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Constant Curvature of A Locally Conformal Almost Cosymplectic Manifold

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Abstract. The purpose of the present paper is to discuss the geometrical properties of a locally conformal almost cosymplectic manifold of constant curvature. In particular, the necessary and sufficient conditions for the aforementioned manifold to be of constant curvature have been determined.
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INTRODUCTION

The concept of a constant curvature is one of the important concepts in contact geometry. Blair [2] established that a cosymplectic manifold of constant curvature is locally flat. Moreover, Goldberg and Yano [5] obtained that an almost cosymplectic manifold of constant curvature is cosymplectic if and only if it is locally flat. Seo [11] exclusively, determined a classification of the translation hypersurfaces with a constant mean curvature in an Euclidean space.

PRELIMINARIES

This section provides a summary of the basic concepts and facts which are related to the discussion of our results.

Definition 0.1 [2] *Let M be a $2n+1$ dimensional smooth manifold , η be a differential 1-form called the contact form, ξ be a vector field called the characteristic vector field, Φ be an endomorphism of the module of the vector fields $X(M)$ called a structure endomorphisim, then the triple (η, ξ, Φ) is called an almost contact structure if the following conditions hold*

(1) $\eta(\xi) = 1$; (2) $\Phi(\xi) = 0$; (3) $\eta \circ \Phi = 0$; (4) $\Phi^2 = -id + \eta \otimes \xi$.

Moreover, if there is a Riemannian metric $g = \langle .,. \rangle$ on M such that $\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y),\ X, Y \in X(M)$, then the set of the tensors (η, ξ, Φ, g) is called an almost contact metric structure. In this case the manifold M equipped with this structure is called an almost contact metric manifold.

Definition 0.2 [9] *At each point $p \in M^{2n+1}$, there is a frame in $T_p^c(M)$ of the form $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$, where $\varepsilon_a = \sqrt{2}\pi(e_a)$, $\varepsilon_{\hat{a}} = \sqrt{2}\bar{\pi}(e_a)$, $\hat{a} = a + n$, $\varepsilon_0 = \xi_p$. The frame $(p, \varepsilon_0, \varepsilon_1, ..., \varepsilon_n, \varepsilon_{\hat{1}}, ..., \varepsilon_{\hat{n}})$ is called an A-frame. The set of such frames defines a G-structure on M with the structure group $1 \times U(n)$. This G-structure is called an adjoined G-structure space.*

Lemma 0.1 [9] *The matrices components of the tensors Φ_p and g_p in an A-frame have the following forms, respectively:* $(\Phi^i_j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & o \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix}$, $(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix}$, *where I_n is the identity matrix of order n.*

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Definition 0.3 [5] *An almost contact metric structure $S = (\eta, \xi, \Phi, g)$ is called an almost cosymplectic structure ($\mathcal{AC}_\int$-structure) if the following conditions hold:*

1. $d\eta = 0$;
2. $d\Omega = 0$.

Definition 0.4 [10] *A conformal transformation of an $\mathcal{AC}$-structure $S = (\eta, \xi, \Phi, g)$ on a manifold is a transformation from S to an $\mathcal{AC}$-structure $\widetilde{S} = (\widetilde{\eta}, \widetilde{\xi}, \widetilde{\Phi}, \widetilde{g})$ such that $\widetilde{\eta} = e^{-\sigma}\eta$, $\quad \widetilde{\xi} = e^{\sigma}\xi$, $\quad \widetilde{\Phi} = \Phi$, $\quad \widetilde{g} = e^{-2\sigma}g$, where σ is a determining function of the conformal transformation.*

Definition 0.5 [10] *An $\mathcal{AC}$-structure S on a manifold M is said to be a locally conformal almost cosymplectic ($\mathcal{LCAC}_\int$-structure) if the restriction of this structure to some neighborhood U of an arbitrary point $p \in M$ admits a conformal transformation of an almost cosymplectic structure. This transformation is called a locally conformal. A manifold M equipped with an $\mathcal{LCAC}_\int$-structure is called a locally conformal almost cosymplectic manifold ($\mathcal{LCAC}_\int$-manifold).*

Lemma 0.2 [6] *In the adjoined G-structure space, the Cartan structural equations of $\mathcal{LCAC}_\int$-manifold have the following forms:*

