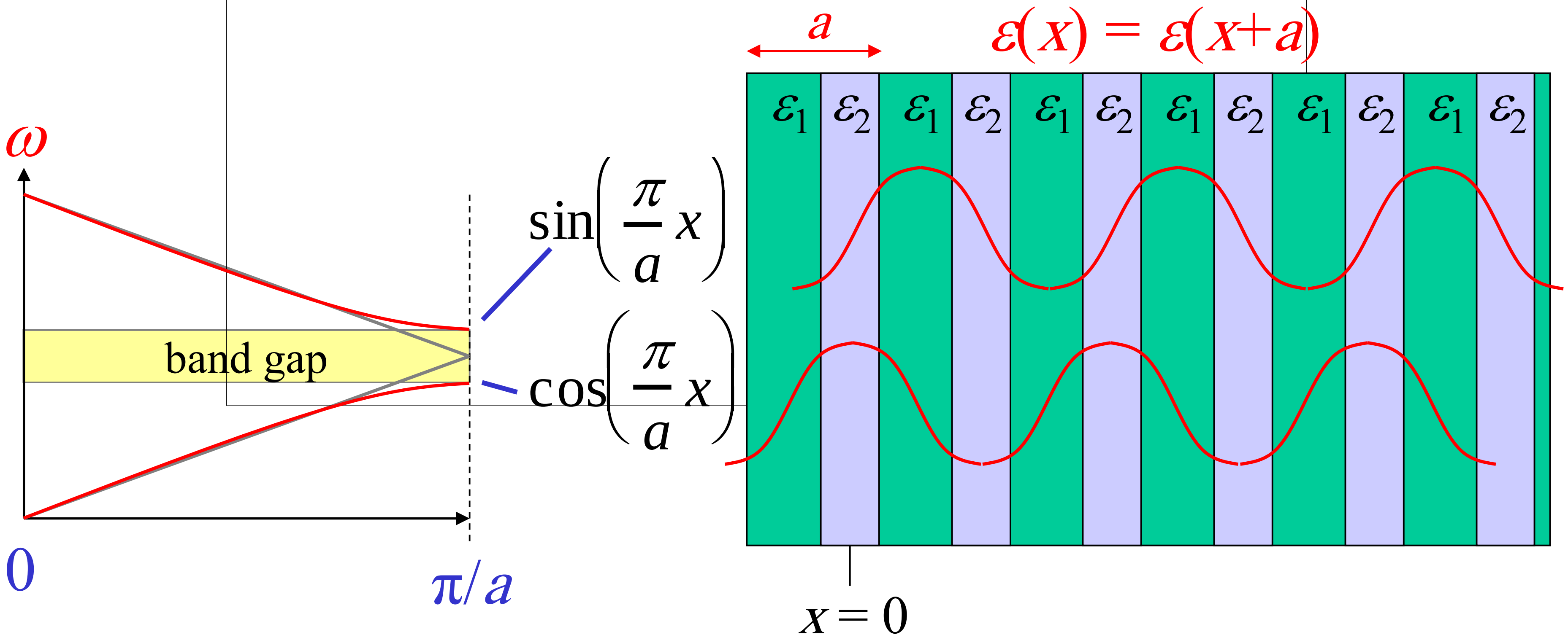
Add a small
“real” periodicity
$\varepsilon_2 = \varepsilon_1 + \Delta\varepsilon$

Splitting of degeneracy:
state concentrated in higher index (ε_2)
has lower frequency



Some 2d and 3d systems have gaps

- In general, eigen-frequencies satisfy Variational Theorem:

$$\omega_1(\vec{k})^2 = \min_{\substack{\vec{E}_1 \\ \nabla\cdot\varepsilon\vec{E}_1=0}} \frac{\int \left|\left(\nabla + i\vec{k}\right)\times \vec{E}_1\right|^2}{\int \varepsilon\left|\vec{E}_1\right|^2} c^2$$

"kinetic"

inverse "potential"

$$\omega_2(\vec{k})^2 = \min_{\substack{\vec{E}_2 \\ \nabla\cdot\varepsilon\vec{E}_2=0 \\ \int \varepsilon E_1^{*}\cdot E_2=0}} \text{"}\cdots\text{"}$$

bands "want" to be in high-ε

…but are forced out by orthogonality

–> band gap (maybe)

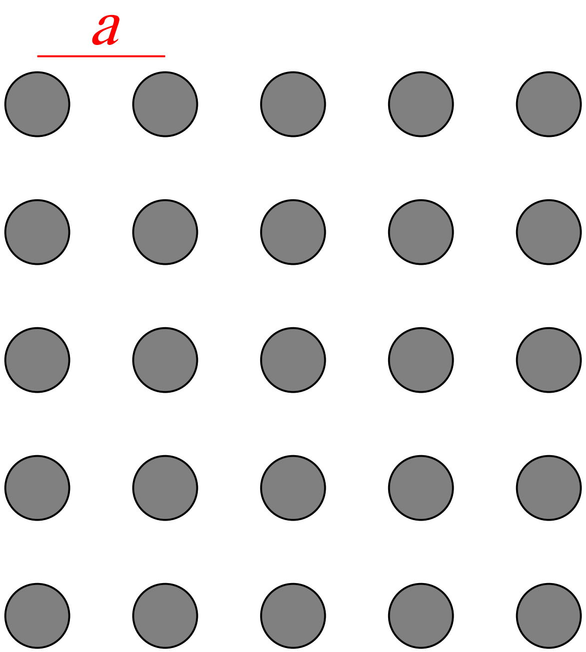
algebraic interlude

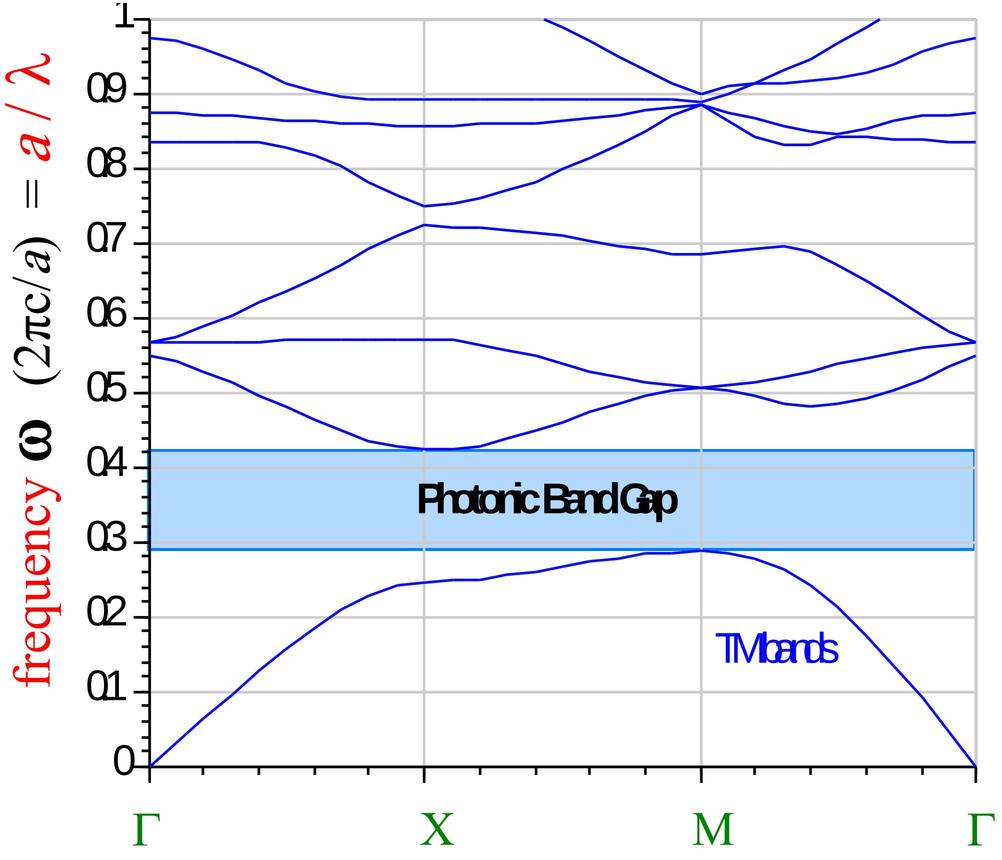
algebraic interlude completed…

… I hope you were taking notes*

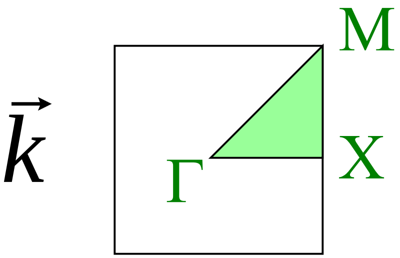
[*if not, see *e.g.*: Joannopoulos, Meade, and Winn, *Photonic Crystals: Molding the Flow of Light*]