1. $d\omega^a = -\omega^a_b \wedge \omega^b + B^{ab}_c \omega^c \wedge \omega_b + B^{abc}\omega_b \wedge \omega_c + B^a_b \omega \wedge \omega^b + B^{ab}\omega \wedge \omega_b$;
2. $d\omega_a = \omega^b_a \wedge \omega_b + B^c_{ab}\omega_c \wedge \omega^b + B_{abc}\omega^b \wedge \omega^c + B^b_a \omega \wedge \omega_b + B_{ab}\omega \wedge \omega^b$;
3. $d\omega = C_b \omega \wedge \omega^b + C^b \omega \wedge \omega_b$;
4. $d\omega^a_b = -\omega^a_c \wedge \omega^c_b + A^{acd}_b \omega_c \wedge \omega_d + A^a_{bcd}\omega^c \wedge \omega^d + A^{ac}_{bd}\omega^d \wedge \omega_c + A^a_{bc0}\omega \wedge \omega^c + A^{ac0}_b \omega \wedge \omega_c$.

Here, B^{abc}, B_{abc}; B^{ab}, B_{ab}; B^a_b, B^b_a; C^{ab}, C_{ab}; C^b, C_b; A^{acd}_b, A^b_{acd}; A^{ac}_{bd}; A^{ac0}_b, A^b_{ac0}; B^{abci}, B_{abci}; D^{abi}, D_{abi} and σ_{ij} are smooth functions in the adjoined G-structure space. B^{abc} are the components of the second structure tensor.

Lemma 0.3 [7] *In the adjoined G-structure space, the components of the Riemannian curvature tensor of $\mathcal{LCAC}_\int$-manifold have the following forms:*

1. $R^a_{bcd} = 2(A^a_{bcd} + 4\sigma^{[a}\delta^{h]}_{[c}B_{d]hb} - \sigma_0 B_{b[d}\delta^a_{c]})$;
2. $R^a_{\hat{b}cd} = 2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} + 2B^{hab}B_{hdc} - \delta^a_{[c}\delta^b_{d]}\sigma^2_0)$;
3. $R^a_{bc\hat{d}} = A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_c \sigma_{[h}\delta^d_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^a_c\delta^d_b\sigma^2_0$;
4. $R^{\hat{a}}_{bcd} = 2(2B_{[c|ab|d]} - 2\sigma_{[a}B_{b]cd} + B_{a[c}B_{d]b})$;
5. $R^a_{0cd} = 2(\sigma_{0[c}\delta^a_{d]} + B^{ab}B_{bcd} - 2\sigma^{[a}\delta^{h]}_{[c}B_{d]h})$;
6. $R^a_{b\hat{c}0} = A^{ac0}_b + \sigma_b B^{ac} - \delta^c_b \sigma_0 \sigma^a$;
7. $R^{\hat{a}}_{bc0} = 2B_{cab0} + 2B_{cab}\sigma_0$;
8. $R^a_{0b0} = -\delta^a_b\sigma_{00} - \delta^a_b\sigma^2_0 - B_{cb}B^{ac} - \sigma^a_b - \sigma^a\sigma_b + 2\sigma^{[a}\delta^{c]}_b\sigma_c$;
9. $R^a_{0\hat{b}0} = 2\sigma_0 B^{ab} - D^{ab0} - \sigma^{ab} - \sigma^a\sigma^b + 2B^{bac}\sigma_c$.

and the other components are conjugate to the above components or can be obtained by the properties of symmetry for R or are equal to zero.

Definition 0.6 [7] *An almost contact manifold is called a Kenmotsu manifold if the equality*

$$\nabla_X(\Phi)Y = \langle X, Y\rangle\xi - \eta(Y)X;$$

holds for each $X, Y \in X(M)$

Definition 0.7 [4] *The Ricci tensor is a tensor of type $(2, 0)$ which is defined by $r_{ij} = -R^k_{ijk}$.*

Lemma 0.4 [1] *In the adjoined G-structure space, the components of the Ricci tensor of $\mathcal{LCAC}_\int$-manifold are given as follows:*

1. $r_{ab} = 2(-2A^c_{(ab)c} - 4(\sigma^{[c}\delta^{h]}_{[b}B_{c]ha} + \sigma^{[c}\delta^{h]}_{[a}B_{c]hb}) + \sigma_0 B_{a[c}\delta^c_{b]} + \sigma_0 B_{b[c}\delta^c_{a]} + 2\sigma_0 B_{ab} - D_{ab0} - \sigma_{ab} - \sigma_a\sigma_b + 2B_{bah}\sigma^h$;