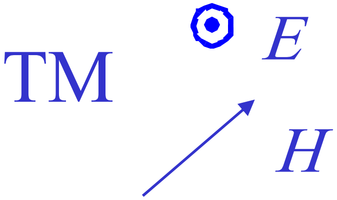
2d periodicity, ε=12:1





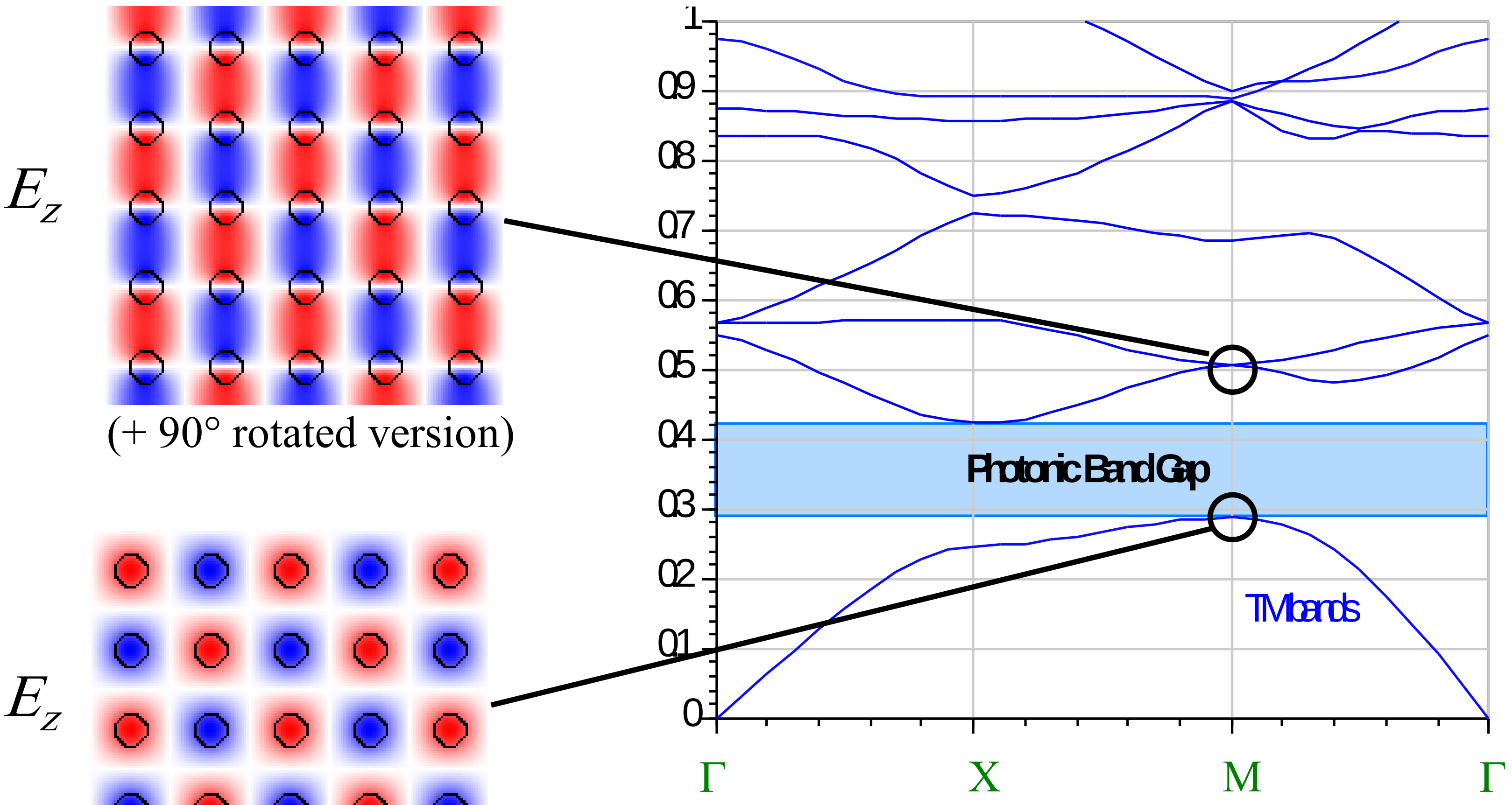
irreducible Brillouin zone

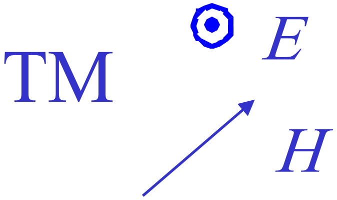




gap for
$n > \sim 1.75:1$

2d periodicity, ε=12:1





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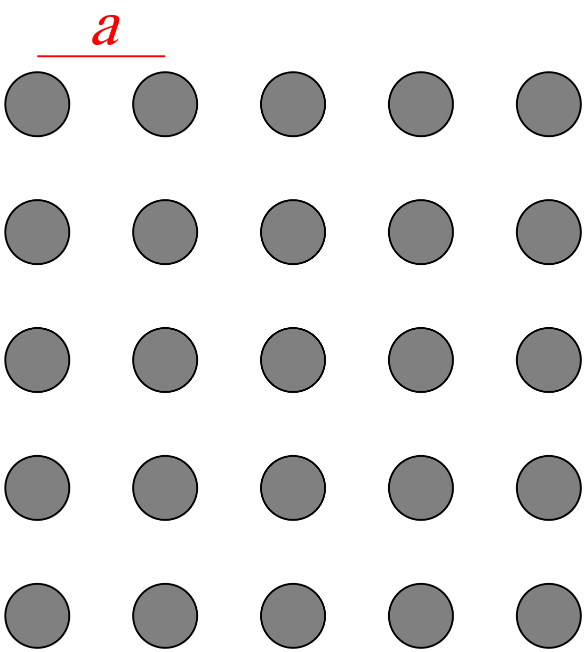

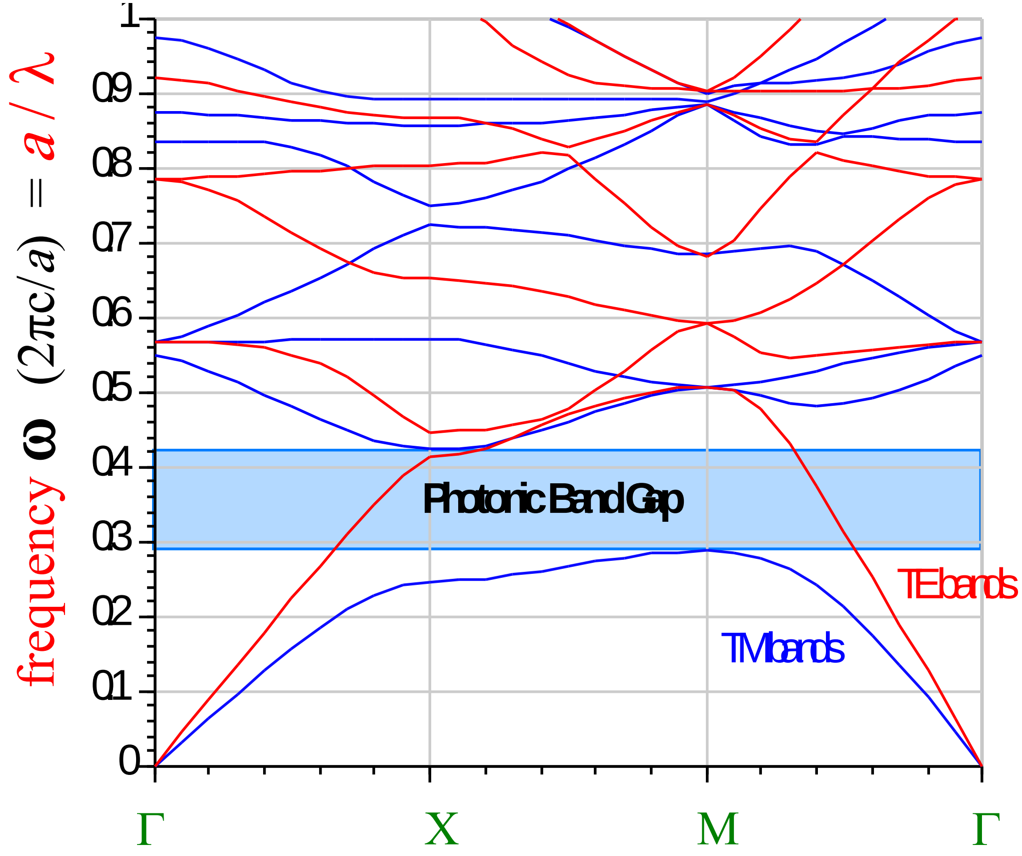

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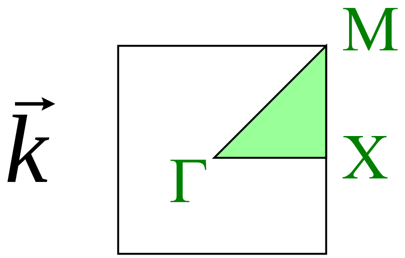
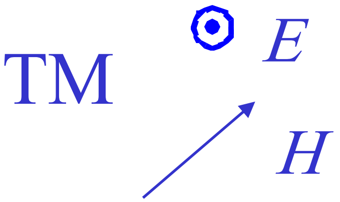
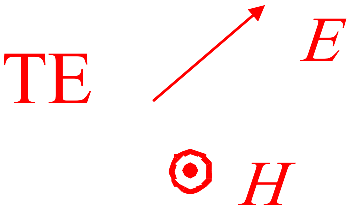
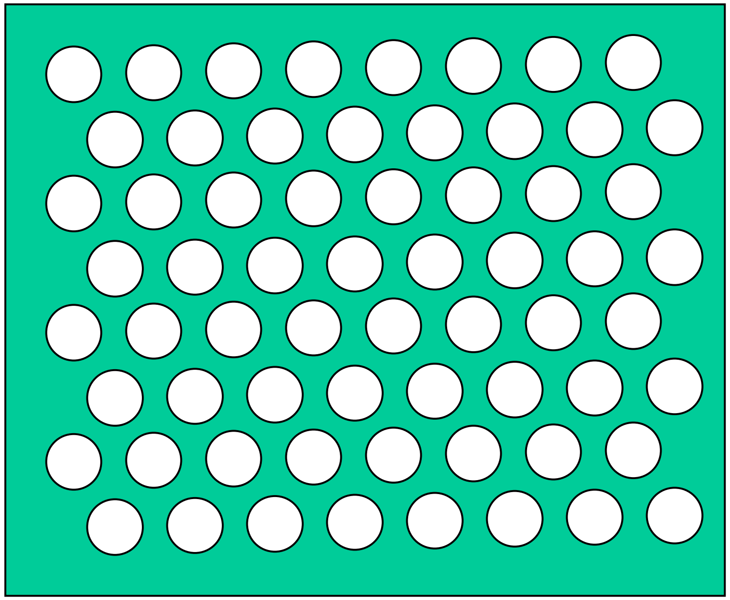