2. $r_{\hat{a}b} = -4(\delta^{[a}_{[b}\sigma^{c]}_{c]} - \sigma_{[c}\delta^{b}_{h]}\sigma^{[h}\delta^{a]}_{c} - \frac{1}{2}\sigma^{[a}\delta^{h]}_{b}\sigma_h + B^{hca}B_{hcb} + B^{bch}B_{cha}) + (B^{cb}B_{ac} - B_{hb}B^{ah}) + A^{cb}_{ac} - \delta^{a}_{b}\sigma_{00} - 2n\sigma^2_0 - \sigma^a_b - \sigma^a\sigma_b$;
3. $r_{a0} = -A^{c}_{ac0} - \sigma^c B_{ac} + n\sigma_0\sigma_a + 2(\sigma_{0[c}\delta^{c}_{a]} + B^{cb}B_{bca} - 2\sigma^{[c}\delta^{h]}_{[c}B_{a]h})$;
4. $r_{oo} = -2n(\sigma_{00} + \sigma^2_0) - 2B_{hc}B^{ch} - 2(\sigma^c_c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta^{h]}_{c}\sigma_h$.

and the other components can be obtain by taking the conjugate operator to the above components.

Definition 0.8 [8] *An $\mathcal{AC}$-manifold is said to be of constant curvature k, if the Riemannian curvature tensor satisfies the relations $R_{ijkl} = k(g_{ik}g_{jl} - g_{il}g_{jk})$.*

Lemma 0.5 [8] *In the G-adjoined structure space, the nonzero components of the Riemannian curvature tensor of a manifold M of constant curvature have the forms*

$$R_{\hat{a}\hat{b}cd} = k\delta^{ab}_{cd} \quad R_{\hat{a}bc\hat{d}} = k\delta^a_c\delta^b_d \quad R_{\hat{a}0c0} = k\delta^a_c \quad .$$

Definition 0.9 [8] *The manifolds of constant curvature are called the special forms. The manifolds of zero constant curvature are said to be planes.*

Definition 0.10 [3] *A pseudo-Riemannian manifold M is called an η-Einstein manifold of type (α, β) if its Ricci tensor satisfies the equation $r = \alpha g + \beta\eta \otimes \eta$, where α and β are suitable smooth functions. If $\beta = 0$, then M is called an Einstein manifold.*

Definition 0.11 *The Riemannian curvature tensor of $\mathcal{LCAC}_f$-manifold has the first special property, if*

$$\eta \circ R(\Phi^2X, \Phi^2Y)\Phi^2Z = \eta \circ (R(\Phi X, \Phi Y)\Phi^2Z + R(\Phi^2X, \Phi Y)\Phi Z + R(\Phi X, \Phi^2Y)\Phi Z);$$

hold for each $X, Y, Z \in X(M)$.

Definition 0.12 *The Riemannian curvature tensor of $\mathcal{LCAC}_f$-manifold has the second special property, if*

$$\eta \circ [R(\Phi^2X, \xi)\Phi^2Y + \eta \circ (R(\Phi X, \xi)\Phi Y] = 0;$$

hold for each $X, Y, \in X(M)$.

THE MAIN RESULTS

Theorem 0.1 *The necessary and sufficient conditioni for a $\mathcal{LCAC}_f$ to be a manifold of constant curvature k is $A^{ad}_{bc} = B^{abc} = B^{ab} = \sigma^a = \sigma_{00} = 0$. Moreover, $k = -\sigma^2_0$.*

Proof. Comparing the components of the Riemannian curvature tensor in the Lemmas 0.3 and 0.5, we have

$$2(2\delta^{[b}_{[c}\sigma^{a]}_{d]} + 2B^{hab}B_{hdc} - \delta^a_{[c}\delta^b_{d]}\sigma^2_0) = k\delta^{ab}_{cd} \quad .$$

Consequently, we get

$$k = -\sigma^2_0 \quad .$$

Moreover, we have

$$A^{ad}_{bc} + 4\sigma^{[a}\delta^{h]}_{c}\sigma_{[h}\delta^{d}_{b]} - 4B^{dah}B_{chb} + B^{ad}B_{bc} - \delta^a_c\delta^d_b\sigma^2_0 = k\delta^a_c\delta^b_d \quad .$$