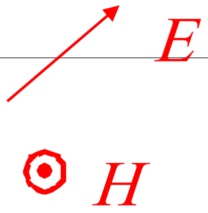
2d photonic crystal: TE gap, ε=12:1



TE bands

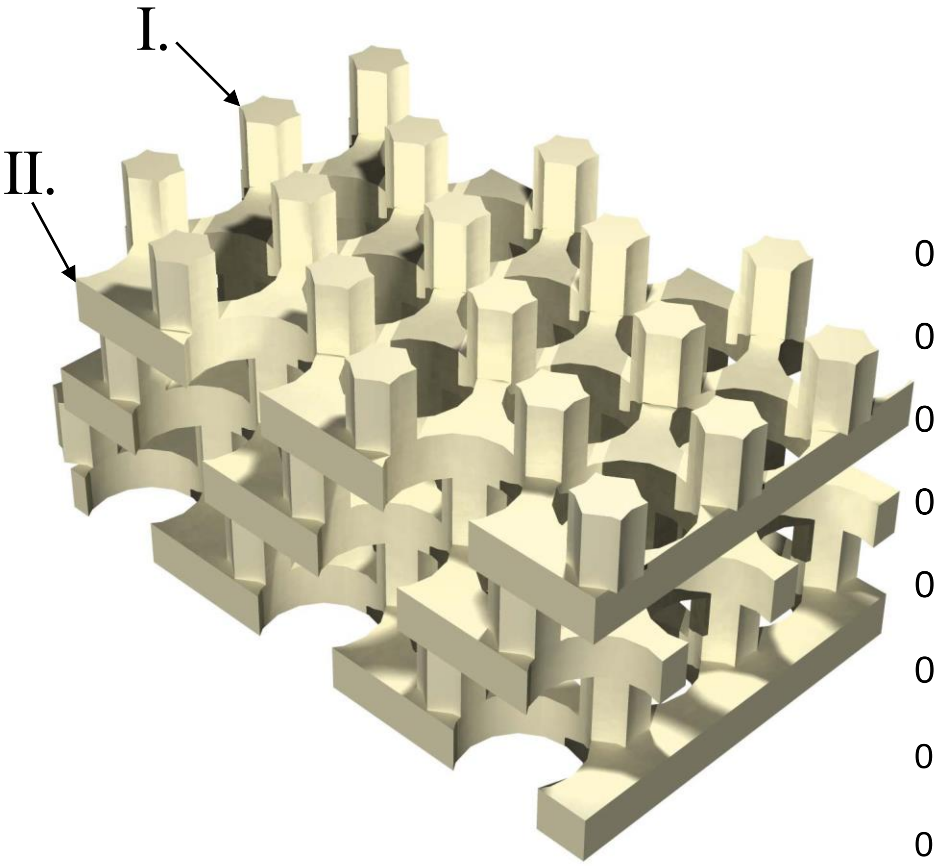
TM bands

TE

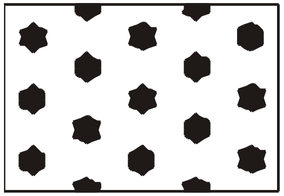


gap for $n > \sim 1.4:1$

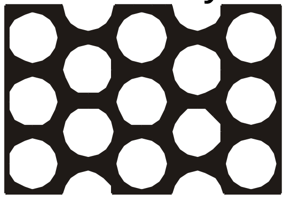
3d photonic crystal: complete gap , ε=12:1