Making use of the equality $R^{\hat{a}}_{bcd} = 0$, consequently we obtain

$$A^{ad}_{bc} = 0 \quad .$$

Now, according to the Lemma 0.3, item 3, it follows that $B^{abc} = 0$.
According to the relation $R^a_{0c0} = R_{\hat{a}0c0}$, we get

$$-\delta^a_c\sigma_{00} - \delta^a_c\sigma^2_0 - B_{hb}B^{ah} - \sigma^a_c - \sigma^a\sigma_c + 2\sigma^{[a}\delta^{h]}_{c}\sigma_h = k\delta^a_c \quad .$$

Now, making use of $R^a_{0\hat{b}0} = 0$, we get $\sigma^a\sigma^b = 0$ and according to the Lemma 0.2, items 9 and 10, we have $\sigma^a_a = 0$, which means

$$\sigma_{00} = 0 \quad .$$

Conversely, we can get the requirement directly from the lemmas 0.2 and 2.3. □

Theorem 0.2 *The $\mathcal{LCAC}_f$-manifold M which is a special form has nonpositive curvature. Moreover,*
(1) *M is a conformally flat Kenmotsu manifold if and only if $k = -1$;*
(2) *If $k = 0$, then M is a locally flat cosymplectic manifold.*

Theorem 0.3 *Suppose that a $\mathcal{LCAC}_f$-manifold M is an η-Einstein manifold of type (α, β), then $\alpha = \frac{1}{n}A^{(ca)}_{ac} - \sigma_{00} - 2n\sigma_0^2 - \frac{1}{n}(\sigma^a_a + \sigma^a\sigma_a)$ and $\beta = -\frac{1}{n}A^{(ca)}_{ac} - (2n-1)\sigma_{00} + (\frac{1}{n} - 2)(\sigma^a_a + \sigma^a\sigma_a)$ hold. In addition, if M is a manifold of constant curvature, then M is an Einstein manifold with a cosmological constant $\alpha = -2n\sigma_0^2$.*

Proof. Comparing the components of the Ricci tensor in the Lemma 0.3 and 0.5 , we have

$$-4(\delta^{[a}_{[b}\sigma^{c]}_{c]} - \sigma_{[c}\delta^b_{h]}\sigma^{[h}\delta^{a]}_c - \frac{1}{2}\sigma^{[a}\delta^{h]}_b\sigma_h + B^{hca}B_{hcb} + B^{bch}B_{cha})$$
$$+(B^{cb}B_{ac} - B_{hb}B^{ah}) + A^{cb}_{ac} - \delta^a_b\sigma_{00} - 2n\sigma_0^2 - \sigma^a_b - \sigma^a\sigma_b = \alpha\delta^a_b \quad . \tag{0.1}$$

Symmrtrizing and antisymmrtrizing (0.1) by the indices (a, h) and then symmrtrizing by the indices (b, c), we get

$$A^{(cb)}_{ac} + B^{cb}B_{ac} - \delta^a_b\sigma_{00} - 2n\sigma_0^2 - \sigma^a_b - \sigma^a\sigma_b = \alpha\delta^a_b \quad . \tag{0.2}$$

Contracting (0.2) by the indices (a, b), we obtain

$$\alpha = \frac{1}{n}A^{(ca)}_{ac} + \frac{1}{n}B^{ca}B_{ac} - \sigma_{00} - 2n\sigma_0^2 - \frac{1}{n}(\sigma^a_a + \sigma^a\sigma_a) \quad .$$

Moreovere, we have

$$-2n(\sigma_{00} + \sigma_0^2) - 2B_{hc}B^{ch} - 2(\sigma^c_c + \sigma^c\sigma_c) + 4\sigma^{[c}\delta^{h]}_c\sigma_h = \alpha + \beta \quad . \tag{0.3}$$

Symmrtrizing and antisymmrtrizing (0.3) by the indices (c, h), we conclude

$$\beta = -\frac{1}{n}A^{(ca)}_{ac} - \frac{1}{n}B^{ca}B_{ac} - (2n-1)\sigma_{00} + (\frac{1}{n} - 2)(\sigma^a_a + \sigma^a\sigma_a) \quad .$$

Now, if M is of constant curvature, then by the Theorem 0.1, directly we get that M is an Einstein manifold with a cosmological constant α. □

Theorem 0.4 *The Riemannian curvature tensor of $\mathcal{LCAC}_f$-manifold M has the first special property, if M has the constant curvature k.*

Theorem 0.5 *The Riemannian curvature tensor of $\mathcal{LCAC}_f$-manifold M has the second special property, if M has the constant curvature $k = 0$.*

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