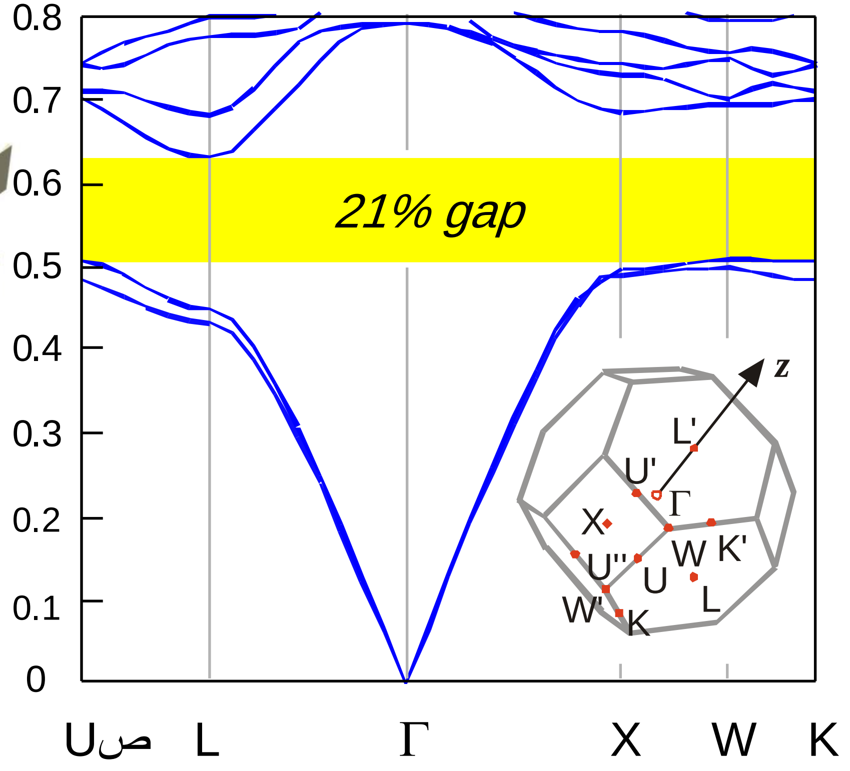


I: rod layer



II: hole layer





gap for n > ~4:1

[S. G. Johnson *et al.*, *Appl. Phys. Lett.* **77**, 3490 (2000)]

You, too, can compute photonic eigenmodes!

MIT Photonic-Bands (MPB) package:

`http://ab-initio.mit.edu/mpb`

on Athena:

```
add mpb
```